On random variables defined on the atom space of an orthomodular atomistic σ-lattice

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1. Introduction

In a previous paper [3] we have given a new generalized axiomatic system for probability theory. We set out from the following conditions:

(I) Let an atomistic orthomodular σ-lattice $\mathscr{L}(\vee, \wedge, \perp, 1, 0)$ be given, calling its elements *propositions*. Denote $\Omega(\mathscr{L})$ the atom space (= the set of all atoms of $\mathscr{L}$).

(II) Denote by $S(\Omega)$ the class of all subsets A of $\Omega(\mathscr{L})$ for which the supremum $\sup A = VA$ exists. Let us suppose that $S(\Omega)$ forms a σ-field of sets.

(III) Let m be a *probability measure* on $\mathscr{L}$, that is $0 \leqq m(a) \leqq 1$, $a \in \mathscr{L}$; $m(1)=1$ and

$$m\left(\bigvee_{i=1}^{\infty} a_i\right) = \sum_{i=1}^{\infty} m(a_i) \quad \text{if} \quad a_i^{\perp} \geqq a_j (i \neq j).$$

We remark that $a_i^{\perp}$ denotes the *orthocomplement* of a_i. It is customary to say that $a, b \in \mathscr{L}$ are *orthogonal* (or disjoint) if $a^{\perp} \geqq b$. In this case we write $a \perp b$.

As is known in the axiomatic quantum mechanics (cf. [1], [2]), such an axiomatic system can be justified on the grounds of physics. Contrary to axiomatic quantum mechanics, where it is supposed that there is a full set of probability measures (states) on $\mathscr{L}$ (cf. [2, p. 72]), we presume the existence of only one probability measure m on $\mathscr{L}$. In our system $\Omega(\mathscr{L})$ may be regarded as the space of elementary outcomes, therefore, by a *random variable* we mean a function f of the atom space $\Omega(\mathscr{L})$ into the real line R^1, such that

$$f^{-1}(B) = \{p \in \Omega(\mathscr{L}) \mid f(p) \in B\} \in S(\Omega) \quad \text{for all} \quad B \in \mathscr{B}(R^1),$$

where $\mathscr{B}(R^1)$ is the set of the Borel subsets of the real line R^1. As we have defined above, the notion of random variable is different from the notion of observable in quantum physics. An observable is defined as a map x from $\mathscr{B}(R^1)$ into $\mathscr{L}$ which satisfies:

(i) $$x(R^1) = 1$$

(ii) $$x(E) \perp x(F) \quad \text{if} \quad E \cap F = \emptyset.$$

(iii) $$x\left(\bigvee_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} x(E_i) \quad \text{if} \quad E_i \cap E_j = \emptyset, \quad i \neq j.$$

This means that x is a σ-homomorphism of $\mathscr{B}(R^1)$ into $\mathscr{L}$. However, it may happen that for an observable x there exists a random variable f such that $x(B)=\sup f^{-1}(B)$, $B\in\mathscr{B}(R^1)$. ($\sup A=VA$, $A\subseteqq\Omega(\mathscr{L})$ is the least upper bound of the elements of A if it exists.) In this case we can say that x is induced by f (or f induces x). Let O denote the set of all random variables f for which f induces an observable, that is $\sup f^{-1}(B)$, $B\in\mathscr{B}(R^1)$ is a σ-homomorphism.

We have already see (cf. [3, Theorem 3.1]) that generally $\sup f^{-1}(B)$, $B\in\mathscr{B}(R^1)$ is not an observable. Furthermore, it is not necessarily an observable induced by a random variable (cf. [3, Theorem 3.6]).

In this paper firstly we shall examine the structure of O. The main result of this paragraph is the following: any Borel-function of n random variables contained in O is in O as well. Moreover, we shall examine the expectation of random variables and it will be proved that the expectation is an additive function on O.

2. Random variables which induce observables

Before we go on we shall recall some more definitions and notations.

A lattice $\mathscr{L}$ with 0 and 1 is called *orthocomplemented* when there is a mapping $a\to a^\perp$ of $\mathscr{L}$ into itself with the following three property:

(i) $a^\perp\wedge a=1$, $a^\perp\vee a=0$,

(ii) $a\leqq b$ implies $a^\perp\geqq b^\perp$,

(iii) $a^{\perp\perp}=a$ for all a.

An orthocomplemented lattice $\mathscr{L}$ is called *orthomodular* when in $\mathscr{L}$ $a\leqq b$ implies $b=a\vee(b\wedge a^\perp)$.

An element p of a lattice $\mathscr{L}$ with o is called an *atom* when $0\leqq b\leqq p$ is not satisfied by any $b\neq 0$, $b\neq p$. We say that $\mathscr{L}$ is *atomistic* when every non-zero element a of $\mathscr{L}$ is the join of atoms contained in a.

Throughout this article we shall suppose that $\mathscr{L}$, $S(\Omega)$, m are such as we have defined in I—II—III.

We have proved in [3, Theorem 3.5] that if the supremum of any two nonzero elements of $\mathscr{L}$ is 1, then $f\in O$ if and only if f is constant on the whole $\Omega(\mathscr{L})$. The question arises: what can we say about O in a more general case?

Let $A, B\subseteqq\Omega(\mathscr{L})$ and $A\cap B=\emptyset$. We say A, B to be *orthogonal* if $A, B\neq\emptyset$ and their elements are pairwise orthogonal: $p\perp q$ if $p\in A$, $q\in B$. A partition γ of $\Omega(\mathscr{L})$ is said to be orthogonal if γ has at least two classes and any two classes A, B of γ are orthogonal.

First we need the following

Lemma 1. *Let* $A, B\in S(\Omega)$. *Then* $\sup A\perp\sup(B)$ *if and only if* $p\perp q$ *for every* $p\in A$, $q\in B$.

Proof. It is trivial.

Theorem 2. *a) If* $\Omega(\mathscr{L})$ *has no orthogonal partition, then an arbitrary random variable* $f\in 0$ *is constant on the whole* $\Omega(\mathscr{L})$.

b) If $\Omega(\mathscr{L})$ has orthogonal partition then there exists $f \in 0$ such that f is not constant on the whole $\Omega(\mathscr{L})$.

PROOF. *a)* If f is constant then evidently $f \in O$. On the other hand let us assume that $\Omega(\mathscr{L})$ has no orthogonal partition and $f \in O$ is not constant. Then $\sup f^{-1}(\alpha) \perp \sup f^{-1}(\beta)$ for $\alpha, \beta \in R^1$, $\alpha \neq \beta$. Hence by Lemma 1. $\Omega(\mathscr{L})$ has orthogonal partition which is a contradiction.

b) If $\Omega(\mathscr{L})$ has orthogonal partition then there exists also an orthogonal partition including only two classes, namely A and B. Let now $f(p)=1$ if $p \in A$ and $f(p)=0$ if $p \in B$. Then it is easy to see that $f \in O$. Thus the theorem is proved.

Let H be a separable complex Hilbert space and let $\mathscr{L}(H)$ denote the class of all closed subspaces of H. Ordering $\mathscr{L}(H)$ by inclusion and defining the complement of a subspace as its orthocomplement one can prove that $\mathscr{L}(H)$ is an atomistic orthomodular lattice.

Theorem 3. *The atom space $\Omega(\mathscr{L}(H))$ has no orthogonal partition.*

PROOF. Let $\{e_1, e_2, \ldots\}$ be a complete orthonormal system in H. Denote by $[a]$ the smallest subspace of $\mathscr{L}(H)$ which contains a $(a \in H)$. Then $[a]$ is an atom in $\mathscr{L}(H)$. Let $\omega_i = [e_i]$ $(i=1, 2, \ldots)$. Suppose now that there exists an orthogonal partition: $A \cup B = \Omega(\mathscr{L}(H))$ $A \cap B = \emptyset$ of $\Omega(\mathscr{L}(H))$. On account of the completeness there exists $\omega_i, \omega_j \in \Omega(\mathscr{L}(H))$ such that $\omega_i \in A, \omega_j \in B$. But $[e_i + e_j]$, $[e_i]$ and $[e_i + e_j]$, $[e_j]$ are not orthogonal pairs which contrasts with our assumption.

We have seen in [3, Theorem 3.11] that if every random variable is in O then $\mathscr{L}$ is distributive. As we can show in Fig. 1, there exists an atomistic orthomodular

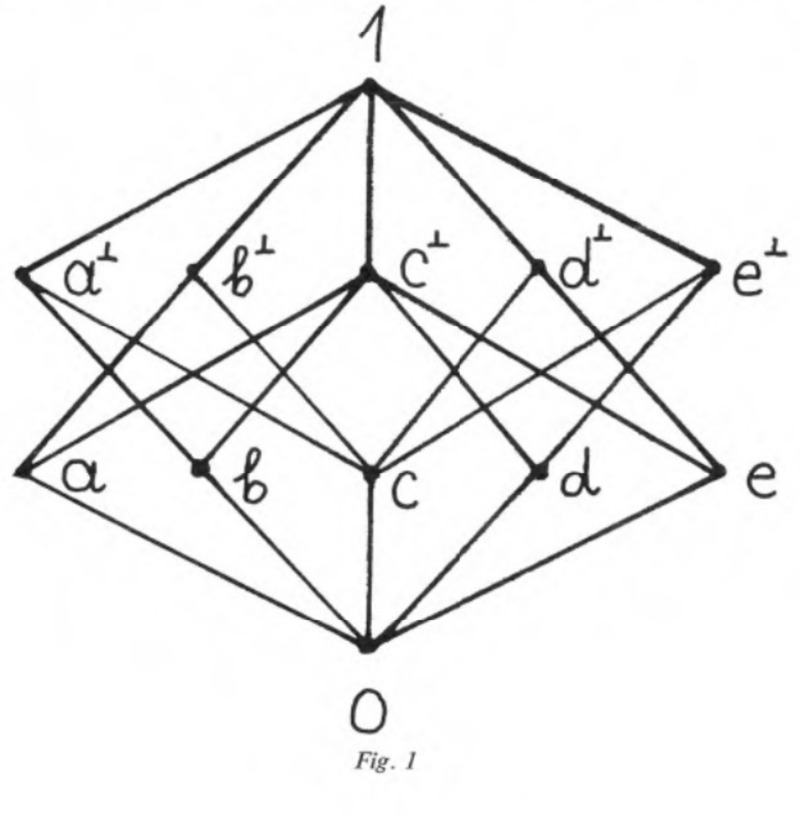

Fig. 1

non-distributive lattice such that its atom space has an orthogonal partition. According to the above mentioned theorem, this means that in order that the atom space should have orthogonal partition, it is not necessary for every random variable to be contained in O.

For example $\{c\}$, $\{a, b, d, e\}$ is an orthogonal partition of the atom space.

Let us define now functions of several random variables. Let $f_1, f_2, \ldots, f_n$ be random variables and let ψ be a Borel function on R^n. We then define $\psi(f_1, f_2, \ldots, f_n)$ as the random variable $p \to \psi(f_1(p), f_2(p), \ldots, f_n(p)) \in R^1$. It is easily checked that $\psi(f_1, f_2, \ldots, f_n)$ is a random variable indeed. Moreover, in the case of $f_1, f_2, \ldots, f_n \in O$ $\psi(f_1, f_2, \ldots, f_n)$ is in O, too, as we shall prove in the following theorem.

Theorem 4. *If* $f_1, f_2, \ldots, f_n \in O$ *and* ψ *is a Borel function on* R^n *then* $\psi(f_1, f_2, \ldots, f_n) \in O$.

PROOF. For the sake of simplicity, we shall restrict our proof to $n=2$. In general we can prove the theorem by a similar method.

Let $\mathscr{R}(f)$ denote the range of an arbitrary random variable f. Let $L=\mathscr{R}(f_1)$, $M=\mathscr{R}(f_2)$, $\psi(f_1, f_2)=h$. We prove that

$$\sup h^{-1}(B_1) \perp \sup h^{-1}(B_2) \quad \text{if} \quad B_1, B_2 \in \tag{4.1}$$

$\in \mathscr{B}(R^1)$, $B_1 \cap B_2 = \emptyset$. Namely,

$$h^{-1}(B_1) = \bigcup_{\substack{l \in L \\ m \in M \\ \Psi(l,m) \in B_1}} \left(f_1^{-1}(l) \cap f_2^{-1}(m)\right) \quad \text{and} \quad h^{-1}(B_2) = \bigcup_{\substack{k \in L \\ n \in M \\ \Psi(k,n) \in B_2}} \left(f_1^{-1}(k) \cap f_2^{-1}(n)\right).$$

Then $h^{-1}(B_1) \cap h^{-1}(B_2) = \emptyset$, so $\psi(l, m) \in B_1$, $l \in L$, $m \in M$ and $\psi(k, n) \in B_2$, $k \in L$, $n \in M$ implies $l \neq k$ or $m \neq n$. To verify assertion (4.1) let us observe that if $a \perp b$, $c \leqq a$, $d \leqq b$ in $\mathscr{L}$, then also $c \perp d$. If $l \neq k$, then $\sup f_1^{-1}(l) \perp \sup f_1^{-1}(k)$. Similarly, if $m \neq n$ $(m, n \in M)$, then $\sup f_2^{-1}(m) \perp \sup f_2^{-1}(n)$. This implies $\sup \left(f_1^{-1}(l) \cap f_2^{-1}(m)\right) \perp \sup \left(f_1^{-1}(k) \cap f_2^{-1}(n)\right)$ if $\psi(l, m) \in B_1$, $l \in L$, $m \in M$ and $\psi(k, n) \in B_2$, $k \in L$, $n \in M$. Now, on the basis of the preceding remarks we have

$$\sup h^{-1}(B_1) = \bigvee_{\substack{l \in L \\ m \in M \\ \Psi(l,m) \in B_1}} \sup \left(f_1^{-1}(l) \cap f_2^{-1}(m)\right),$$

$$\sup h^{-1}(B_2) = \bigvee_{\substack{k \in L \\ n \in M \\ \Psi(k,n) \in B_2}} \sup \left(f_1^{-1}(k) \cap f_2^{-1}(n)\right)$$

and $\sup h^{-1}(B_1) \perp \sup h^{-1}(B_2)$. Since $\sup h^{-1}(R^1)=1$, $\sup h^{-1}(\emptyset)=0$ hold and $\sup h^{-1}(\bigcup_i E_i) = \bigvee_i \sup h^{-1}(E_i)$, $E_i \in \mathscr{B}(R^1)$ is fulfilled for every random variable h (cf. [3, Theorem 3.4]), so $h \in O$ indeed. Thus the theorem is proved.

Corollary 5. *If* $f \in O$ *and* φ *is a Borel function on* R^1 *then* $\varphi(f) \in O$ *as well.*

PROOF. Let g be in O and let $\psi(x, y) = \varphi(x)$. If $B \in \mathscr{B}(R^1)$ then $\psi^{-1}(B) = \varphi^{-1}(B) \times R^1 \in \mathscr{B}(R^2)$, so $\psi(x, y)\colon R^2 \to R^1$ is a Borel function of two variables. Thus by the preceding theorem $\varphi(f) = \psi(f, g) \in O$ which was to be proved.

3. *The expectation of real random variables*

Let f be a random variable and let $F(x)=m(\sup f^{-1}(-\infty, x))$. $F(x)$ is called the distribution function of f. One can verify (cf. [3. Theorem 4.3]) that $F(x)$ is monotone increasing and continuous from the left. This allows one to define the expectation of a random variable f as a Lebesque—Stieltjes integral

$$E(f) = \int_{-\infty}^{\infty} x\, dF(x).$$

In this paragraph we shall deal with the expectation of random variables which are contained in O. This is the most important class of random variables because the inverse map of these random variables induces an observable.

We prove first some theorems for discrete random variables. A real random variable f is called to be discrete if its range $\mathscr{R}(f)=\{x_1, x_2, \dots\}$ is denumerable.

Theorem 6. *If f is a discrete real random variable with a range $\mathscr{R}(f)=\{x_1, x_2, \dots\}\subset R^1$ and there exists $E(f)$ then*

$$E(f) = \sum_i x_i p_i^*,$$

where

$$p_i^* = m\{\sup f^{-1}(-\infty, x_i]\wedge(\sup f^{-1}(-\infty, x_i))^{\perp}\} = $$
$$= m\{\sup f^{-1}(-\infty, x_i]-\sup f^{-1}(-\infty, x_i)\}.$$

PROOF. It is clear that $F(x)=m(\sup f^{-1}(-\infty, x))$ is a step-function and its discontinuity qoints are in $\mathscr{R}(f)$. Furthermore, $x_i\in\mathscr{R}(f)$ is a discontinuity point of $F(x)$ if and only if $p_i^*>0$ and in case of $p_i^*>0$ the jump in x_i is $F(x_i+o)-F(x_i)=p_i^*$.

Corollary 7. *If $f\in O$, $\mathscr{R}(f)=\{x_1, x_2, \dots\}$ and $E(f)$ exists, then*

$$E(f) = \sum_i x_i p_i,$$

where $p_i=m(\sup f^{-1}(x_i))$.

PROOF.

$$p_i^* = m(\sup f^{-1}(-\infty, x_i]-\sup f^{-1}(-\infty, x_i)) =$$
$$= m((\sup f^{-1}(-\infty, x_i)\vee \sup f^{-1}(x_i))-\sup f^{-1}(-\infty, x_i)) =$$
$$= m(\sup f^{-1}(-\infty, x_i))+m(\sup f^{-1}(x_i))-m(\sup f^{-1}(-\infty, x_i)) =$$
$$= m(\sup f^{-1}(x_i)) = p_i.$$

This proves our statement.

Remark. The following example demonstrates that in the preceding corollary we can not omit condition $f\in O$.

Let $\mathscr{L}$ be the lattice of Fig. 2. and let $a_1^{\perp}=a_2$, $a_2^{\perp}=a_1$, $a_3^{\perp}=a_4$, $a_4^{\perp}=a_3$, $0^{\perp}=1$, $1^{\perp}=0$; $m(a_i)=\frac{1}{2}(i=1, 2, 3, 4)$; $f(a_i)=i$ $(i=1, 2, 3, 4)$.

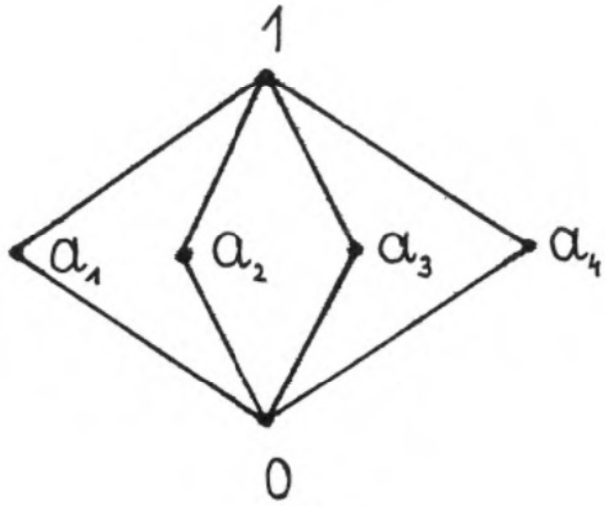

Fig. 2

In this case $f \notin O$ and $E(f) = 1{,}5 \neq \sum_{i=1}^{4} \frac{1}{2} i$.

In order to prove the main theorem of this section we require the following two lemmas.

Lemma 8. *If $f, g \in O$ are discrete random variables with $\mathscr{R}(f) = \{x_1, x_2, \dots\}$, $\mathscr{R}(g) = \{y_1, y_2, \dots\}$ and $E(f)$, $F(g)$ are finite, then there exists also $E(f+g)$ and*

$$E(f+g) = E(f) + E(g).$$

Proof. Now by Theorem 4. $f+g \in O$. Combining this fact with Corollary 7. $E(f+g)$ is of the form

$$E(f+g) = \sum_i \sum_j (x_i + y_j) r_{ij} \tag{8.1}$$

where $r_{ij} = m(\sup (f+g)^{-1}(z_{ij}))$, $\mathscr{R}(f+g) = \{z_{ij} = x_i + y_j;\ i, j = 1, 2, 3, \dots\}$. If $z_{i_1 j_1}$ is of the form $z_{i_1 j_1} = x_{i_1} + y_{j_1} = x_{i_2} + y_{j_2} = x_{i_3} + y_{j_3} = \dots$, then

$$r_{i_1 j_1} = m(\sup (f+g)^{-1}(z_{i_1 j_1})) = (\sup \bigcup_k (f^{-1}(x_{i_k}) \cap g^{-1}(y_{j_k}))) =$$

$$= m\big(\bigvee_k \sup (f^{-1}(x_{i_k}) \cap g^{-1}(y_{j_k}))\big).$$

Moreover, it can be shown (see the proof of Theorem 4) that

(8.2)

$$\sup (f^{-1}(x_i) \cap g^{-1}(y_i)) \perp \sup (f^{-1}(x_i') \cap g^{-1}(y_j')) \quad \text{if} \quad x_i \neq x_i' \quad \text{or} \quad y_j \neq y_j'.$$

This implies that in case of $k \neq l$

$$\sup (f^{-1}(x_{i_k}) \cap g^{-1}(y_{j_k})) \perp \sup (f^{-1}(x_{i_l}) \cap g^{-1}(y_{j_l})).$$

Therefore

$$r_{i_1 j_1} = \sum_k m\,(\sup (f^{-1}(x_{i_k}) \cap g^{-1}(y_j))) = \sum_k r'_{i_k j_k}, \tag{8.3}$$

where $r'_{ij}=m(\sup(f^{-1}(x_i)\cap g^{-1}(y_j)))$. By using (8.3) one can easily see that

$$\sum_i\sum_j(x_i+y_j)r_{ij}=\sum_i\sum_j(x_i+y_j)r'_{ij}.$$

Furthermore, by virtue of (8.2)

$$\sum_j r'_{ij}=\sum_j m\left(\sup(f^{-1}(x_i)\cap g^{-1}(y_j))\right)=$$
$$=m\left(\sup\left(\bigcup_j(f^{-1}(x_i)\cap g^{-1}(y_j))\right)\right)=m\left(\sup f^{-1}(x_i)\right).$$

Returning now to (8.1), these imply that

$$E(f+g)=\sum_i\sum_j(x_i+y_j)r'_{ij}=\sum_i x_i\sum_j r'_{ij}+\sum_j y_j\sum_i r'_{ij}=$$
$$=\sum_i x_i m\left(\sup f^{-1}(x_i)\right)+\sum_j y_j m\left(\sup g^{-1}(y_j)\right)=E(f)+E(g),$$

which was to be prove.

Lemma 9. *Let $f_n, f\in O$ having finite expectations $E(f_n), E(f)$ $(n=1,2,\ldots)$. If $0\leqq f(\omega)-f_n(\omega)<1/2^n$ holds for every $\omega\in\Omega(\mathscr{L})$ and $n=1,2,\ldots,$ then*

$$\lim_{n\to\infty}E(f_n)=E(f).$$

Proof. To prove this lemma we first observe that for arbitrary integer l and natural numbers n, k hold the following inclusion relations:

$$f_n^{-1}\left[\frac{l}{2^n},\frac{l+k}{2^n}\right)\subseteqq f^{-1}\left[\frac{l}{2^n},\frac{l+k+1}{2^n}\right) \tag{9.1}$$

$$f^{-1}\left[\frac{l}{2^n},\frac{l+k}{2^n}\right)\subseteqq f_n^{-1}\left[\frac{l-1}{2^n},\frac{l+k}{2^n}\right) \tag{9.2}$$

Let $E_n=\sum_{l=-\infty}^{\infty}\frac{l}{2^n}m\left(\sup f^{-1}\left[\frac{l}{2^n},\frac{l+1}{2^n}\right)\right)$. Then $\lim_{n\to\infty}E_n=E(f)$. Since $f_n, f\in O$ we have

$$\sum_{l=-\infty}^{\infty}\frac{l}{2^n}m\left(\sup f^{-1}\left[\frac{l}{2^n},\frac{l+k}{2^n}\right)\right)=kE_n-\sum_{i=0}^{k-1}\frac{i}{2^n} \tag{9.3}$$

$$\sum_{l=-\infty}^{\infty}\frac{l}{2^n}m\left(\sup f_n^{-1}\left[\frac{l}{2^n},\frac{l+k}{2^n}\right)\right]=kE(f_n)-\sum_{i=0}^{k-1}\frac{i}{2^n} \tag{9.4}$$

By means of (9.1)—(9.4)

$$\sum_{l=-\infty}^{\infty}\frac{l}{2^n}m\left(\sup f_n^{-1}\left[\frac{l}{2^n},\frac{l+k}{2^n}\right)\right)\leqq\sum_{l=-\infty}^{\infty}\frac{l}{2^n}m\left(\sup f^{-1}\left[\frac{l}{2^n},\frac{l+k+1}{2^n}\right)\right)\leqq$$
$$\leqq\sum_{l=-\infty}^{\infty}\frac{l}{2^n}m\left(\sup f_n^{-1}\left[\frac{l-1}{2^n},\frac{l+k+1}{2}\right)\right).$$

Hence

$$kE(f_n)-\sum_{i=0}^{k-1}\frac{i}{2^n}\leqq (k+1)E_n-\sum_{i=0}^{k}\frac{i}{2^n}\leqq (k+1)E(f_n)-\sum_{i=0}^{k-1}\frac{i}{2^n}+\frac{1}{2^n},$$

$$\frac{k}{k+1}E(f_n)\leqq E_n-\frac{k}{k+1}\frac{1}{2^n}\leqq E(f_n)+\frac{1}{(k+1)2^n}.$$

Tending with k and n to infinite we find $\lim_{n\to\infty} E(f_n)\leqq \lim_{n\to\infty} E_n=E(f)\leqq \lim_{n\to\infty} E(f_n)$, that is $\lim_{n\to\infty} E(f_n)=E(f)$. Thus the lemma is proved.

Theorem 10. *If $f, g\in O$ and f, g have finite expectations $E(f), E(g)$ then there exists $E(f+g)$ and $E(f+g)=E(f)+E(g)$.*

Proof. *Let $f_n(\omega)=\frac{l}{2^n}$ if $f(\omega)\in\left[\frac{l}{2^n},\frac{l+1}{2^n}\right)$ and $g_n(\omega)=\frac{l}{2^n}$ if $g(\omega)\in\left[\frac{l}{2^n},\frac{l+1}{2^n}\right)$, where $l=0, \pm 1, \pm 2, \dots$; $n=1, 2, \dots$.* Then $f_n, g_n\in O$ and $o\leqq f(\omega)-f_n(\omega)<\frac{1}{2^n}$, $0\leqq g(\omega)-g_n(\omega)<\frac{1}{2^n}$ for all $\omega\in\Omega(\mathscr{L})$.

By virtue of the preceding theorem now we have

$$\lim_{n\to\infty} E(f_n) = E(f),\ \lim_{n\to\infty} E(g) = E(g). \tag{10.1}$$

Since $f_n, g_n\in O$ and they are discrete random variables, on the basis of Lemma 8 we have

$$E(f_n+g_n) = E(f_n)+E(g_n) \tag{10.2}$$

Because of (10.1) and (10.2) it will be sufficient to show that $E(f+g)$ exists and $\lim_{n\to\infty} E(f_n+g_n)=E(f+g)$. Let us employ the following notations:

$$h_n=f_{n+1}+g_{n+1},\quad h=f+g\quad (n=1, 2, \dots)$$

In this case $h_n, h\in O$ and $0\leqq h(\omega)-h_n(\omega)<1/2^n$ $(n=1, 2, \dots)$. On the other hand $E(f+g)=E(h)$ exists and is finite because

$$h^{-1}\left[\frac{l}{2^n},\frac{l+1}{2^n}\right)\subseteqq h_n^{-1}\left[\frac{l-1}{2^n},\frac{l+1}{2^n}\right),$$

that is

$$\sum_l \frac{|l|}{2} m\left(\sup h^{-1}\left[\frac{l}{2^n},\frac{l+1}{2^n}\right)\right)\leqq \sum_l \frac{|l|}{2^n} m\left(\sup h_n^{-1}\left[\frac{l-1}{2^n},\frac{l+1}{2^n}\right)\right),$$

where the right side is convergent and equals $2E(h_n)+\frac{1}{2^n}$. By the preceding theorem now we get $\lim_{n\to\infty} E(h_n)=E(h)$. This completes the proof.

References

1] G. Birkhoff—J. von Neumann, The logic of quantum mechanics, *Ann. of Math.* **37** (1936), 823—843.

[2] S. Gudder, Axiomatic quantum mechanics and generalized probability theory, in Probabilistic Methods in Applied Mathematics, Vol. 2, 53—129, *Academic Press, New York* (1970).

[3] I. G. Kalmár, Atomistic orthomodular lattices and a generalized probability theory, *Publ. Math. (Debrecen)* **25** (1978).

[4] F. Maeda—S. Maeda, Theory of symmetric lattices, *Springer Verlag, Berlin* (1970)

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