On finiteness of near-rings

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Prompted by a recent paper of PUTCHA and YAQUB [6], we resume the study of finiteness in near-rings, which was pursued by various authors several years ago (see [1], [3], [4] and [5]). Our results, most of which are for distributive near-rings, depend on the following results from [1]:

(I) Every infinite distributive near-ring contains an infinite subring or an infinite subnear-ring with trivial multiplication.

(II) Every infinite nil ring contains an infinite zero ring.

We deal with left near-rings N, denoting the additive group by N^+, the centralizer of an element a in the group N^+ by $C_N(a)$, and the derived group of N^+ by N'. For $S \subseteq N$, the symbols $A_l(S)$, $A_r(S)$ and $A(S)$ denote the left, right, and two-sided annihilators of S; and the symbol $\langle S \rangle$ denotes the subnear-ring generated by S. In one of our theorems we shall speak of FC-groups, defined as those in which each element has only finitely many conjugates, or equivalently as those in which the centralizer of each element has finite index. We shall also mention *Tarski groups,* defined as infinite groups in which every proper subgroup has order p, where p is an odd prime. The existence of Tarski groups has only recently been announced; and, as far as I am aware, a proof has not yet appeared in the literature.

1. Near-rings with finitely many non-nilpotent elements

Theorem 1. *If N is a non-nil distributive near-ring having only a finite number of non-nilpotent elements, then N is finite.*

PROOF. Note that N contains non-zero idempotents; indeed, each non-nilpotent element has an idempotent power. If there exists a non-zero *central* idempotent e, then for each nilpotent element u, $e+u$ is non-nilpotent; otherwise $e=e+u-u$ would be the difference of two commuting nilpotent elements, hence a nilpotent element. Thus $\{e+u \mid u \text{ nilpotent}\}$ is finite; hence N has only finitely many nilpotent elements, so N is finite. In particular, N is finite if it has a multiplicative identity.

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We make our proof by induction on the number $n(N)$ on non-zero idempotents. If $n(N)=1$, let e be the unique non-zero idempotent and let x be any element of N. Then $e+xe-exe$ and $e+ex-exe$ are both non-zero idempotents, hence both equal to e. It follows that e is central, hence N is finite.

Now suppose that our theorem is true for all N_1 with $n(N_1)<k$, and consider N with $n(N)=k$. Let e be a non-zero idempotent, which we may assume to be non-central, and write

$$N = Ne + A_l(e). \tag{1}$$

If $A_l(e)$ is non-nil, then $n(Ne)<k$ and $n(A_l(e))<k$, hence finiteness of N follows by the inductive hypothesis; thus, we assume $A_l(e)$ is nil and let A be any sub-near-ring of $A_l(e)$ with trivial multiplication. Then for $u\in A$ we see that $(e+u)^2=$ $=e+eu$ is a non-zero idempotent, hence $e+u$ is not nilpotent. Thus, $\{e+u\,|\,u\in A\}$ is finite, which forces A to be finite; and an appeal to (I) and (II) shows that $A_l(e)$ is finite. Now write $Ne=eNe+(Ne\cap A_r(e))$, and repeat the above argument to show the second summand is finite. But eNe is finite because it has a multiplicative identity; hence Ne is finite, and by (1), N is finite. This completes the induction.

Applying this theorem yields an extension of Theorems 2.1 and 2.2 of [5]; the proof is omitted, since it is the same as the proof of the ring-theoretic version in [2].

Theorem 2. *Let N be a distributive near-ring having only a finite number $n\geqq 1$ of non-nilpotent zero divisors. Then N is finite.*

As past experience predicts, it is difficult to extend these results even to distributively-generated near-rings; however, our next theorem is one such extension. Recall that a near-ring N is zero-commutative if $ab=0$ implies $ba=0$, and N is an *IFP* near-ring if $ab=0$ implies $axb=0$ for all $x\in N$.

Theorem 3. *Let N be a non-nil, distributively-generated, and zero-commutative near-ring. If N has only finitely many non-nilpotent elements, then N is finite.*

PROOF. Let D be any set of distributive elements generating N^+, and let $-D=\{-d\,|\,d\in D\}$. Note that if $d_1, \dots, d_k$ are in $D\cup(-D)$ and n is a positive integer, then $(d_1+d_2+\dots+d_k)^n$ is a sum of terms each of which is a product of n of the d_i. Now the hypothesis that N is zero-commutative implies that N is an IFP near-ring; hence if $d_1, \dots, d_k$ are nilpotent elements belonging to $D\cup(-D)$, $d_1+\dots+d_k$ is also nilpotent. Thus, our hypothesis that N is non-nil guarantees that D contains a non-nilpotent element, hence a non-zero idempotent e. Since N is zero-commutative, e must be central.

Let u be an arbitrary nilpotent element of N. If we can show that $e+u$ is non-nilpotent, then finiteness of N follows as in the first paragraph of the proof of Theorem 1. Now if $e+u$ were nilpotent, then e would be of the form u_1-u_2, with u_1 and u_2 nilpotent and u_1-u_2 central. However, for any pair $u_1, u_2\in N$ with u_1-u_2 central, we can show by induction on n that $(u_1-u_2)^n$ is a sum of terms of form $\pm v_1v_2\dots v_n$, where each v_i is either u_1 or u_2; and it follows by IFP that if u_1 and u_2 are nilpotent, then so is u_1-u_2. Thus, our proof is complete.

2. Near-rings with chain conditions on non-nil subnear-rings

In [2], there appears the following result, which we shall refer to as Proposition R_4: if the ring R is not a nil ring and has both ascending chain condition and descending chain condition on non-nil subrings, then R is finite. We shall not attempt to prove the corresponding result for distributive near-rings, for — assuming that Tarski groups do in fact exist — the direct sum of a field $GF(p)$ and a trivial near-ring on a Tarski group is a counterexample. However, under suitable restrictions on the additive group, Proposition R_4 does generalize to distributive near-rings.

Before proceeding, we note that if N is any distributive near-ring, N^2 is a ring, and N' is an ideal contained in $A(N)$.

Theorem 4. *Let N be a non-nil distributive near-ring having both ascending chain condition and descending chain condition on non-nil subnear-rings. If N^+ is an FC-group, then N is finite.*

PROOF. The factor near-ring $\bar{N}=N/N'$ is a ring; and since $N'\subseteqq A(N)$, $\bar{N}$ inherits the hypotheses of Theorem 4, hence by Proposition R_4 must be finite. It follows that N is periodic, and must therefore contain non-zero idempotents. If it happens that N contains a regular idempotent, then N has 1 and is consequently a ring; thus, we may assume that N contains a non-zero idempotent e which is a left zero divisor, and we may write

$$N = eN + A_r(e). \tag{2}$$

Assume that N is infinite. Since N^2 is a ring, it is necessarily finite; and since $e^2+xy-e^2-xy=0$ for all $x, y\in N$, we have $eN\subseteqq N^2\subseteqq C_N(e)$. The additive group $C_N(e)$ must be infinite, since it is of finite index in N^+; hence, (2) and the finiteness of eN imply that $S=C_N(e)\cap A_r(e)$ is infinite. In fact, the observation that $N^2 \subseteqq C_N(e)$ guarantees that S is a subnear-ring; and applying (I) shows that S contains either an infinite near-ring with trivial multiplication or an infinite ring having a.c.c. and d.c.c. on non-nil subrings. In the latter case, (II) and Proposition R_4 imply that S contains an infinite zero ring.

Let B be an infinite subnear-ring of S with trivial multiplication. Now Be, being contained in N^2, must be finite; hence, by the first isomorphism theorem for groups, $B_1=B\cap A_l(e)$ must be infinite. Moreover, each element of B_1 is of finite additive order, for if $u\in B_1$ had infinite order, u and e would generate a counterexample to Proposition R_4. Since B_1^+ is an FC-group, we see that for each finite subset $\{u_1, \ldots, u_k\}$ of B_1, $\bigcap_{i=1}^{k} B_1\cap C_N(u_i)$ has finite index in B_1, hence is infinite; therefore, we can construct inductively an infinite sequence $u_1, u_2, \ldots$ of pairwise-additively-commutative elements of B_1, which together with e generate an infinite ring — a counterexample to Proposition R_4. Thus, the assumption that N is infinite must be false.

Theorem 5. *Let N be a non-nil distributive near-ring with ascending chain condition and descending chain condition on non-nil subnear-rings. If N^+ is nilpotent, then N is finite.*

PROOF. Let $N=N_1\supseteqq N_2\supseteqq N_3\supseteqq\ldots\supseteqq N_r=\{0\}$ be the lower central series of N^+; and let m be the smallest positive integer for which N_m is finite. If $m=1$, there is nothing to prove. If $m=2$, N' is finite; and since N/N' is finite by Proposition R_4, N must be finite. Thus, we need only consider the case $m\geqq 3$.

Let e be a non-zero idempotent, the existence of which follows from the initial steps of the proof of Theorem 4. Since N is periodic, e must have finite additive order; and we now show that elements of N_{m-1} also have finite additive order. Accordingly, for fixed $u\in N_{m-1}$, consider the descending chain $\langle e, u\rangle\supseteqq\langle e, 2u\rangle\supseteqq\langle e, 4u\rangle\supseteqq\ldots$. Since it must become stationary at some point, there exists k such that $2^k u\in\langle e, 2^{k+1}u\rangle$; and since $N_{m-1}\subseteqq N'\subseteqq A(N)$, we have integers h, j and an element $c_1\in[N_{m-1}, N]=N_m$ for which $2^k u=he+j2^{k+1}u+c_1$. Left-multiplying this equation by e yields $he=0$, hence

$$2^k u = j\,2^{k+1}u+c_1. \tag{3}$$

Substituting (3) into itself, we get $2^k u=2j(2^k u)+c_1=2j(j2^{k+1}u+c_1)+c_1=(2j)^2 2^k u+c_2$ for some $c_2\in N_m$; and continuing inductively, we obtain a sequence $c_1, c_2, c_3, \ldots$ of elements of N_m such that $2^k u=(2j)^s 2^k u+c_s$ for $s=1, 2, \ldots$. The finiteness of N_m yields distinct positive integers p, q for which $c_p=c_q$, and therefore $((2j)^p-(2j)^q)2^k u=0$; thus, provided $j\neq 0$, u is now seen to have finite additive order. But if $j=0$, then $2^k u\in N_m$; and again the finiteness of N_m guarantees that $2^k u$, and hence u, has finite order.

Consider the collection $\mathscr{C}$ of all subnear-rings of form $\langle e, u_1, \ldots, u_k\rangle$, where $u_1, \ldots, u_k\in N_{m-1}$. The ascending chain condition on non-nil subnear-rings yields a maximal member of $\mathscr{C}$, say E, which clearly contains N_{m-1}. But $E=\langle e, u_1, \ldots, u_k\rangle$ is generated *as an additive group* by $e, u_1, \ldots, u_k$; and in a nilpotent group, any subgroup generated by finitely many elements of finite order must be finite. Thus, N_{m-1} is finite, contrary to the minimality of m; consequently, the case $m\geqq 3$ cannot occur, and N is finite.

References

[1] H. E. BELL, Infinite subrings of infinite rings and near-rings. *Pacific J. Math.* **59** (1975), 345—358.

[2] H. E. BELL, Some conditions for finiteness of a ring. *Submitted for publication.*

[3] H. E. BELL and S. LIGH, On finiteness conditions for near-rings. *Publ. Math. (Debrecen)* **22** (1975), 35—40.

[4] H. E. HEATHERLY, Distributive near-rings. *Quarterly J. Math. Oxford* (2) **24** (1973), 63—70.

[5] S. LIGH and J. J. MALONE, Zero divisors and finite near-rings. *J. Austral. Math. Soc.* **11** (1970), 374—378.

[6] M. S. PUTCHA and A. YAQUB, Rings with a finite set of nonnilpotents. *Internat. J. Math. Math. Sci.* **2** (1979), 121—126.

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