Extendibility of *-representations from *-ideals of *-semigroups

By ZOLTÁN SEBESTYÉN (Budapest)

In a previous paper [3] the author proved a result concerning extendibility of a *-representation (on Hilbert space) of a *-ideal to the whole *-algebra under a relatively simple condition known from the same problem with respect to C^*-seminorms [4].

The aim of this note is to extend the extendibility question to a more general case. First we shall prove that a *-representation of a *-ideal of a *-semigroup is extendible to the whole semigroup if (and only if) the preceding condition ((1) below) holds. The case when our *-semigroup is embedded in a Banach *-algebra the extendibility problem is known from the author's investigations concerning moment type problems [5].

Let G be a (multiplicative, associative) semigroup with involution (*), briefly a *-semigroup, introduced by SZ.-NAGY [2] concerning dilation problems. A subsemigroup J of G is a *-ideal if it is closed under the involution and $GJ\subset G$ (hence $JG\subset G$). A *-homomorphism of J into $B(H)$, the C^*-algebra of all bounded linear operators on a (complex) Hilbert space H, is called a *-representation of J on H.

Lemma. *Let T be a *-representation of J on the Hilbert space H and let $H_0=\{x\in H: T_h x=0$ for each $h\in J\}$. Each $x\in H$ orthogonal to H_0 belongs to the closed subspace in H spanned by $\{T_h x\colon h\in J\}$, which is orthogonal to H_0.*

PROOF. For a fixed $0\neq x\in H_0^\perp$, where $H_0^\perp$ denote the orthogonal complement to H_0 in H, let

$$H_x := \vee\{T_h x\colon h\in J\}$$

be the closed subspace of H spanned by elements $T_h x, h$ running through J. If y denotes the unique element of H_x with the property that $x-y$ is orthogonal to H_x we have to show $x=y$. First of all $x-y\in H_0$ because

$$\|T_h(x-y)\|^2 = (T_{h^*h}(x-y), x-y) = -(T_{h^*h}y, x-y) = 0$$

holds for any h in J since $y\in H_x$ immediately implies $T_{h^*h}y\in H_x$. Hence we have that

$$\|x-y\|^2 = (x-y, x-y) = (x, x-y) = 0$$

indeed (because x is orthogonal to H_0, hence to $x-y$). The proof of Lemma is complete.

Theorem 1. *Let T be a *-representation of the *-ideal J of the *-semigroup G on a Hilbert space H. There exists a *-representation S of G on H extending T if and only if*

$$p(g) := \inf\{M>0\colon \|T_{gh}\| \leqq M\|T_h\|(h\in J)\} < \infty \quad (g\in G) \tag{1}$$

Proof. Assuming such an S the condition (1) is obviously satisfied because of $gh\in J$ provided $g\in G$, $h\in J$ and because

$$\|T_{gh}\| = \|S_{gh}\| \leqq \|S_g\|\|S_h\| = \|S_g\|\|T_h\|.$$

To prove the sufficiency of (1) we take first S to be zero on $H_0 := \{x\in H\colon T_h x=0$ for each $h\in J\}$, a closed subspace of H on which T is also trivial. For if $x\in H_0^\perp$, i.e. x is orthogonal to H_0, we know from the Lemma that x belongs to $H_x := \vee\{T_h x\colon h\in J\}$, the closed subspace spanned by $T_h x$'s as h runs through J. It is obvious that for any x in $H_0^\perp$ which is of the form $\sum_h T_h x_h$ (finite sum) with $h\in J$, $x_h\in H$, we have to define Sg ($g\in G$) as follows

$$S_g x := \sum_g T_{gh} x_h \quad \text{for any} \quad g \quad \text{in} \quad G. \tag{2}$$

Since such elements are dense in $H_0^\perp$ by Lemma, we have only to prove that (2) is correct for S and gives a bounded operator on $H_0^\top$. The remainder of the proof is then clear, namely that S establishes a *-representation of G on H. But we have

$$\Big\|\sum_h T_{gh} x_h\Big\|^2 = \Big(\sum_h T_{g^*gh} x_h, \sum_h T_h x_h\Big) \leqq \Big\|\sum_h T_{g^*gh} x_h\Big\| \, \Big\|\sum_h T_h x_h\Big\|$$

so that by induction we have also that

$$\Big\|\sum_h T_{gh} x_h\Big\|^{2^{n+1}} \leqq \Big\|\sum_h T_{(g^*g)^{2n}} x_h\Big\| \Big\|\sum_h T_h x_h\Big\|^{2^{n+1}-1} \leqq$$

$$\leqq \sum_h \|T_{(g^*g)^{2^n}h}\| \, \|x_h\| \Big\|\sum_h T_h x_h\Big\|^{2^{n+1}-1} \leqq$$

(by (1))

$$\leqq \sum_h p\big((g^*g)^{2^n}\big)\|T_h\| \, \|x_h\| \Big\|\sum_h T_h x_h\Big\|^{2^{n+1}-1} \; (n=0,1,2,\dots)$$

and hence that

$$\Big\|\sum_h T_{gh} x_h\Big\| \leqq \varliminf_{n\to\infty} p\big((g^*g)^{2^n}\big)^{2^{-n-1}} \Big\|\sum_h T_h x_h\Big\|. \tag{3}$$

We need properties for p as follows

$$p(gg_1) \leqq p(g)p(g_1) \quad (g, g_1\in G) \tag{4}$$

$$p(g^*g) = p(g)^2 \quad (g\in G) \tag{5}$$

The proof of (4) follows by the definition of p since

$$\|T_{gg_1h}\| \leqq p(g)\|T_{g_1h}\| \leqq p(g)p(g_1)\|T_h\|$$

holds for each $g, g_1 \in G$, $h \in J$. But the estimates

$$\|T_{gh}\|^2 = \|T_{h^* g^* gh}\| \leqq \|T_{h^*}\| \|T_{g^* gh}\| \leqq p(g^* g)\|T_h\|^2$$

for $g \in G$, $h \in J$ imply

$$p(g)^2 \leqq p(g^* g) \leqq p(g^*)p(g),$$

$$p(g) \leqq p(g^*) \leqq p((g^*)^*) = p(g)$$

for each $g \in G$ and hence (5) since thus

$$p(g)^2 \leqq p(g^* g) \leqq p(g)^2$$

holds also for any $g \in G$.

To end the proof we have by (2), (3), (4), (5) that

$$\Big\|\sum_h T_{gh} x_h\Big\| \leqq p(g) \Big\|\sum_h T_h x_h\Big\|$$

hence that

$$\|S_g\| \leqq p(g)$$

holds for each $g \in G$ indeed.

Theorem 2. *Let G be a *-semigroup in a Banach *-algebra A such that G generates a norm dense *-subalgebra in A. Let further T be a *-representation of a *-ideal J of G on a Hilbert space H. There exists a *-representation S of A extending T on the same space H if and only if*

$$M := \sup\Big\{\Big\|\sum_g c_g T_{gh}\Big\| \colon h \in J, \ \|T_h\| \Big\|\sum_g c_g g\Big\| \leqq 1\Big\} < \infty \tag{6}$$

holds, where $\{C_g\}$ is any finite set of complex numbers indexed by elements of G.

Proof. The necessity of the condition (6) is obvious since any *-representation of a Banach *-algebra on Hilbert space is automatically continuous (see [1]).

We shall prove that (6) ensures the desired *-representation S on H. Since (6) clearly implies (1) we have by Theorem 1 a *-representation S of G on H such that

$$\|S_g\| \leqq M\|g\| \quad (g \in G) \tag{7}$$

We need an extension to (7) for the *-subalgebra $A(G)$ in A generated by G as follows

$$\|S_a\| \leqq C\|a\| \quad (a \in A(G)) \tag{8}$$

where for an $a = \sum_g c_g g \in A(G)$ the representing operator on H is S_a and $C > 0$ is a constant depending only on A.

$$S_a = \sum_g c_g S_g \quad \Big(a = \sum_g c_g g \in A(G)\Big) \tag{9}$$

is thus well defined. To prove (8) we shall use the same argument as before: for $x=\sum_h T_h x_h$ $(h\in J,\ x_h\in H)$

$$\|S_a x\|^2 = (S_{a^*a}x, x) \leqq \|S_{a^*a}x\|\,\|x\|,$$

$$\|S_a x\|^{2^{n+1}} \leqq \|S_{(a^*a)^{2^n}}x\|\,\|x\|^{2^{n+1}-1} \leqq \sum_h \|S_{(a^*a)^{2^n}}T_h x_h\|\,\|x\|^{2^{n+1}-1} =$$

$$= \sum_h \Big\|\sum_s \lambda_s T_{g_s h} x_h\Big\|\,\|x\|^{2^{n+1}-1} \leqq \sum_h M\Big\|\sum_s \lambda_s g_s\Big\|\,\|T_h\|\,\|x_h\|\,\|x\|^{2^{n+2}-1} =$$

$$= M\|(a^*a)^{2^n}\|\,\|x\|^{2^{n+1}-1}\sum_h \|T_h\|\,\|x_h\|$$

and thus also that

$$\|S_a x\| \leqq \lim \|(a^*a)^{2^n}\|^{2^{-n-1}}\|x\| = r(a^*a)^{1/2}\|x\|,$$

where $r(a^*a)$ denotes the spectral radius of a^*a in A, and $\sum_s \lambda_s g_s$ stands for $(a^*a)^{2^n}$.

By the way

$$r(a^*a)^{1/2} \leqq c\|a\| \quad (a\in A) \tag{10}$$

holds true, as is known for Banach *-algebras from [1, *p.* 196], implying (8) indeed.

Since we have thus really a *-representation S of $A(G)$ on H which is continuous with respect to the norm of A, the norm denseness of $A(G)$ in A implies the statement of the theorem; only (unique) continuous extension of S is needed to A. The proof is complete.

References

[1] F. F. BONSALL—J. DUNCAN, Complete Normed Algebras, Erg. Math. Band 80, *Springer (Berlin—Heidelberg—New York)*, 1973.

[2] F. RIESZ—B. SZ.-NAGY, Functional Analysis, *New York*, 1960.

[3] Z. SEBESTYÉN, On extendibility of *-representations from *-ideals, *Acta Sci. Math. (Szeged)*, **40** (1978) 169—174.

[4] Z. SEBESTYÉN, Every C^*-seminorm is automatically submultiplicative, *Periodica Math. Hungarica*, **10** (1979), 1—8.

[5] Z. SEBESTYÉN, Moment problem for dilatable semigroup of operators, *Acta Sci. Math. (Szeged)*, **45** (1983), 365—376.

DEPARTMENT OF MATH. ANALYSIS II.
EÖTVÖS LORÁND UNIVERSITY
MÚZEUM KÖRÚT 6—8.
1088 BUDAPEST, HUNGARY

(Received July 26, 1982.)