The neutrix distribution product $x_+^\lambda \circ x_+^\mu$

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In the following we define the ordinary locally summable functions x_+^λ, x_-^λ for $\lambda > -1$ by

$$x_+^\lambda = \begin{cases} x^\lambda & \text{for } x > 0, \\ 0 & \text{for } x < 0, \end{cases}$$

$$x_-^\lambda = \begin{cases} 0 & \text{for } x > 0, \\ (-x)^\lambda & \text{for } x < 0. \end{cases}$$

We define the distributions x_+^λ, x_-^λ for $\lambda < -1$ and $\lambda \neq -2, -3, \ldots$ inductively by

$$x_+^\lambda = (\lambda+1)^{-1}(x_1^{\lambda+1})',$$

$$x_-^\lambda = -(\lambda+1)^{-1}(x_-^{\lambda+1})'.$$

We define the distributions $|x|^\lambda$, $\operatorname{sgn} x \cdot |x|^\lambda$ for $\lambda \neq -1, -2, \ldots$ by

$$|x|^\lambda = x_+^\lambda + x_-^\lambda, \quad \operatorname{sgn} x . |x|^\lambda = x_+^\lambda - x_-^\lambda.$$

It follows that

$$(|x|^\lambda)' = \lambda \operatorname{sgn} x . |x|^{\lambda-1},$$

$$(\operatorname{sgn} x . |x|^\lambda)' = \lambda |x|^{\lambda-1}.$$

Further, if $\lambda > -r-1, \lambda \neq -1, -2, \ldots, -r$ and φ is an arbitrary test function with compact support, then

$$(x_+^\lambda, \varphi) = \int_0^1 x^\lambda \left[\varphi(x) - \sum_{m=0}^{r-1} \frac{\varphi^{(m)}(0)}{m!} x^m\right] dx + \int_1^\infty x^\lambda \varphi(x)\, dx + \sum_{m=0}^{r-1} \frac{\varphi^{(m)}(0)}{m!(\lambda+m+1)}$$

see GELFAND and SHILOV [7].

The following definition was given in [2].

Definition 1. Let f and g be two distributions for which on the open interval (a, b), f is the r-th derivative of an ordinary summable function F in $L^p(a, b)$ and $g^{(r)}$ is an ordinary summable function in $L^q(a, b)$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then the product $fg = gf$ of f and g is defined on the interval (a, b) by

$$fg = gf = \sum_{i=0}^{r} \binom{r}{i} (-1)^i [Fg^{(i)}]^{(r-i)},$$

where

$$\binom{r}{i} = \frac{r!}{i!(r-i)!}.$$

The next definition was given by van der CORPUT [1].

Definition 2. A neutrix N is a commutative additive group of functions $\nu(\xi)$ defined on a domain N' with values in an additive group N'', where further if for some ν in N, $\nu(\xi)=\gamma$ for all ξ in N', then $\gamma=0$. The functions in N are called negligible functions. Now let N' be a set contained in a topological space with a limit point b which does not belong to N'. If $f(\xi)$ is a function defined on N' with values in N'' and it is possible to find a constant β such that $f(\xi)-\beta$ is negligible in N, then β is called the neutrix limit of f as ξ tends to b and we write

$$N-\lim_{\xi\to\infty} f(\xi) = \beta,$$

where the limit β must be unique if it exists.

Now let ϱ be a fixed infinitely differentiable function having the properties

(i) $$\varrho(x) = 0 \quad \text{for} \quad |x| \geqq 1,$$

(ii) $$\varrho(x) \geqq 0,$$

(iii) $$\varrho(x) = \varrho(-x),$$

(iv) $$\int_{-1}^{1} \varrho(x)\, dx = 1.$$

We define the function δ_n by

$$\delta_n(x) = n(\varrho n x)$$

for $n=1, 2, \ldots$. It is obvious that $\{\delta_n\}$ is a regular sequence of infinitely differentiable functions converging to the Dirac delta-distribution δ. For an arbitrary distribution g we define the function g_n by

$$g_n(x) = g * \delta_n(x) = \int_{-1/n}^{1/n} g(x-t)\delta_n(t)\, dt$$

for $n=1, 2, \ldots$. The sequence $\{g_n\}$ is regular and converges to g.

The following definition now extends definition 1 to a wider class of distributions and was given in [4].

Definition 3. Let f and g be arbitrary distributions and let

$$g_n = g * \delta_n.$$

We say that the neutrix product *fog* of f and g exists and is equal to h on the open interval (a, b) if

$$N-\lim_{n\to\infty} (fg_n, \varphi) = N-\lim_{n\to\infty} (f, g_n\varphi) = (h, \varphi)$$

for all test functions φ with compact support contained in the interval (a, b), where N is the neutrix having domain $N'=\{1, 2, \ldots, n, \ldots\}$ and range N'' the real numbers with negligible functions linear sums of the functions

$$n^\lambda \ln^{r-1} n, \quad \ln^r n$$

for $\lambda>0$ and $r=1, 2, \ldots$ and all functions of n that converge to zero as n tends to infinity.

The following theorems are immediate consequences of theorems given in [3] and [4].

Theorem 1. *Let f and g be distributions. If the product fg exists on the open interval (a, b) then the neutrix products $f\circ g$ and $g\circ f$ exist and*

$$f\circ g = g\circ f = fg$$

on this interval.

Theorem 2. *Let f and g be distributions and suppose that the neutrix products $f\circ g$ and $f\circ g'$ (or $f'\circ g$) exist on the open interval (a, b). Then the neutrix product $f'\circ g$ (or $f\circ g'$) exists and*

$$(f\circ g)' = f'\circ g + f\circ g'$$

on this interval.

The next theorem was proved in [5].

Theorem 3. *The neutrix products $x_+^\lambda \circ x_-^\mu$ and $x_-^\lambda \circ x_+^\mu$ exist and*

$$x_+^\lambda \circ x_-^\mu = x_-^\lambda \circ x_+^\mu = 0$$

for $\lambda+\mu\neq -1, -2, \ldots$.

We now prove the following theorem.

Theorem 4. *The neutrix product $x_+^\mu \circ x_+^\lambda$ exists and*

(1) $$x_+^\lambda \circ x_+^\mu = x_+^{\lambda+\mu}$$

for $\lambda, \mu, \lambda+\mu\neq -1, -2, \ldots$.

Proof. We will first of all suppose that $\lambda>-1$ and put

$$(x_+^\mu)_n = x_+^\mu * \delta_n(x).$$

Then it follows that

$$\prod_{i=1}^{r} (\mu+i)(x_+^\mu)_n = x_+^{\mu+r} * \delta_n^{(r)}(x) =$$

$$= \begin{cases} \int\limits_{-1/n}^{1/n} (x-t)^{\mu+r} \delta_n^{(r)}(t)\, dt & \text{for} \quad x > 1/,n \\ \int\limits_{-1/n}^{x} (x-t)^{\mu+r} \delta_n^{(r)}(t) dt & \text{for} \quad -1/n \leqq x \leqq 1/n, \\ 0 & \text{for} \quad x < -1/n, \end{cases}$$

where r is a non-negative integer chosen so that $\mu+r>0$. Thus

$$\prod_{i=1}^{r}(\mu+i)\int_0^1 x^{\lambda+m}(x_+^\mu)_n\,dx = \int_0^{-1/n} x^{\lambda+m}\int_{-1/n}^{x}(x-t)^{\mu+r}\delta_n^{(r)}(t)\,dt\,dx+$$

$$+\int_{1/n}^{1} x^{\lambda+m}\int_{-1/n}^{1/n}(x-t)^{\mu+r}\delta_n^{(r)}(t)\,dt\,dx =$$

$$= n^{-\lambda-\mu-m-1}\int_0^1 u^{\lambda+m}\int_{-1}^{u}(u-v)^{\mu+r}\varrho^{(r)}(v)\,dv\,du+$$

$$+n^{-\lambda-\mu-m-1}\int_1^n u^{\lambda+m}\int_{-1}^{1}(u-v)^{\mu-r}\varrho^{(r)}(v)\,dv\,du = I_1+I_2,$$

where the substitutions $nx=u$ and $nt=v$ have been made.

Since $\lambda+\mu\neq-1,-2,\ldots,$ we see immediately that I_1 is negligible for $m=0,1,2,\ldots$. Further

$$\int_1^n u^{\lambda+m}\int_{-1}^{1}(u-v)^{\mu+r}\varrho^{(r)}(v)\,dv\,du = \int_{-1}^{1}\varrho^{(r)}(v)\int_1^n u^{\lambda+\mu+r+m}(1-v/u)^{\mu+r}\,du\,dv$$

and

$$\int_1^n u^{\lambda+\mu+r+m}(1-v/u)^{\mu+r}\,du =$$

$$= \int_1^n u^{\lambda+\mu+r+m}\left[1-(\mu+r)v/u+\frac{(\mu+r)(\mu+r-1)}{2!}\frac{v^2}{u^2}-\ldots\right]du =$$

$$= n^{\lambda+\mu+r+m+1}\left[\frac{1}{\lambda+\mu+r+m+1}-\frac{(\mu+r)v}{(\lambda+\mu+r+m)n}+\frac{(\mu+r)(\mu+r-1)v^2}{2!(\lambda+\mu+r+m-1)n^2}-\ldots\right]-$$

$$-\left[\frac{1}{\lambda+\mu+r+m+1}-\frac{(\mu+r)v}{\lambda+\mu+r+m}+\ldots\right].$$

It follows that

$$N-\lim_{n\to\infty} I_2 = (-1)^r\prod_{i=1}^{r}(\mu+i)[r!\,(\lambda+\mu+m+1)]^{-1}\int_{-1}^{1}v^r\varrho^{(r)}(v)\,dv =$$

$$= \prod_{i=1}^{r}(\mu+i)/(\lambda+\mu+m+1)$$

and so

$$\int_0^1 x^{\lambda+m}(x_+^\mu)_n\,dx = (\lambda+\mu+m+1)^{-1}$$

for $m=0,1,2,\ldots$.

Now let φ be an arbitrary test function with compact support contained in the interval (a, b). Then

$$(x_+^\lambda, (x_+^\mu)_n\varphi) = \int_0^\infty x^\lambda (x_+^\mu)_n \varphi(x)\,dx = \int_0^1 x^\lambda (x_+^\mu)_n \left[\varphi(x) - \sum_{m=0}^{r-1} \frac{\varphi^{(m)}(0)}{m!} x^m\right] dx +$$

$$+ \sum_{m=1}^{r-1} \frac{\varphi^{(m)}(0)}{m!} \int_0^1 x^{\lambda+m} (x_+^\mu)_n\,dx + \int_1^\infty x^\lambda (x_+^\mu)_n \varphi(x)\,dx.$$

By Taylor's theorem

$$\varphi(x) - \sum_{m=0}^{r-1} \frac{\varphi^{(m)}(0)}{m!} x^m = \frac{x^r}{r!} \varphi^{(r)}(\xi x),$$

where $0 \leqq \xi \leqq 1$ and so

$$(x_+^\lambda, (x_+^\mu)_n\varphi) = \frac{1}{r!} \int_0^1 x^\lambda [x^r (x_+^\mu)_n] \varphi^{(r)}(\xi x)\,dx +$$

$$+ \sum_{m=0}^{r-1} \frac{\varphi^{(m)}(0)}{m!} \int_0^1 x^{\lambda+m} (x_+^\mu)_n\,dx + \int_1^\infty x^\lambda (x_+^\mu)_n \varphi(x)\,dx.$$

Since the sequence of continuous functions $\{x^r (x_+^\mu)_n\}$ converges uniformly to the continuous function $x^{\mu+r}$ $(\mu + r > 0)$ on the closed interval $[0, 1]$ and the sequence of continuous functions $\{(x_+^\mu)_n\}$ converges uniformly to the continuous function x^μ on the closed interval $[a, b] \cap [1, \infty)$, it follows that

$$N - \lim_{n\to\infty} (x_+^\lambda, (x_+^\mu)_n\varphi) = \lim_{n\to\infty} \frac{1}{r!} \int_0^1 x^\lambda [x^r (x_+^\mu)_n] \varphi^{(r)}(\xi x)\,dx +$$

$$+ N - \lim_{n\to\infty} \sum_{m=0}^{r-1} \frac{\varphi^{(m)}(0)}{m!} \int_0^1 x^{\lambda+m} (x_+^\mu)_n\,dx + \lim_{n\to\infty} \int_1^\infty x^\lambda (x_+^\mu)_n \varphi(x)\,dx =$$

$$= \frac{1}{r!} \int_0^1 x^{\lambda+\mu} x^r \varphi^{(r)}(\xi x)\,dx + \sum_{m=0}^{r-1} \frac{\varphi^{(m)}(0)}{m!(\lambda+\mu+m+1)} + \int_1^\infty x^{\lambda+\mu} \varphi(x)\,dx =$$

$$= \int_0^1 x^{\lambda+\mu} \left[\varphi(x) - \sum_{m=0}^{r-1} \frac{\varphi^{(m)}(0)}{m!} x^m\right] dx + \int_1^\infty x^{\lambda+\mu} \varphi(x)\,dx +$$

$$+ \sum_{m=0}^{r-1} \frac{\varphi^{(m)}(0)}{m!(\lambda+\mu+m+1)} = (x_+^{\lambda+\mu}, \varphi).$$

This proves that the neutrix product $x_-^\lambda \circ x_-^\mu$ exists and

$$x_+^\lambda \circ x_+^\mu = x_+^{\lambda+\mu}$$

for $\lambda > -1$ and $\mu, \lambda+\mu \neq -1, -2, \ldots$.

Now suppose that the neutrix product $x_+^\lambda \circ x_+^\mu$ exists and satisfies equation (1) for $-k-1 < \lambda - k$ and $\mu, \lambda+\mu \neq -1, -2, \ldots$, where k is a positive integer. Then it follows from theorem 2 that the neutrix product $x_+^\lambda \circ x_+^\mu$ exists and satisfies equation (1) for $-k-2<\lambda<-k-1$ and $\mu, \lambda+\mu\neq -1, -2, \ldots$. Since equation (1) is certainly satisfied for $-1<\lambda<0$ and $\mu, \lambda+\mu\neq -1, -2, \ldots$ the result of the theorem follows by inducfion. This completes the proof of the theorem.

Corollary 4.1. *The neutrix product* $x_-^\lambda \circ x_-^\mu$ *exists and*

$$x_-^\lambda \circ x_-^\mu = x_-^{\mu+\mu}$$

for $\lambda, \mu, \lambda+\mu\neq -1, -2, \ldots$.

PROOF. The result follows immediately from the theorem on replacing x in equation (1) by $-x$.

Corollary 4.2. *The neutrix products* $|x|^\lambda \circ |x|^\mu$, $(\operatorname{sgn} x\cdot|x|^\lambda)\circ(\operatorname{sgn} x\cdot|x|^\mu)$, $|x|^\lambda\circ$ $\circ(\operatorname{sgn} x\cdot|x|^\mu)$ *and* $(\operatorname{sgn} x\cdot|x|^\lambda)\circ|x|^\mu$ *exist and*

$$|x|^\lambda\circ|x|^\mu = (\operatorname{sgn} x.|x|^\lambda)\circ(\operatorname{sgn} x.|x|^\mu) = |x|^{\lambda+\mu},$$

$$|x|^\lambda\circ(\operatorname{sgn} x.|x|^\mu) = (\operatorname{sgn} x.|x|^\lambda)\circ|x|^\mu = \operatorname{sgn} x.|x|^{\lambda+\mu}$$

for $\lambda, \mu, \lambda+\mu\neq -1, -2, \ldots$.

PROOF. Since the neutrix product is obviously distributive with respect to addition we have

$$|x|^\lambda\circ|x|^\mu = (x_+^\lambda+x_-^\lambda)\circ(x_+^\mu+x_-^\mu) =$$

$$= x_+^\lambda\circ x_+^\mu + x_+^\lambda\circ x_-^\mu + x_-^\lambda\circ x_+^\mu + x_-^\lambda\circ x_-^\mu = |x|^{\lambda+\mu}$$

on using theorems 3 and 4 and corollary 4.1.

The other results follows similarly.

Corollary 4.3. *The neutrix products* $(x+i0)^\lambda\circ(x+i0)^\mu$ *and* $(x-i0)^\lambda\circ(x-i0)^\mu$ *exist and*

$$(x+i0)^\lambda\circ(x+i0)^\mu = (x+i0)^{\lambda+\mu},$$

$$(x-i0)^\lambda\circ(x-i0)^\mu = (x-i0)^{\lambda+\mu}$$

for $\lambda, \mu, \lambda+\mu\neq -1, -2, \ldots$.

PROOF. The distributions $(x+i0)^\lambda$ and $(x-i0)^\mu$ are defined by

$$(x+i0)^\lambda = x_+^\lambda + e^{i\lambda\pi}x_-^\lambda,$$

$$(x-i0)^\lambda = x_+^\lambda + e^{-i\lambda\pi}x_-^\lambda$$

for $\lambda\neq -1, -2, \ldots$, see GELFAND and SHILOV [7]. The results follow immediately from theorems 3 and 4 and corollary 4.1.

Corollary 4.4. *The neutrix products* $x_\lambda^+\circ\delta^{(r)}$ *and* $\delta^{(r)}\circ x_+^\lambda$ *exist and*

$$x_+^\lambda\circ\delta^{(r)} = \delta^{(r)}\circ x_+^\lambda = 0$$

for $\lambda\neq 0, 1, \ldots, r, -1, -2, \ldots$ *and* $r=0, 1, 2, \ldots$.

PROOF. From the theorem we have

$$x_+^\lambda\circ H = x_+^\lambda$$

for $\lambda \neq -1, -2, \ldots,$ where $H = x_+^0$. It follows from theorems 2 and 4 that the neutrix product $x_+^\lambda \circ \delta$ exists and

$$x_+^\lambda \circ \delta = 0$$

for $\lambda \neq -1, -2, \ldots$. A simple induction argument now proves that

$$x_+^\lambda \circ \delta^{(r)} = 0$$

for $\lambda \neq -1, -2, \ldots$ and $r = 0, 1, 2, \ldots$.

The existence of the neutrix product $\delta^{(r)} \circ x_+^\lambda$ follows similarly.
The result of this corollary was given in [6].

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