Localization in duo rings

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1. *Introduction.* A duo ring is a ring in which every one sided ideal is two sided. Such rings are called two-sided rings in [8] and [9]. Duo rings were so named by FELLER in [2]. Several examples of noncommutative duo rings were given in [4]. In [1], SCHILLING first studied noncommutative valuation rings and lemma 2 of [1] shows that they are duo rings. Since every noncommutative totally ordered group is the value group of some valuation [5], the collection of noncommutative valuation rings is a large class of duo rings.

It is easy to see that R is a duo ring if and only if $aR=Ra$ for all $a\in R$. Thus when R is an integral domain, R has a (left and right) division ring of quotients $D=\{a^{-1}b \mid a, b\in R,\ a\neq 0\}=\{ba^{-1} \mid a, b\in R,\ a\neq 0\}$. In [1], Schilling showed that if R is valuation ring with unique maximal ideal P then $x^{-1}Px=P$ and $x^{-1}Rx=R$ for all $x\in D^*=D\ \{0\}$. P and R are said to be invariant under inner automorphisms of D^*. Schilling used the invariance of R and P to construct a quotient ring R_P with properties similar to those in the case where R is commutative.

In this paper we extend Schillings ideas to integral domains which are duo rings.

Let R be a duo ring with identify $1\neq 0$. A subset S of R is called a multiplicative system in R if $0\notin S$ and $s_1s_2\in S$ for all $s_1, s_2\in S$. The following theorem shows that prime ideals in duo rings have the same characterization as prime ideals in commutative rings. Thus let P be an ideal of R.

Theorem. *The following are equivalent in* R.
(i) P *is a prime ideal of* R.
(ii) *For* $a, b\in R$, *if* $ab\in P$, *then* $a\in P$ *or* $b\in P$.
(iii) $R\setminus P$ *is a multiplicative system in* R.

PROOF. (i)⇔(ii) is found in [3]. (ii)⇔(iii) is clear.

In this paper all rings are integral domains with identity $1\neq 0$. Section 2 characterizes duo rings in terms of groups of divisibility. Section 3 treats the problem of localization with respect to multiplicative systems.

2. *Groups of divisibility in integral domains which are duo rings.*

Let R be an integral domain which is a duo ring and let D denote the division ring of quotients of R. U denotes the multiplicative group of units of R and D^* denotes the multiplicative group of nonzero elements of D. This section characterizes duo domains in terms of groups of divisibility.

Theorem 2.1. U *is a normal subgroup of* D^*.

PROOF. Let $x\in D^*$, $u\in U$. Then $x=a^{-1}b=ca^{-1}$ for some $a, b, c\in R$. Thus $xux^{-1}=a^{-1}bub^{-1}a=ca^{-1}uac^{-1}$. So it is sufficient to show that $a^{-1}ua\in U$, $aua^{-1}\in U$ for all $a\in R$. So let $a\in R$, $a\neq 0$, and let $u\in U$. Then $a^{-1}ua=a^{-1}ar=r\in R$, where $u^{-1}a=as$. So $(a^{-1}ua)(a^{-1}u^{-1}a)=1=rs=sr$. Thus $r=a^{-1}ua\in U$. Similarly, $aua^{-1}\in U$ for all $a\in R$, $a\neq 0$ and U is normal in D^*.

Definition 2.2. D^*/U is called the group of divisibility of R.

Proposition 2.3. *D^*/U is a partially ordered group where $xU\leqq yU \Leftrightarrow x^{-1}y\in R$.*

The proof is straight forward and is omitted.

Let v: $D^*\to D^*/U$ denote the natural map. Then

(1) $v(xy)=xyU=(xU)(yU)=v(x)v(y)$.

(2) If $x, y\in D^*$ and $x+y\neq 0$, then $v(x+y)\geqq v(t)$ for any $t\in D^*$ such that $v(t)\leqq v(x)$ and $v(t)\leqq v(y)$.

For suppose $v(t)\leqq v(x)$ and $v(t)\leqq v(y)$. Then $t^{-1}x, t^{-1}y\in R$ and so $t^{-1}x+t^{-1}y=t^{-1}(x+y)\in R$. Then $tU\leqq (x+y)U$, i.e., $v(t)\leqq v(x+y)$.

Definition 2.4 [6, pg. 10]. A partially ordered (p.o.) group is said to be directed if any pair $a, b\in G$ has a lower bound (l.b.) $c\in G$ (equivalently, if any pair $a, b\in G$ has an upper bound).

Proposition 2.5. *With R, D, D^*, U as above, D^*/U is a directed group.*

PROOF. Let $xU, yU\in D^*/U$. Then $x=a^{-1}b, y=c^{-1}d$ for some $a, b, c, d\in R$. Then $a^{-1}U\leqq xU$ and $c^{-1}\leqq yU$. This gives $(ac)^{-1}U\leqq a^{-1}U$ and $(ac)^{-1}U\leqq c^{-1}U$. For $aca^{-1}=c_0aa^{-1}=c_0$, where $ac=c_0a$ for some $c_0\in R$, and $acc^{-1}=a\in R$. Thus $(ac)^{-1}U$ is a lower bound for xU and yU and D^*/U is directed.

Now, let D be a division ring and let G be a directed group. Let v: $D\to G\cup\{\infty\}$ be a function satisfying (1). (2) of Proposition 2.3 and (3) $v(0)=\infty$ and v maps D onto $G\cup\{\infty\}$. Let $R=\{x\in D \mid v(x)\geqq e\}$. The following extends Schilling [1] and YaKabe [7]. We assume without loss of generality that G is generated by $U(e)=\{x\in G \mid x\geqq e\}$ [6].

Proposition 2.6. (i) *R is a subring of D;* (ii) *R is a duo ring;* (iii) *D is the ring of quotients of R.*

PROOF. (i) Clearly $v(1)=e$. Since $(-1)^2=1$, this gives $v(-1)=v(-1)^{-1}$. Let $a\in G$ be a lower bound for $v(-1)=v(-1)^{-1}$ and e. Then $a\leqq v(-1)$ and $a\leqq e$. This gives $a\leqq v(-1)=v(-1)^{-1}\leqq a^{-1}\leqq e$ so $v(-1)\leqq e$ and $v(-1)=v(-1)^{-1}\geqq e$, giving $v(-1)=e$. It follows that $v(x)=v(-x)$ for all $x\in D$, and that R is a subring of D. Let $a\in R$, and let A $A=\{x\in R \mid v(x)\geqq v(a)\}$. Clearly $aR=A=Ra$ and R is duo. That D is the ring of quotients of R follows from the fact that v maps D onto $G\cup\{\infty\}$.

We combine the above to get the following.

Theorem 2.7. *Let D be a division ring. A subring R of D is a duo ring with D as ring of quotients if and only if there is a directed group $(G, \leqq)$ and a map from D onto $G\cup\{\infty\}$ satisfying properties* (1), (2), (3).

3. *Multiplicative systems and localization in duo domains.*

Let R be a duo domain with D as division ring of quotients. Let T be a multiplicative system in R. The collection $\mathcal{S}$ of ideals A of R such that $A\cap T=\emptyset$ is not empty and Zorn's lemma shows that $\mathcal{S}$ contains maximal elements. The argument that the maximal elements of $\mathcal{S}$ are prime ideals of R is similar to the commutative case using the fact that $Ra=aR$ for all $a\in R$. Let P_λ denote the collection of maximal elements of $\mathcal{S}$, and let $S=R\setminus(\bigcup_\lambda P_\lambda)=\bigcap_\lambda(R\setminus P_\lambda)$. Then S is a multiplicative system in in R and $T\subset S$. When R is commutative, S is called the saturation of T and $R_T=R_S$ (see [10], pg. 13).

Lemma 3.1. *Let S be a saturated multiplicative system in R, and let $s_1,s_2\in S$. Then $s_1s_2=s_2s_1=s_2s$, for some $s_1, s_2\in S$.*

PROOF. $s_1s_2=s_2s_2'$ for some $s_1'\in R$. If $s_1'\notin S=\bigcap_\lambda(R\setminus P_\lambda)$ then $s_1'\in P_\lambda$ for some λ and then $s_1s_2\in S\cap P_\lambda=\emptyset$, a contradiction. So $s_1'\in S$. Similarly $s_2'\in S$.

We observe that the above lemma holds when R is not an integral domain.

Proposition 3.2. *Let S be a saturated multiplicative system in R. Then R_S is a subring of D, where $R_s=\{s^{-1}a_S|s\in S, a\in R\}$.*

PROOF. It is straight forward to show that R_S is closed under multiplication. To see that R_S is closed under addition, let $s^{-1}a, t^{-1}b\in R_S$. Then $s^{-1}a+t^{-1}b=$ $=(s't)^{-1}(ta+sb)$, where $ts=s't$. The above lemma show that $s'\in S$.

Definition 3.3. If T is a multiplicative system in R, we define $R_T=R_S$, where S is the saturation of T.

In the case where R is a noncommutative valuation ring with unique maximal ideal P, SCHILLING [1] showed that P is invariant under all inner automorphisms, i.e., that $xPx^{-1}=P$ for all $x\in D^*$. It follows that in a duo domain R, a prime ideal P is invariant if and only if $S=R\setminus P$ is invariant.

Definition 3.4. Let T be a saturated multiplicative system in R. T is said to be invariant iff $xTx^{-1}=T$ for all $x\in D^*$.

The following characterizes invariant multiplicative systems.

Theorem 3.5. *Let $T=\bigcap_\lambda(R\setminus P_\lambda)$ be a saturated multiplicative system in R, where $\{P_\lambda\}=\mathcal{S}$ is a collection of prime ideals of R. Then T is invariant iff $xP_\lambda x^{-1}\in\mathcal{S}$ for all $x\in D^*$, $P_\lambda\in\mathcal{S}$.*

PROOF. It is easy to see that if $x\in D^*$, $P_\lambda\in\mathcal{S}$, then $xP_\lambda x^{-1}$ is a prime idealof R.

Suppose that $\mathcal{S}$ is closed under inner automorphisms. Let $t\in T, x\in D^*$. If $xtx^{-1}\notin T$, then $xtx^{-1}\in P_\lambda$ for some λ. Say $xtx^{-1}=a\in P_\lambda$. Then $t=x^{-1}ax\in$ $\in x^{-1}P_\lambda x\in\mathcal{S}$. Then $t\in T\cap(\bigcup P_\lambda)=[R\setminus(\bigcup P_\lambda)]\cap(\bigcup P_\lambda)=\emptyset$, a contradiction. So $xTx^{-1}\subseteq T$ for all $x\in D^*$. It follows that $xTx^{-1}=T$ for all $x\in D^*$.

On the other hand, if T is invariant, then $xtx^{-1}\in T$ for all $x\in D^*, t\in T$. So $T=xTx^{-1}\cap P_\lambda=\emptyset$ for all $P_\lambda\in\mathcal{S}$. Thus $T\cap x^{-1}P_\lambda x=\emptyset$ for all $P_\lambda\in\mathcal{S}$ and $\mathcal{S}$ is closed under inner automorphisms.

In general, if T is a saturated multiplicative system in R, it is not always true that R_T is a duo domain. The following gives a sufficient condition for R_T to be a duo domain.

Proposition 3.6. *Let T be a saturated multiplicative system in R. If T is invariant, then R_T is a duo ring.*

PROOF. Suppose T is invariant. Then for $t\in T$, $a\in R^*$, we have $a^{-1}ta=t'\in T$, so $ta=at_1$, and $ata^{-1}=t_2\in T$, i.e., $at=t_2a$. Let $x=s^{-1}b$, $y=t^{-1}c$ be arbitrary nonzero elements of R_M. Then $xy=(s^{-1}b)(t^{-1}c)=s^{-1}(bt^{-1})c$. Since T is invariant, $bt^{-1}=t_1^{-1}b$ for some $t_1\in T$. Then $xy=s^{-1}(t_1^{-1}b)c$. Now, $bc=c'b$ for some $c'\in R$ and since M is saturated, $s^{-1}t_1^{-1}=t_2^{-1}s^{-1}$ for some $t_2\in T$. Then $xy=t_x^{-1}s^{-1}c'b$. We have $s^{-1}c'=c''s^{-1}$ for some $c''\in R$, so $xy=(t_x^{-1}c'')s^{-1}b=y'x$, where $y'=t_2^{-1}c$. Thus $xR_T\subseteqq R_Tx$. Similarly $R_Tx\subseteqq xR_T$ and R_T is a duo ring.

Lemma 3.7. (1) *If $\{T_\lambda\}$ is a collection of subsets of R then for any $x\in D^*$, $x^{-1}(\bigcap T_\lambda)x=\bigcap x^{-1}T_\lambda x$.*

(2) *If S is a multiplicative system in R, then for any $x\in D^*$, $x^{-1}Sx$ is a multiplicative system in R and $x^{-1}R_Sx=R_{x^{-1}Sx}$.*

PROOF. (1) is straight forward. It is also easy to show that $x^{-1}Sx$ is a multiplicative system if S is a multiplicative system. So let $y\in x^{-1}R_Sx$. Then $y=$ $=x^{-1}(s^{-1}r)x=(x^{-1}s^{-1}x)(x^{-1}rx)=(x^{-1}sx)^{-1}(x^{-1}rx)\in R_{x^{-1}Sx}$ since $x^{-1}rx\in R$. On the other hand, let $Z\in R_{x^{-1}xS}$, say $Z=(x^{-1}sx)^{-1}r$. Since $R=x^{-1}Rx$, we have $r=x^{-1}tx$ for some $t\in R$. Then $Z=(x^{-1}sx)^{-1}r=(x^{-1}s^{-1}x)x^{-1}tx=x^{-1}(s^{-1}t)x\in$ $\in x^{-1}R_Sx$, and we have equality.

Now, let S be a saturated multiplicative system in R, say $S=\bigcap\{R-P\mid P\in\mathscr{F}\}$, where $\mathscr{F}$ is the collection of prime ideals of R which are maximal with respect to the property that $P\cap S=\emptyset$. If R_S is a duo ring, then $R_S=x^{-1}R_Sx=R_{x^{-1}xS}=$ $=R_{\cap x^{-1}(R\setminus P)x}$. Thus $\mathscr{F}$ is closed under inner automorphisms and S is invariant by 3.5 above. This together with 3.7 above gives

Theorem 3.8. *Let S be a saturated multiplicative system in R. Then R_S is a duo ring $\Longleftrightarrow$ S is invariant.*

Let S be a saturated multiplicative system in R. $R_S=\{s^{-1}r\mid s\in S, r\in R\}$. It can be easily shown that $R_S=\{s^{-1}r\mid s\in S, r\in R\}=\{rs^{-1}\mid s\in S, r\in R\}$. Let A be an ideal of R. The right extension of A to R_S is

$$A_r^e=AR_S=\left\{\sum a_ir_is_i^{-1}\middle| a_i\in A,\ r_i\in R,\ s_i\in S\right\}=\{as^{-1}\mid a\in A,\ s\in S\}.$$

Similarly, the left extension of A to R_S is

$$A_l^e=R_SA=\left\{\sum s_i^{-1}r_ia_i\middle| s_i\in S,\ r_i\in R,\ a_i\in A\right\}=\{s^{-1}a\mid s\in S,\ a\in A\}.$$

Proposition 3.9. (1) *The right extension of A to R_S is a right ideal of R_S* (2) *The left extension of A to R_S is a left ideal of R.*

PROOF. (1) Suppose $as^{-1}\in A_r^e$ and $rt^{-1}\in R_S$. Then $(as^{-1})(rt^{-1})=a(s^{-1}rs)s^{-1}t^{-1}$. Then $s^{-1}rs\leftarrow R$, so $a(s^{-1}rs)=a'\in A$, and so $(as^{-1})(rt^{-1})=a'(ts)^{-1}\in AR_S$. Similarly, R_SA is a left ideal of R_S.

Definition 3.10. Let A be an ideal of R, S a saturated multiplicative system in R. A is said to be S-invariant if $s^{-1}As=A$ for all $s\in S$.

Clearly, if A is invariant, then A is S-invariant for any S.

Let P be any prime ideal of R, and let $S(P)=R\setminus P$. Then P is $S(P)$ invariant. For let $s\in S=S(P)$ and $a\in P$. Then if $s^{-1}as=t\in S$, then $as=st\in P$ and $s\notin P$, $t\notin P$ a contradiction. Thus $s^{-1}Ps\subseteq P$, and P is $S(P)$-invariant.

Proposition 3.11. *Let A be an ideal of R, S a saturated multiplicative system in R, with $A\cap S=\emptyset$. If A is S-invariant, then $AR_S=R_SA$ is an ideal of R.*

PROOF. Let $as^{-1}\in AR_S$. Then $as^{-1}=s^{-1}(sas^{-1})$ and $sas^{-1}\in A$ since A is S-invariant. Thus $AR_S\subseteq R_SA$. Similarly $R_SA\subseteq AR_S$.

It is clear that AR_S is a proper right ideal (1.8) of $R_S\Leftrightarrow A\cap S=\emptyset$; and similarly for R_SA.

As is the case when R is commutative, if P is a prime ideal of R and $S=R\setminus P$, we write R_P for R_S.

Corollary 3.12. *For any prime ideal P of R, R_P is a local ring with unique maximal ideal PR_P.*

Now, let A, B be ideals of R.

Definition [13, pg. 254]. $A\,.\!\cdot B=\{d\in R\,|\,Bd\subseteq A\}$. $A\,.\!\cdot B$ is an ideal of R [13].

Let $\{M_\lambda\}$ denote the collection of maximal ideals of R. The next lemma is similar to [12, pg. 94].

Lemma 3.13. *Let A be an ideal of R, and let $x\in R$. If $x\in AR_{M_L}$ for all λ, then $x\in A$.*

PROOF. If $x\in AR_{M_\lambda}$ then $x=at_m^{-1}$, where $a\in A$, $t_m\in R\setminus M_\lambda$. Then $xt_m=a\in A$, and so $A\,R_x\not\subseteq M_\lambda$ since $t_m\in R\setminus M_\lambda$. Thus $A\,.\!\cdot R_x$ is contained in no maximal ideal of R and $A\,.\!\cdot R_x=R$. Thus $1\in A\,.\!\cdot R_x$ and $x\in A$.

Corollary 3.14. $R=\bigcap\limits_\lambda R_{M_\lambda}$.

PROOF. Clearly $R\subseteq\bigcap\limits_\lambda R_{M_\lambda}$. Let $z=y^{-1}x\in\bigcap\limits_\lambda R_{M_\lambda}$ for some $x, y\in R$. Then $x\in(Ry)R_{M_\lambda}=$ for all λ. Thus $x\in Ry=yR$, and $y^{-1}x=z\in R$, and $R=\bigcap\limits_\lambda R_{M_\lambda}$.

Now, let S be an invariant multiplicative system in R so that R_S is a duo ring. Let A be an ideal of R and let $\mathscr{A}$ be an ideal of R_S.

Definition 3.15 [11, pg. 218].

(i) The extension of A to R_S is $A^e=AR_S=R_SA$.

(ii) The contraction of $\mathscr{A}$ to R is $\mathscr{A}^c=\mathscr{A}\cap R$.

Then as in [11, pg. 219] we have the following

Lemma 3.16. *Let $\mathscr{A}, \mathscr{B}$ be ideals of R_S, A, B ideals of R.*

$$\mathscr{A}\subseteq B\Rightarrow\mathscr{A}^c\subseteq\mathscr{B}^c;\ A\subseteq B\Rightarrow A^e\subseteq B^e \tag{1}$$

$$\mathscr{A}^{ce}\subseteq\mathscr{A};\ A^{ec}\supseteq A \tag{2}$$

$$\mathscr{A}^{cec}=\mathscr{A}^c;\ A^{ece}=A^e \tag{3}$$

Corollary 3.17. [11, p. 223]. *Let $\mathscr{A}$ be an ideal of R_S. Then $\mathscr{A}=\mathscr{A}^{ce}$, i.e., every ideal of R_S is an extended ideal of R.*

PROOF. Let $a'\in\mathscr{A}$. Then $a'=s^{-1}$, for some $s\in S$, $r\in R$. Then $r\in\mathscr{A}^c$, and $a'\in\mathscr{A}^{ce}$, and $\mathscr{A}\subseteqq\mathscr{A}^{ce}$. Clearly $\mathscr{A}^{ce}\subseteqq\mathscr{A}$.

It is clear that (i) if A is an invariant ideal of R, then $A^e=AR_S$ is an invariant ideal of R_S: and (ii) if $\mathscr{A}$ is an invariant ideal of R_S, then $\mathscr{A}^c=\mathscr{A}\cap R$ is an invariant ideal of R.

Theorem 3.18. *Let $\mathscr{P}$ be an invariant prime ideal of R_S. Then $\mathscr{P}=PR_S$ where $P=\mathscr{P}\cap\mathscr{R}=\mathscr{P}^c$ is an invariant prime of R and $R_p=(R_S)_{PR_S}$.*

PROOF. Let $T=R\setminus P$. Then $S\subseteqq T$ since $P\cap S=\emptyset$. Let $x\in R_p$. Then $x=t^{-1}r$, $t\in T$, $r\in R$. We have $\mathscr{P}=PR_S$ by corollary 3.17 above, and so $PR_S=\{s^{-1}a|s\in S,\ a\in P\}$. Then $R_S\setminus PR_S=\{s^{-1}t|s\in S,\ t\in P=\{s^{-1}t|s\in S,\ t\in T.$ If $x\in(R_S)_{PR_S}$, then $x=(s^{-1}t)^{-1}(s_1^{-1}r)$, where $t\in T, s_1s_1\in S$. Then $x=t^{-1}ss_1^{-1}r=t^{-1}s_1^{-1}(s_1ss_1^{-1})r=(s_1t)^{-1}(s''r)\in R_P$. So $(R_S)_{PR_S}\subseteqq R_P$. On the other hand, $T\subseteqq R_S-PR_S$ and $R\subseteqq R_S$, so $R_P=R_T\subseteqq(R_S)_{PR_S}$.

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