On the focal locus of a submanifold in euclidean space

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The concept of focal point of a submanifold in Euclidean space was applied in several cases; e.g. by MILNOR [1] in his lecture on Morse theory and SZENTHE [2] studied focal points of a principal orbit. It has been shown that along any normal line at a point of a submanifold of dimension m embedded in Euclidean space, there are at most m focal points ([1], pp. 34). In the present paper the author will present some results on such focal points. Here we shall consider the locus of all focal points of an m-dimensional submanifold and obtain conditions under which it is the union of m hypersurfaces, the so called sheets.

1. Preliminaries

Let $f: M \to R^n$ be an immersion of an m-dimensional manifold M into a Euclidean space R^n of dimension n. Let the normal bundle $T(M)^\perp$ of M be defined by

$$T(M)^\perp = \{(p, \omega): p \in M, \omega \text{ normal to } M \text{ at } p\}.$$

Obviously $T(M)^\perp \subset M \times R^n$, is an n-dimensional bundle space differentiably embedded in $R^{2n} = TR^n$, the tangent bundle of R^n.

Consider the end point map

$$\Phi: T(M)^\perp \to R, \quad \text{defined by}$$

$$\Phi(p, \omega) = p + \omega, \quad (p, \omega) \in T(M)^\perp.$$

Then a point $s \in R^n$ is called a focal point of M if $s = p + \omega$, is a critical value of the end point map.

Intuitively, a focal point of M is a point in R^n where nearby normals intersect.

Let $k_1, k_2, \ldots, k_m$ be the principal curvatures of M at p in the normal direction ω. The reciprocals $k_1^{-1}, k_2^{-1}, \ldots, k_m^{-1}$ of these principal curvatures are called principal radii of curvature.

Consider the normal line L containing all focal points $p + R\omega$, where ω is a fixed unit vector orthogonal to M at p and let R be the radius of curvature at the point on M along ω. Then the locus of these focal points of M will be called focal locus of M.

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The following well known lemma ([1], pp 34) plays a central role in this paper.

Lemma. *The focal points of* (M, p) *along* L *are precisely the points* $p+k_i^{-1}\omega$, *where* $1\leqq i\leqq m$, $k_i\neq 0$. *Thus there are at most* m *focal points of* (M, p) *along* L.

Since there are m focal points of M along L, therefore the locus of all these m points will determine m sheets. Our aim in this paper is to find the conditions under which m sheets are hypersurfaces.

Before deriving conditions, we shall extend the concept of Weingarten map from hypersurfaces to submanifolds and generalize the Rodrigues formula from hypersurfaces to submanifolds.

2. Weingarten map and Rodrigues formula in case of submanifolds

Let ∇' be the natural connection on R^n, and X be a vector field tangent to the submanifold M defined on a neighborhood of p. Let $\nabla'_X\omega$ denote the covariant derivative of ω in the direction of X. Then we can decompose $\nabla'_X\omega$ as

$$\nabla'_X\omega=(\nabla'_X\omega)_T+(\nabla'_X\omega)_N,$$

where $(\nabla'_X\omega)_T$ denotes the tangential and $(\nabla'_X\omega)_N$ denotes the normal part of $\nabla'_X\omega$.

Now we define the linear map

$$A\omega(X)=(\nabla'_X\omega)_T.$$

Since ∇' is linear, therefore $A\omega$ is linear too. The vector $A\omega(X)$ obviously lies in the tangent space $T_p(M)$ of M at p. This map $A\omega\colon T_p(M)\to T_p(M)$ is called the Weingarten map.

Our next object is to show that $A\omega$ is self-adjoint or symmetric; i.e. if X and Y are in $T_p(M)$, then

$$\langle A\omega(X), Y\rangle=\langle X, A\omega(Y)\rangle.$$

To do this we have the following theorem.

Theorem (2.1). *The Weingarten map is self-adjoint.*

Proof. Let X and Y are in $T_p(M)$. Imbed X and Y in C^∞ fields on a special coordinate neighborhood U of p, and extend X and Y to C^∞ fields $\overline{X}$ and $\overline{Y}$ on associated coordinate neighborhood $\overline{U}$ of p in R^n. Then

$$\langle A\omega(X), Y\rangle-\langle X, A\omega(Y)\rangle=\langle\nabla'_X\omega, Y\rangle_T-\langle X, \nabla'_Y\omega\rangle_T=$$

$$=\langle\nabla'_{\overline{X}}\overline{\omega}, \overline{Y}\rangle_{pT}-\langle\overline{X}, \nabla'_{\overline{Y}}\overline{\omega}\rangle_{pT}=\overline{X}_p\langle\overline{\omega}, \overline{Y}\rangle-\langle\overline{\omega}, \nabla'_{\overline{X}}\overline{Y}\rangle_{pT}-\overline{Y}_p\langle\overline{\omega}, \overline{X}\rangle+\langle\overline{\omega}, \nabla'_{\overline{Y}}\overline{X}\rangle_{pT}=$$

$$=\langle\nabla'_{\overline{Y}}\overline{X}-\nabla'_{\overline{X}}\overline{Y}, \overline{\omega}\rangle_{pT}=\langle[\overline{Y}, \overline{X}], \overline{\omega}\rangle_p=\langle[Y, X]_p, \omega_p\rangle=0,$$

since $\overline{X}_p\langle\overline{\omega}, \overline{Y}\rangle=X_p\langle\omega, Y\rangle=0=Y_p\langle\omega, X\rangle$ and $\nabla'_Y X-\nabla'_X Y=[Y, X]$ [3].

This result also follows from proposition (3.3) pp. 14 and Weingarten formula (II) pp. 15 of [3] Volume II.

Since Weingarten map is self-adjoint, therefore the eigenvalue of the corresponding matrix will define principal curvature and eigen vector as principal vector [4].

If X is a principal vector, then the Weingarten map says that

$$\nabla'_X \omega = -kX,$$

where k is a principal curvature. This equality is classically called the formula of Rodrigues for submanifold M.

3. To find the condition for a sheat of the focal locus to be a hypersurface

Let $\varphi^1, \dots, \varphi^{n-m-1}$ be spherical coordinate system on a unit sphere in R^{n-m} with base $(e_1, \dots, e_{n-m})$. Let $\xi^i(\varphi^1, \dots, \varphi^{n-m-1})$ be the component of a unit vector. Then this vector will be given by $\sum_{i=1}^{n-1} \xi^i(\varphi^1, \dots, \varphi^{n-m-1})e_i$. Let $u^1, \dots, u^m$ be a local coordinate system in a neighborhood U of $p \in M$. Then a local coordinate system $(u^1, \dots, u^m; \varphi^1, \dots, \varphi^{n-m-1})$ can be defined in a neighborhood V of q, where q is a point of the manifold of unit normals to M. Let $\omega_1, \dots, \omega_{n-m}$ be vector fields defined in a neighborhood of p which form an orthonormal base for the normal space $T_p(M)^\perp$ of M at every point of the neighborhood.

Consider the isomorphism from the normal space $T_p(M)^\perp$ to the space R^{n-m} s.t. $\omega_i \to e_i$. Then the unit vector $\omega(u^1, \dots, u^m; \varphi^1, \dots, \varphi^{n-m-1})$ at point p of M will be given by

$$\omega = \sum_{i=1}^{n-m} \xi^i \omega_i,$$

where $\xi^i(\varphi^1, \dots, \varphi^{n-m-1})$ are components of its image vector in R^{n-m}.

Let R_1 be the first radius of curvature at the point r on M along ω. Then the corresponding focal point q' on the first sheet of focal locus is given by

$$r'(u^1, \dots, u^m; \varphi^1, \dots, \varphi^{n-m-1}) = r(u^1, \dots, u^m) +$$
$$+ R_1(u^1, \dots, u^m; \varphi^1, \dots, \varphi^{n-m-1})\,\omega\,(u^1, \dots, u^m; \varphi^1, \dots, \varphi^{n-m-1}).$$

Taking the partial derivative of this expression with respect to u^l and φ^a respectively we get

$$\partial_l r' = \partial_l r + (\partial_l R_1)\omega + R_1 \partial_l \omega; \ \partial_l = \partial/\partial u^l, \quad l = 1, \dots, m$$

and $\partial'_a r' = (\partial' a R_1)\omega + R_1(\partial' a\omega)$; $\partial' a = \partial/\partial\varphi^a$, $a = 1, \dots, n-m-1$, since $\partial'_a r = 0$.

If $\partial_l r$ is in principal direction, then by Rodrigues formula

$$\partial_l \omega = -k_l \partial_l r,$$

where k_l are m principal curratures at p of the submanifold; i.e. $k_l = \frac{1}{R_l}$. Therefore

$$\partial_l r' = (1 - k_l R_1)\,\partial_l r + (\partial_l R_1)\omega.$$

We are going to find the general condition for the jth sheets to be a hypersurface. Here $R_j = 1/k_j$, where j is fixed and R_j denotes jth radius of curvature along ω.

In this case we have

$$\partial_j r' = \partial_j \left(\frac{1}{k_j}\right) \omega$$

and $\partial_1 r' = \left(1 - \frac{k_l}{k_j}\right) \partial_l r + \partial_l \left(\frac{1}{k_j}\right) \omega$; $l = 1, 2, \ldots, j-1, j+1, \ldots, m$. Taking the wedge product of $\partial_l r'$ and $\partial' a^{r'}$, we have

$$\partial_1 r' \wedge \ldots \wedge \partial_m r' \wedge \partial_1' r' \wedge \ldots \wedge \partial_{n-m-1}' r' =$$

$$= \left\{\partial_j \left(\frac{1}{k_j}\right) \frac{1}{k_j^{n-m-1}} \left[\left(1 - \frac{k_1}{k_j}\right) \ldots \left(1 - \frac{k_m}{k_j}\right)\right]\right\} \times$$

$$\times \partial_1 r \wedge \ldots \wedge \partial_{j-1} r \wedge \omega \wedge \partial_{j+1} r \wedge \ldots \wedge \partial_m r \wedge \partial_1' \omega \wedge \ldots \wedge \partial_{n-m-1}' \omega; \quad \text{as} \quad \omega \wedge \omega = 0.$$

Since $\partial_1 r, \ldots, \partial_{j-1} r, \omega, \partial_{j+1} r, \ldots, \partial_m r$; $\partial_1' \omega, \ldots, \partial_{n-m-1}' \omega$ are $n-1$ linearly independent vectors, thus their wedge product is non zero.

Thus we have

Theorem (3.1). *The sufficient condition for the* $j-th$ $(1 \leqq j \leqq m)$ *sheet of focal locus to be a hypersurface is that*

$$\frac{1}{k_j^{n-m-1}} \partial_j \left(\frac{1}{k_j}\right) \left\{\left(1 - \frac{k_1}{k_j}\right) \ldots \left(1 - \frac{k_m}{k_j}\right)\right\} \neq 0.$$

We shall call these hypersurfaces as focal hypersurfaces for further studies.

From Theorem (3.1) we conclude that all the m sheets are hypersurfaces if no of the $k_1, \ldots, k_m$ is constant, and they are all different.

The question concerning the independence of the above conditions requires some additional results and it will be studied elsewhere.

References

[1] J. MILNOR, Lectures on morse theory, *Princeton,* 1963.
[2] J. SZENTHE, Sur les voisinages invariants des orbits d'actions orthogonales, *Publ. Math. (Debrecen) (to appear).*
[3] S. KOBAYASHI and K. NOMIZU, Foundations of differential geometry I, II, *New York,* 1963, 1969.
[4] M. SPIVAK, A comprehensive introduction to differential geometry IV. *Berkeley,* 1979.
[5] N. J. HICKS, Notes on differential geometry, *Princeton,* 1965.

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