Optimal control for linear systems
described by m-times integrated semigroups

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Abstract. In this paper we consider the question of existence of optimal controls and necessary conditions of optimality for a general class of linear functional differential equations described by m-times integrated semigroups and m-times integrated solution family. For linear systems with quadratic cost functionals this generalizes similar results of the author [8] for ordinary functional differential equations (equations governed by 0-times integrated solution family) on Banach spaces.

1. Introduction

In this paper we consider optimal control for a class of evolution equations where the infinitesimal generators are not of Hille-Yosida type. These are the generators of m-times integrated semigroups and m-times integrated solution family covering the so called distribution semigroups of Lion [see FATTORINI 7]. C_0-semigroups are covered by m-times integrated semigroups which in turn are covered by the so called m-times integrated solution family. In section 3 we give a brief review of some of these results as required in the paper. In section 4 we formulate a linear quadratic control problem and prove the existence of optimal controls and the necessary conditions of optimality. In section 5 we present a computational algorithm.

2. m-times integrated solution family

Let us consider the following integro-differential equation

(2.1)
$$(d/dt)x = \int_0^t da(s)Ax(t-s)ds, \quad t \geq 0,$$
$$x(0) = \zeta,$$

where A is generally an unbounded linear operator in a Banach space X.

Definition 2.1. A strongly continuous operator valued function $S(t)$, $t \geq 0$, in X is said to be the solution operator of the Cauchy problem (2.1) if

(i) $S(0) = I$ (identity operator)

(ii) there exist constants $\omega \in R$ and $M \geq 1$ such that

$$\|S(t)\|_{\mathcal{L}(X)} \leq Me^{\omega t}, \quad \text{for } t \geq 0.$$

(iii) For $\zeta \in D(A)$, $S(.)\zeta \in C([0,T], X) \cap C^1((0,T), X)$, $S(t)$ commutes with A on $D(A)$ and satisfies equation (2.1) for all $t \in I$.

Thus the solution of equation (2.1) is given by $x(t) = S(t)\zeta$, $t \geq 0$. We call the pair (A, a) the (infinitesimal) generator of a strongly continuous solution family $S(t)$, $t \geq 0$, if it generates the solution operator for the homogeneous Cauchy problem (2.1).

Note that if $a(t) \equiv 1$, for $t \geq 0$; and $a(t) \equiv 0$ for $t < 0$, then the system (2.1) reduces to a differential equation and $S(t)$, $t \geq 0$, is a C_0-semigroup with infinitesimal generator given by A. If $a(t) = t$, then the system (2.1) is equivalent to a second order evolution equation

$$(d^2/dt^2)y = Ay, \quad t \geq 0, \ y(0) = 0, \ \dot{y}(0) = \zeta.$$

In case of viscoelastic problems the operator A is given by the restriction of the laplacian on divergence free vector fields in a suitable L_2 space and the function a is given by

$$a(t) = a_0 + a_1 t + \int_0^t a_2(s)ds$$

with $a_0 \geq 0$, representing the Newtonian viscosity, $a_1 \geq 0$ representing the elasticity modulus and $a_2(t) \geq 0$, $t \geq 0$, is a nonincreasing function with $\lim_{t \to \infty} a_2(t) = 0$ representing the second law of thermodynamics [see 8].

A result characterizing the generators of solution operators for integro-differential equations of the form (2.1), generalizing Hille-Yosida theorem, is due to DA PRATO and IANELLI [5].

Using the solution operator corresponding to the generator, (A, a), one can then construct the mild solution of the nonhomogeneous equation:

$$\dot{x}(t) = \int_0^t da(s)Ax(t-s) + f(t), \quad t \in I,$$
(2.2)
$$x(0) = \zeta,$$

as

(2.3)
$$x(t) = S(t)\zeta + \int_0^t S(t-s)f(s)ds, \quad t \in I,$$

exactly as in the case of differential equations. If $\zeta \in D(A)$ and $f \in C^1(I, X)$, then x, given by expression (2.3), is a classical solution satisfying the equation (2.2).

Optimal control problems for such systems including semilinear ones have been considered in the literature [8].

Recently ARENDT and KELLERMAN [6] has generalised the result of DA PRATO and IANELLI [5] to m-times integrated solution family. This is in the same spirit as the generalization of the theory of classical C_0-semigroups to m-times integrated semigroups.

For convenience of notation define

$$R(\lambda) \equiv (\lambda - \hat{a}(\lambda)A)^{-1}$$
$$R_m(\lambda) \equiv R(\lambda)/\lambda^m \equiv (\lambda - \hat{a}(\lambda)A)^{-1}/\lambda^m, m \in N_0, \lambda > \omega.$$

Definition 2.2. A family of strongly continuous operator valued functions, $S(t), t \geq 0$, in X is said to be an m-times integrated solution family for the Cauchy problem (2.1), for some $m \in \mathbb{N}_0$, if

(i): There exist $M > 0$, $\omega \in \mathbb{R}$ such that $\|S(t)\| \leq Me^{\omega t}$, for all $t \geq 0$,

(ii): $S(0) = I$ for $m = 0$, $S(0) = 0$ for $m > 0$,

(iii): $R_m(\lambda)\xi = \int_0^\infty e^{-\lambda t}S(t)\xi dt$ for all $\lambda > \omega$ and $\xi \in X$.

The following result, essentially due to ARENDT and KELLERMAN [6], generalizes the generation theorem due to DA PRATO and IANELLI [5]. The work of ARENDT and KELLERMAN [6] is based on the theory of vector valued Laplace transforms due to ARENDT [3]. For details see also [2,4].

Lemma 2.3. *Necessary and sufficient conditions for the pair (A, a) to be the generator of an $m + 1$-times integrated solution family $S(t), t \geq 0$, are*

(1): *A is a closed operator with domain and range in X and $a \in BV_{\text{loc}}(R^+)$ satisfying $\int_0^\infty e^{-\omega t}|da(t)| < \infty$, $\hat{a}(\lambda) \neq 0, (\lambda/\hat{a}(\lambda)) \in \rho(A)$, for $\lambda > \omega$.*

(2): *There exists a number $M \geq 0$ and $\omega \in \mathbb{R}$, such that for $\lambda > \omega$,*

$$\|(\lambda - \omega)^{n+1} R_m^{(n)}(\lambda)/n!\| \leq M \quad \text{for all} \quad n \in \mathbb{N}.$$

If A is also densely defined then the pair (A, a) is the infinitesimal generator of an m-times instead of $m + 1$-times integrated solution family.

PROOF. See [6; 1].

Similar to the result for m-times integrated semigroups [2, Theorem 2.5.12, p. 59], one has the following result for m-times integrated solution family.

Consider the nonhomogeneous system (2.2) and suppose that the pair (A, a) is the generator of an m-times integrated solution family $S(t)$, $t \geq 0$. Define

$$(2.4) \qquad y(t) \equiv S(t)\zeta + \int_0^t S(t - s)f(s)ds, \quad t \in I.$$

Then, the system (2.2) has a classical solution if, and only if, $y \in C^{m+1}(I, X)$ and in that case the solution x is given by, $x = D^m y$, where D^m denotes the m-th derivative with respect to $t \in I$.

A set of sufficient conditions that guarantee the smoothness requirement of y, given by the expression (2.4), and hence the existence of a classical solution of the Cauchy problem (2.2), is given in the following theorem.

Theorem 2.4. *Consider the Cauchy problem (2.2) and suppose that the pair (A, a) is the generator of an m-times integrated solution family $S(t)$, $t \geq 0$. Then, the system (2.2) has a unique classical solution, $x \in C(I, X) \cap C^1((0, T), X)$, if $\zeta \in D(A^{m+1})$ and $f \in C^{m+1}(I, X)$ satisfying the condition $f^{(k)}(0) \in D(A^{m-k})$ for $0 \leq k \leq m - 1$*

PROOF. [See 6, Theorem 1.8, p. 30].

3. Generalized solutions

Note that for existence of solution, according to Theorem 2.4, the smoothness requirement of the data (ζ, f) is rather too severe and therefore very limited for application. For applications to control problems one would like to consider solutions of (2.2) for more general data like $\zeta \in X$ and $f \in L_1(I, X)$. For this one must must generalize the notion of solution. A notion of generalized solution was recently introduced by the author in [1] for the Cauchy problem (2.2) as follows.

Let X be a separable reflexive Banach space with dual X^*. Let

$$(3.1) \qquad W^{m,1}(X^*) \equiv \{\phi \in L_1(I, X^*) : D^k\phi \in L_1(I, X^*), 0 \le k \le m\}.$$

The space $W^{m,1}(X^*)$, furnished with the norm topology given by

$$\|\phi\|_{W^{m,1}(X^*)} \equiv \sum_{k=0}^{m} \|D^k\phi\|_{L_1(I,X^*)},$$

is a Banach space. Let $\partial I \equiv \{0, T\}$ denote the two end points of the interval I and

$$W_0^{m,1}(X^*) \equiv \{\phi \in W^{m,1}(X^*) : D^k\phi|_{\partial I} = 0, 0 \le k \le m-1\}$$

denote the completion in the topology of $W^{m,1}(X^*)$ of the vector space $C_0^m((0,T), X^*)$ of m-times differentiable functions on $(0,T)$ with compact supports. Clearly the dual of the Banach space $W_0^{m,1}(X^*)$ is given by $W^{-m,\infty}(X)$.

Consider the Cauchy problem (2.2). Suppose $\zeta \in X$, $f \in L_1(I, X)$ and the pair (A, a) is the generator of an m-times integrated solution family, $S(t)$, $t \ge 0$. Define

$$(3.2) \qquad\qquad y(t) = S(t)\zeta + \int_0^t S(t-s)f(s)ds$$

Definition 3.1. A (generalized) function x mapping I to X is said to be a generalized solution of the Cauchy problem (2.2) if

(i): $x(0) = \zeta$ and

(ii): $\int_I \langle x(t), \phi(t)\rangle_{X,X^*} dt = (-1)^m \int_I \langle y(t), D^m\phi(t)\rangle_{X,X^*} dt$, for all $\phi \in W_0^{m,1}(X^*)$, where D^m denotes the distributional derivative of order m with respect to time $t \in I$.

We need the following result which appeared recently in [1, Theorem 4.2, p. 55].

Theorem 3.2. *Consider the system* (2.2) *and suppose that the pair* (A, a) *is the generator of an m-times integrated solution family for some* $m \in \mathbb{N}_0$ *with the corresponding solution operator* $S(t)$, $t \geq 0$. *Suppose* $\overline{D(A^{m+1})} = X$. *Then, for each* $\zeta \in X$ *and* $f \in L_1(I, X)$, *the system* (2.2) *has a unique generalized solution* $x \in W^{-m,\infty}(X)$.

This result has also been extended by the author to stochastic systems [1, Theorem 5.4, p. 60]. Our major concern here is optimal control for deterministic systems described by m-times integrated solution family.

4. Optimal control: existence and necessary conditions

In this section we shall make use of the above results to formulate and solve the classical linear quadratic regulator problem for the system (2.2) which is governed by the infinitesimal generator (A, a) of m-times integrated solution family. We wish to note that up to the present time no such results exist in the literature. In this sense the results presented here are new.

The very first question is how do we formulate a regulator problem when the solutions are not ordinary vector-valued functions but instead vector-valued generalized functions? Here, we shall first give a reasonable formulation. Consider the control system:

$$(d/dt)x = \int_0^t da(s) Ax(t - s) + B(t)u(t), \quad t \in I$$
$$x(0) = \zeta,$$

(4.1)

where B is a suitable operator valued function and u is the control policy.

Let Y be another separable reflexive Banach space with dual Y^*, $\mathcal{U}_{ad} \subset L_p(I, Y)$ the class of admissible controls, and $B \in L_q(I, \mathcal{L}(Y, X))$ with $2 \leq p \leq \infty$ satisfying $1/p + 1/q = 1$ where $q \geq 1$ is the conjugate of p. For simplicity of notation, set $W_0^{m,1}(X^*) \equiv \mathcal{X}$. Since X is a reflexive Banach space the dual of $\mathcal{X}$ is given by $\mathcal{X}^* = W^{-m,\infty}(X)$. Let $Q \in \mathcal{L}_n^+(\mathcal{X}^*, \mathcal{X})$ where $\mathcal{L}_n^+(\mathcal{X}^*, \mathcal{X})$ is the space of positive nuclear operators from $\mathcal{X}^*$ to $\mathcal{X}$ and let $\mathcal{R}$ be a positive operator from $L_p(I, Y) \equiv L_p(Y)$ to $L_q(I, Y^*) \equiv L_q(Y^*)$. For the cost functional we take

(4.2) $J(u) \equiv (1/2) \langle Q(x - x_d), x - x_d \rangle_{\mathcal{X}, \mathcal{X}^*} + (1/2) \langle \mathcal{R}u, u \rangle_{L_q(Y^*), L_p(Y)},$

where $x_d \in \mathcal{X}^*$ is the desired trajectory. The objective is to find a control policy that imparts a minimum to the functional (4.2) subject to the dynamic constraint (4.1). It is well known that every such nuclear operator

has the representation

$$Q = \sum_{i \geq 1} \lambda_i \phi_i \otimes \phi_i$$

where $\{\phi_i\} \in \mathcal{X}, i \in \mathbb{N}$, is a suitable basis. Thus for every $\zeta \in \mathcal{X}^*$

$$\langle Q\zeta, \zeta \rangle = \sum \lambda_i \langle \phi_i, \zeta \rangle^2_{\mathcal{X}, \mathcal{X}^*}.$$

Without any loss of generality we may assume that $\|\phi_i\|_{\mathcal{X}} = 1$ for all $i \in \mathbb{N}$. Since in our case Q is positive, $\lambda_i \geq 0$ and it follows from the representation that

$$\langle Q\zeta, \zeta \rangle = \sum \lambda_i \langle \phi_i, \zeta \rangle^2_{\mathcal{X}, \mathcal{X}^*} \leq \operatorname{Tr} Q \|\zeta\|^2_{\mathcal{X}^*},$$

where $\operatorname{Tr} Q = \sum \lambda_i < \infty$.

Since $x_d \in \mathcal{X}^*$ is fixed, for simplicity of presentation but without any loss of generality, we may drop this term and consider instead the cost functional given by :

$$J(u) \equiv (1/2) \langle Qx, x \rangle_{\mathcal{X}, \mathcal{X}^*} + (1/2) \langle \mathcal{R}u, u \rangle_{L_q(Y^*), L_p(Y)}$$

(4.3)
$$= (1/2) \sum \lambda_i \langle \phi_i, x \rangle^2_{\mathcal{X}, \mathcal{X}^*} + (1/2) \int_0^T \langle R(t)u, u \rangle_{Y^*, Y} \, dt,$$

where we have assumed that $\mathcal{R}$ is a multiplication operator given by $(\mathcal{R}u)(t) = R(t)u(t)$ where $R \in L_s(I, \mathcal{L}(Y, Y^*))$ with $s = (p/(p-2))$.

First we give a result on the existence of optimal controls.

Theorem 4.1 (Existence of optimal control). *Consider the control problem (4.1)–(4.3) and suppose the assumptions of* Theorem 3.2 *hold. Let Y be a reflexive Banach space, $B \in L_q(I, \mathcal{L}(Y, X))$, $\mathcal{U}_{ad} = L_p(I, Y)$ with $2 \leq p$ and $q \geq 1$ such that $(1/p) + (1/q) = 1$, and there exists a constant $\beta > 0$ such that $\langle \mathcal{R}v, v \rangle_{L_q(Y^*), L_p(Y)} \geq \beta \|v\|^2_{L_p(Y)}$. Then there exists an optimal control.*

PROOF. First note that, under the given assumptions $Bu \in L_1(I, X)$ for each $u \in \mathcal{U}_{ad}$. Hence, by Theorem 3.2, for $\zeta \in X$ and each $u \in \mathcal{U}_{ad}$, equation (4.1) has a unique generalized solution $x(u) \equiv x(\,.\,, u) \in \mathcal{X}^* = W^{-m, \infty}(X)$. Thus the the cost functional (4.3) can be rewritten as

(4.4) $$J(u) = J_1(u) + J_2(u).$$

where

$$J_1(u) \equiv (1/2) \sum \lambda_i \left(\int_I \langle y(t, u), D^m \phi_i(t) \rangle_{X, X^*} \, dt \right)^2$$

$$J_2(u) \equiv \int_I \langle R(t)u(t), u(t) \rangle_{Y^*, Y} \, dt,$$

and D^m represents the m-th order distributional derivative with respect to time and y is given by

$$(4.5) \qquad y(t, u) = S(t)\zeta + \int_0^t S(t - s)B(s)u(s)ds, \quad t \in I.$$

Let

$$\text{Inf}\{J(u), u \in \mathcal{U}_{ad}\} = \eta,$$

and $\{u_n\}$ a minimizing sequence so that

$$(4.6) \qquad \lim_{n \to \infty} J(u_n) = \eta.$$

Since the first term in (4.4) is non negative and R is coercive, the sequence $\{u_n\}$ is a bounded sequence and hence due to reflexivity of $L_p(Y)$, there exist a subsequence, relabeled as $\{u_n\}$, and a control $u_0 \in \mathcal{U}_{ad}$ so that $u_n \xrightarrow{w} u_0$ in $L_p(Y)$. Since $B \in L_q(I, \mathcal{L}(Y, X))$ and $\mathcal{U}_{ad} = L_p(Y)$ and Y is reflexive, it follows from (4.5) that

$$y(u_n) \equiv y(\,.\,, u_n) \xrightarrow{w^*} y(\,.\,, u_0) \equiv y(u_0)$$

in $L_\infty(I, X)$. This, combined with the fact that every nuclear operator is compact, implies that the first term, $J_1(u)$, of the cost functional is weakly continuous on $L_p(Y)$. By virtue of positivity of the operator R, the second term in the cost functional is weakly lower semi continuous and hence J is weakly lower semicontinuous. Consequently

$$J(u_0) \leq \operatorname*{Lim\,inf}_{n \to \infty} J(u_n).$$

Combining this with (4.6), we have $J(u_0) = \eta$ proving that u_0 is the optimal control.

Remark. Note that if $\mathcal{U}_{ad}$ is a closed bounded convex subset of $L_p(I, Y)$, the coercivity condition for $\mathcal{R}$ is not necessary.

Now we present necessary conditions of optimality for the problem (4.1)–(4.3).

Theorem 4.2 (Necessary conditions of optimality). *Let $\mathcal{U}_{ad}$ be a closed convex subset of $L_p(Y)$ and suppose $u_0 \in \mathcal{U}_{ad}$ and $x_0 \equiv x(u_0)$ is the corresponding trajectory (generalized solution of (4.1)). Then, in order that the pair $\{u_0, x_0\}$ be optimal it is necessary that there exists a nontrivial*

$\psi_0 \in C(I, X^*) \cap L_\infty(I, X^*)$ such that the following inequality and equations hold:

(4.7)

$$(1) : \int_I \langle R(t)u_0(t) + B^*(t)\psi_0(t), u(t) - u_0(t)\rangle_{Y^*,Y} \, dt \geq 0,$$

$$\text{for all} \quad u \in \mathcal{U}_{ad},$$

$$(2) : -(d/dt)\psi_0 = \int_0^{T-t} da(r)A^*\psi_0(t+r) + \gamma^0(t), \quad \psi_0(T) = 0,$$

$$(3) : (d/dt)x_0(t) = \int_0^t da(r)Ax_0(t-r) + B(t)u_0(t), \quad x(0) = \zeta,$$

where
$$\gamma^0(t) \equiv (Qx_0)(t) = \sum \lambda_i \langle x_0, \phi_i\rangle_{\mathcal{X}^*,\mathcal{X}} \, \phi_i(t).$$

PROOF. First we prove that $u \longrightarrow x(u)$ is a Lipschitz continuous map from $L_p(Y)$ to $\mathcal{X}^*$ and that it is Frechet differentiable. Indeed let $u, v \in L_p(Y)$ and $x(u), x(v)$ the corresponding (generalized) solutions of equation (4.1). Then it is easy to verify that for any $\Theta \in \mathcal{X}$ we have

(4.8)

$$\left| \int_I \langle x(u) - x(v), \Theta\rangle_{X,X^*} \, dt \right|$$

$$= \left| (-1)^m \int_I \langle y(u) - y(v), D^m\Theta\rangle_{X,X^*} \, dt \right|$$

$$\leq \tilde{M}_T(\|B\|_{L_q(I,\mathcal{L}(Y,X))})(\|D^m\Theta\|_{L_1(I,X^*)})(\|(u-v)\|_{L_p(Y)}),$$

where $\tilde{M}_T$ is a suitable constant depending on T and the constants M, ω appearing in the inequality

$$\|S(t)\| \leq Me^{\omega t}.$$

Hence

(4.9)
$$\|(x(u) - x(v))\|_{\mathcal{X}^*} \leq (\tilde{M}_T\|B\|)\|(u-v)\|_{L_p(Y)}.$$

This shows that the map $u \longrightarrow x(u)$ is Lipschitz and Frechet differentiable as stated. From this fact it is easy to deduce that $u \longrightarrow J(u)$ is Frechet differentiable. Consider the pair $\{u_0, x_0\}$ as described in the statement of the theorem. By convexity, for any $u \in \mathcal{U}_{ad}$ and $0 < \epsilon < 1$, $u^\epsilon \equiv u_0 + \epsilon(u - u_0) \in \mathcal{U}_{ad}$. Let x^ϵ denote the solution of (4.1) corresponding to u^ϵ. Then by direct computation one can deduce that the Gateaux

differential of J at u_0 in the direction $u - u_0$ is given by

(4.10)
$$J^1(u_0, u - u_0) \equiv \int_I \Big\{ \langle R(t)u_0(t), u(t)$$
$$-u_0(t) \rangle_{Y^*,Y} + \langle \gamma^0(t), z_0(t) \rangle_{X^*,X} \Big\} dt,$$

where, as defined in the statement of the theorem, $\gamma^0 \in \mathcal{X}$ and z_0 is the Frechet differential of x at u_0 in the direction $u - u_0$ which is the (generalized) solution of equation

(4.11) $\quad (d/dt)z_0(t) = \int_0^t da(r)Az_0(t - r) + B(t)(u(t) - u_0(t)), \quad z_0(0) = 0.$

Since $\gamma^0 \in \mathcal{X}$, the adjoint equation given by

(4.12) $\qquad -(d/dt)\psi = \int_0^{T-t} da(r)A^*\psi(t + r) + \gamma^0(t), \quad \psi(T) = 0,$

has a unique solution $\psi_0 \in C(I, X^*) \cap L_\infty(I, X^*)$. This is (2) of equation (4.7). We shall justify this later. Using this fact in (4.10) and recalling that u_0 is optimal, we obtain

$$J^1(u_0, u - u_0) = \int_I \langle R(t)u_0(t) + B^*(t)\psi_0(t), u(t) - u_0(t) \rangle_{Y^*,Y} \geq 0$$

(4.13) $\qquad\qquad\qquad$ for all $\quad u \in \mathcal{U}_{ad}.$

This proves (1) of equation (4.7). Thus we have derived all the necessary conditions of optimality as stated in the theorem. To complete the proof, we must justify the existence of a solution of the adjoint equation (4.12). We show that the function

(4.14) $\qquad \psi(t) \equiv (-1)^m \int_t^T S^*(\theta - t)D^m\gamma^0(\theta)d\theta, \quad t \in I,$

is a solution of equation (4.12). Recall that $\gamma^0 \in \mathcal{X}$ and hence $D^m\gamma^0 \in L_1(I, X^*)$ and the integral is well defined since $S^*(t)$, $t \geq 0$, is a bounded operator valued function in X^*. It is also clear from this expression that $\psi \in C(I, X^*) \cap L_\infty(I, X^*)$. For $\zeta \in D(A)$, and $m \geq 0$, it can be shown [see 6, 1] that

$$S(t)\zeta = \begin{cases} (t^m/m!)\zeta + \int_0^t \left(\int_0^s da(r)AS(s - r)\zeta \right) ds, & m \geq 1, \\ \zeta + \int_0^t \left(\int_0^s da(r)AS(s - r)\zeta \right) ds, & m = 0. \end{cases}$$

Hence $S(.)\zeta \in C^1((0,T),X)$ and

$$(4.15) \quad (d/dt)S(t)\zeta = \begin{cases} (t^{m-1}/(m-1)!)\zeta + \int_0^t da(s)S(t-s)A\zeta, & m \geq 1 \\ \int_0^t da(s)S(t-s)A\zeta, & m = 0. \end{cases}$$

It follows from equation (4.14) and (4.15) that

$$(d/dt)\langle\psi(t),\zeta\rangle = (-1)^{m+1}\int_t^T \langle D^m\gamma^0(\theta),(\theta-t)^{m-1}/(m-1)!\zeta\rangle\, d\theta$$

$$(4.16) \quad +(-1)^{m+1}\int_t^T \left\langle D^m\gamma^0(\theta),\int_0^{\theta-t} da(r)S(\theta-t-r)A\zeta\right\rangle d\theta.$$

Recalling that $D^k\gamma^0(t)|_\partial = 0$ for all $0 \leq k \leq m-1$, it follows from elementary computation that

$$(4.17) \quad \int_t^T \langle D^m\gamma^0(\theta),(\theta-t)^{m-1}/(m-1)!\zeta\rangle\, d\theta = (-1)^m \langle\gamma^0(t),\zeta\rangle.$$

By integration by parts and using Fubini's theorem and (4.14), one can verify that

$$(4.18) \quad \begin{aligned} &\int_t^T \left\langle D^m\gamma^0(\theta),\int_0^{\theta-t} da(r)S(\theta-t-r)A\zeta\right\rangle d\theta \\ &= (-1)^m \int_0^{T-t} da(r)\langle\psi(t+r),A\zeta\rangle. \end{aligned}$$

Substituting (4.17) and (4.18) into equation (4.16) we obtain, for all $t \in I$,

$$(4.19) \quad (d/dt)\langle\psi(t),\zeta\rangle = -\int_0^{T-t} da(r)\langle\psi(t+r),A\zeta\rangle - \langle\gamma^0(t),\zeta\rangle,$$

$$\text{for all} \quad \zeta \in D(A).$$

Since $\zeta \in D(A)$ is arbitrary and, by our assumption, $D(A^{m+1})$ and hence $D(A)$ is dense in X, it follows that ψ given by (4.14) is the unique solution of the adjoint equation (4.12) and hence $\psi^0 = \psi$ is the unique solution of (4.7)(2). This completes the proof.

5. A computational algorithm

In this section we briefly present an algorithm for computation of optimal controls. Let ν denote the duality map:

$$\nu : L_q(Y^*) \setminus \{0\} \longrightarrow L_p(Y)$$

satisfying
$$(\nu(f), f)_{L_p(Y), L_q(Y^*)} = \|f\|_{L_q}^2 = \|\nu(f)\|_{L_p}^2.$$

If Y is strictly convex and $1 < p < \infty$ then $L_p(Y)$ is also a strictly convex Banach space and in that case the duality map ν is single valued and demicontinuous. Define the sequence of controls as follows

(4.20) $$u^{n+1} \equiv u^n - \epsilon\nu(J^1(u^n)), n \geq 1,$$

where J^1 denotes the Frechet derivative of J, and ϵ is a small positive number. The first term of the sequence can be chosen arbitrarily from the admissible class. Then one can verify using Lagrange formula (mean value theorem) that

(4.21)
$$J(u^{n+1}) = J(u^n) - \epsilon(J^1(u^n), \nu(J^1(u^n))) + o(\epsilon)$$
$$= J(u^n) - \epsilon\|J^1(u^n)\|_{L_q(Y^*)}^2 + o(\epsilon).$$

The algorithm can then be summarized as follows. Given u^n solve equation (4.7)(3) to obtain x^n. Using x^n compute γ^n replacing x_0 by x^n in (4.7). Using this γ^n solve the adjoint equation (4.7)(2) to obtain ψ^n. Using the pair $\{\psi^n, u^n\}$ compute $J^1(u^n) \equiv Ru^n + B^*\psi^n$ and define u^{n+1} using (4.20) and $J(u^{n+1})$ by (4.21). Stop if, for given tolerance, say $\delta > 0$, $|J(u^{n+1}) - J(u^n)| \leq \delta$; if not continue the process.

Remark. Here we have studied only linear problem. It would be interesting to study the possiblity of extension of these results to semilinear problems. In case $m = 0$, results for semilinear problems are available in [8]. For $m > 0$, the author is not aware of any such result.

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