On equivalence of variational problems subject to constraints

By MAGDALEN SZ. KIRKOVITS (Sopron)

1 §. Introduction

Let us consider two n-dimensional parameter-invariant variational problems subject to constraints. The fundamental functions are of the form $F(x, \dot{x})$ and $F^*(x, \dot{x})$; $x:=(x^1, x^2, \ldots, x^n)$, $\dot{x}:=(\dot{x}^1, \dot{x}^2, \ldots, \dot{x}^n)$.

The variational problems subject to constraints are formulated as follows: it is required to find curves C and C^* respectively joining two given fixed points. These curves must satisfy some equations of constraint (see [4]) of the form [1])

$$\underset{(\varrho)}{G}(x, \dot{x}) := \underset{(\varrho)}{A_k}(x)\dot{x}^k = 0 \quad (\varrho = 1, 2, \ldots, m). \tag{1.1}$$

Furthermore C must afford an extreme value to the integral $\mathfrak{J} = \int_{t_0}^{t_1} F(x, \dot{x})\,dt$ relative to other curves joining the same points which also satisfy the conditions (1.1). C^* must do the same with respect to $\mathfrak{J}^* = \int_{t_0}^{t_1} F^*(x, \dot{x})\,dt$. It is known (cf. H. Rund [3] page 338) that the extremals of these problems are those solutions of the Euler—Lagrange equations

$$\text{(a) } \mathscr{E}_i(F + \lambda^\varrho \underset{(\varrho)}{G}) = 0 \quad \text{and} \quad \text{(b) } \mathscr{E}_i(F^* + \lambda^{*\varrho} \underset{(\varrho)}{G}) = 0 \tag{1.2}$$

respectively, which satisfy the conditions (1.1). In these equations λ^ϱ and $\lambda^{*\varrho}$ are unknown constants.

Definition. Two variational problems subject to constraints (1.1) with fundamental functions $F(x, \dot{x})$ and $F^*(x, \dot{x})$ respectively, are called equivalent, if the relations

$$\mathscr{E}_i(F^* + \lambda^{*\varrho} \underset{(\varrho)}{G}) - \mu(x, \dot{x})\mathscr{E}_i(F + \lambda^\varrho \underset{(\varrho)}{G}) \equiv \Phi_i^\varrho(x, \dot{x}) \underset{(\varrho)}{G}(x, \dot{x}); \tag{1.3}$$

$$\Phi_i^\varrho(x, \dot{x}) \neq 0, \quad \mu(x, \dot{x}) \neq 0$$

---

[1]) Here and in the following Greek indices run from 1 to m, and Einstein-summation convention is applied for Latin and Greek indices too.

hold identically in $(x^i, \dot{x}^i, \ddot{x}^i)$ with constants $\lambda^\varrho, \lambda^{*\varrho}, \mathscr{E}_i$ being the Euler—Lagrange operators:

(1.3a) $$\mathscr{E}_i := \partial_i - \frac{d}{dt}\dot{\partial}_i \quad \left(\partial_i := \frac{\partial}{\partial x^i};\ \dot{\partial}_i := \frac{\partial}{\partial \dot{x}^i}\right).$$

$\mu(x, \dot{x})$ and $\Phi_i^\varrho(x, \dot{x})$ have to be homogeneous functions of degree zero in $\dot{x}^i$.

It is clear from the definition of the equivalence, that if a curve $x(t)$ satisfies $\underset{(\varrho)}{G}(x, \dot{x})=0$ and $\mathscr{E}_i(F+\lambda^\varrho \underset{(\varrho)}{G})=0$, then $\mathscr{E}_i(F^*+\lambda^{*\varrho}\underset{(\varrho)}{G})=0$ holds also, and conversely. In this paper we investigate the form of the functions $\mu(x, \dot{x})$ and $F(x, \dot{x})$, $F^*(x, \dot{x})$ satisfying (1.3). At the end of the paper we make some geometrical remarks.

The author wishes to express her gratitude to Professor A. Moór for his kind advices and encouragements.

2§. The independence of μ of $\dot{x}^i$

The relation (1.3) has the explicit form

(2.1) $$\partial_i F^* - \frac{d}{dt}\dot{\partial}_i F^* + \lambda^{*\varrho}(\partial_i \underset{(\varrho)}{A}_k - \partial_k \underset{(\varrho)}{A}_i)\dot{x}^k - \mu(x, \dot{x}) \times$$
$$\times \left[\partial_i F - \frac{d}{dt}\dot{\partial}_i F + \lambda^\varrho(\partial_i \underset{(\varrho)}{A}_k - \partial_k \underset{(\varrho)}{A}_i)\dot{x}^k\right] \equiv \Phi_i^\varrho(x, \dot{x}) \underset{(\varrho)}{A}_k \dot{x}^k.$$

Performing the derivations with respect to t and multiplying with (-1) we get:

(2.2) $$\left(\dot{\partial}_i\dot{\partial}_k F^* - \mu(x, \dot{x})\dot{\partial}_i\dot{\partial}_k F\right)\ddot{x}^k +$$
$$+\left[\dot{\partial}_i\partial_k F^* - \mu(x, \dot{x})\dot{\partial}_i\partial_k F - (\lambda^{*\varrho} - \mu(x, \dot{x})\lambda^\varrho)(\partial_i \underset{(\varrho)}{A}_k - \partial_k \underset{(\varrho)}{A}_i) + \Phi_i^\varrho \underset{(\varrho)}{A}_k\right]\times$$
$$\times \dot{x}^k - \partial_i F^* + \mu(x, \dot{x})\partial_i F \equiv 0.$$

In the case without constraints H. Rund has already proved that if the relations $\mathscr{E}_i(F^*(x, \dot{x})) \equiv \mu(x, \dot{x})\mathscr{E}_i(F(x, \dot{x}))$ hold identically, then μ is necessarily independent of $\dot{x}^j$ (see [2]). Since in the identity (2.2) the coefficient of $\ddot{x}^k$ is the same as in the case without constraints, so we obtain analogously to H. Rund [2] the relations

(2.3) $$F^*(x, \dot{x}) = \mu(x, \dot{x})F(x, \dot{x}) + \psi(x, \dot{x}),$$

where

(2.3a) $$\dot{\partial}_i\dot{\partial}_k\psi \equiv -\left(F\dot{\partial}_i\dot{\partial}_k\mu + (\dot{\partial}_k\mu)\dot{\partial}_i F + (\dot{\partial}_i\mu)\dot{\partial}_k F\right).$$

It is evident that $\psi(x, \dot{x})$ must be homogeneous of first degree in $\dot{x}^i$. The method of H. Rund used in [2] can also be applied for equations of type (2.1). Let us substitute (2.3) into (2.1). With respect to (2.3a) it reduces to

(2.4) $$F\partial_i\mu + \partial_i\psi -$$
$$-\left[F\dot{\partial}_i\partial_k\mu + (\dot{\partial}_i\mu)\partial_k F + (\partial_k\mu)\dot{\partial}_i F + \dot{\partial}_i\partial_k\psi - (\lambda^{*\varrho} - \mu(x, \dot{x})\lambda^\varrho)(\partial_i \underset{(\varrho)}{A}_k - \partial_k \underset{(\varrho)}{A}_i)\right]\dot{x}^k \equiv$$
$$\equiv \Phi_i^\varrho \underset{(\varrho)}{A}_k \dot{x}^k.$$

From the homogenity of ψ it follows

$$\psi(x,\dot{x}) = \psi_j(x,\dot{x})\dot{x}^j \quad (\psi_j := \dot{\partial}_j\psi(x,\dot{x})), \tag{2.5}$$

which also implies that

$$\partial_i\psi = \partial_i\psi_j\dot{x}^j. \tag{2.6}$$

With the aid of (2.5) and (2.6) we can write (2.4) in the following form:

$$\dot{x}^k(\partial_i\psi_k - \partial_k\psi_i) \equiv \Phi_i^\varrho \underset{(\varrho)}{A}_k \dot{x}^k + \tag{2.7}$$

$$+[F\dot{\partial}_i\partial_k\mu + (\dot{\partial}_i\mu)\partial_k F + (\partial_k\mu)\dot{\partial}_i F]\dot{x}^k - (\lambda^{*\varrho} - \mu(x,\dot{x})\lambda^\varrho)(\partial_i \underset{(\varrho)}{A}_k - \partial_k \underset{(\varrho)}{A}_i)\dot{x}^k - F\partial_i\mu.$$

Let us differentiate this with respect to $\dot{x}^j$. It follows with respect to $\dot{\partial}_j \underset{(\varrho)}{A}_k \equiv 0$

$$(\partial_i\psi_j - \partial_j\psi_i) + \dot{x}^k(\partial_i\dot{\partial}_j\psi_k - \partial_k\dot{\partial}_j\psi_i) \equiv \dot{\partial}_j\Phi_i^\varrho \underset{(\varrho)}{A}_k \dot{x}^k + \Phi_i^\varrho \underset{(\varrho)}{A}_j + \tag{2.8}$$

$$+[F\dot{\partial}_i\partial_k\dot{\partial}_j\mu + (\dot{\partial}_j F)\dot{\partial}_i\partial_k\mu + (\dot{\partial}_i\dot{\partial}_j\mu)\partial_k F + (\dot{\partial}_i\mu)\partial_k\dot{\partial}_j F + (\partial_k\dot{\partial}_j\mu)\dot{\partial}_i F + (\partial_k\mu)\dot{\partial}_i\dot{\partial}_j F]\dot{x}^k +$$

$$+F\dot{\partial}_i\partial_j\mu + (\dot{\partial}_i\mu)\partial_j F + (\partial_j\mu)\dot{\partial}_i F - (\lambda^{*\varrho} - \mu(x,\dot{x})\lambda^\varrho)(\partial_i \underset{(\varrho)}{A}_j - \partial_j \underset{(\varrho)}{A}_i) +$$

$$+\lambda^\varrho\dot{\partial}_j\mu(\partial_i \underset{(\varrho)}{A}_k - \partial_k \underset{(\varrho)}{A}_i)\dot{x}^k - F\partial_i\dot{\partial}_j\mu - (\partial_i\mu)\dot{\partial}_j F.$$

We differentiate (2.3a) with respect to x^j and we calculate the identity $\dot{\partial}_i\partial_k\dot{\partial}_j\psi \equiv \equiv \dot{\partial}_j\partial_k\psi_i$. Substituted this in (2.8); we have

$$\begin{aligned}(\dot{\partial}_i\mu)\partial_j F + \dot{x}^k[(\partial_k\mu)\dot{\partial}_i\dot{\partial}_j F - (\dot{\partial}_j\mu)\dot{\partial}_i\partial_k F] +\\ +\lambda^\varrho\dot{\partial}_j\mu(\partial_i \underset{(\varrho)}{A}_k - \partial_k \underset{(\varrho)}{A}_i)\dot{x}^k + \Phi_i^\varrho \underset{(\varrho)}{A}_j + \dot{\partial}_j\Phi_i^\varrho \underset{(\varrho)}{A}_k \dot{x}^k \equiv\\ \equiv \partial_i\psi_j - \partial_j\psi_i + (\partial_i\mu)\dot{\partial}_j F - (\partial_j\mu)\dot{\partial}_i F - F(\dot{\partial}_i\partial_j\mu - \dot{\partial}_j\partial_i\mu) +\\ +(\lambda^{*\varrho} - \mu(x,\dot{x})\lambda^\varrho)(\partial_i \underset{(\varrho)}{A}_j - \partial_j \underset{(\varrho)}{A}_i).\end{aligned} \tag{2.9}$$

The right hand side is skew-symmetric in (i, j), thus the symmetric part in (i, j) of the left hand side is identically zero:

(2.10)

$$\frac{1}{2}[(\dot{\partial}_i\mu)\partial_j F + (\dot{\partial}_j\mu)\partial_i F] + \dot{x}^k(\partial_k\mu)\dot{\partial}_i\dot{\partial}_j F - \frac{\dot{x}^k}{2}[(\dot{\partial}_j\mu)\dot{\partial}_i\partial_k F + (\dot{\partial}_i\mu)\dot{\partial}_j\partial_k F] +$$

$$+\frac{\lambda^\varrho}{2}\dot{x}^k[\dot{\partial}_j\mu(\partial_i \underset{(\varrho)}{A}_k - \partial_k \underset{(\varrho)}{A}_i) + \dot{\partial}_i\mu(\partial_j \underset{(\varrho)}{A}_k - \partial_k \underset{(\varrho)}{A}_j)] +$$

$$+\frac{1}{2}(\Phi_i^\varrho \underset{(\varrho)}{A}_j + \Phi_j^\varrho \underset{(\varrho)}{A}_i) + \frac{1}{2}(\dot{\partial}_j\Phi_i^\varrho + \dot{\partial}_i\Phi_j^\varrho)\underset{(\varrho)}{A}_k \dot{x}^k \equiv 0.$$

After multiplication by 2, we write (2.10) in the following form

$$(2.11)\qquad +\dot\partial_j\mu[\partial_i F-\dot\partial_i\partial_k F\dot x^k-\dot\partial_i\dot\partial_k F\ddot x^k+\lambda^\varrho(\partial_i \underset{(\varrho)}{A}_k-\partial_k \underset{(\varrho)}{A}_i)\,\dot x^k]+$$

$$+2(\dot x^k\partial_k\mu+\ddot x^k\dot\partial_k\mu)\,\dot\partial_i\dot\partial_j F+\Phi_i^\varrho \underset{(\varrho)}{A}_j+\Phi_j^\varrho \underset{(\varrho)}{A}_i+(\dot\partial_j\Phi_i^\varrho+\dot\partial_i\Phi_j^\varrho)\underset{(\varrho)}{A}_k\,\dot x^k\equiv$$

$$\equiv-\ddot x^k[(\dot\partial_i\mu)\,\dot\partial_j\dot\partial_k F+(\dot\partial_j\mu)\,\dot\partial_i\dot\partial_k F-2(\dot\partial_k\mu)\,\dot\partial_i\dot\partial_j F].$$

$$\equiv-\ddot x^k[(\dot\partial_i\mu)\,\dot\partial_j\dot\partial_k F+(\dot\partial_j\mu)\,\dot\partial_i\dot\partial_k F-2(\dot\partial_k\mu)\,\dot\partial_i\dot\partial_j F].$$

This can be expressed as

$$(2.12)\qquad \dot\partial_i\mu\,\mathscr{E}_j(F+\lambda^\varrho \underset{(\varrho)}{G})+\dot\partial_j\mu\,\mathscr{E}_i(F+\lambda^\varrho \underset{(\varrho)}{G})+$$

$$+2\frac{d\mu}{dt}\dot\partial_i\dot\partial_j F+\Phi_i^\varrho \underset{(\varrho)}{A}_j+\Phi_j^\varrho \underset{(\varrho)}{A}_i+(\partial_j\Phi_i^\varrho+\dot\partial_i\Phi_j^\varrho)\underset{(\varrho)}{A}_k\dot x^k\equiv$$

$$\equiv-\ddot x^k[(\dot\partial_i\mu)\,\dot\partial_j\dot\partial_k F+(\dot\partial_j\mu)\,\dot\partial_i\dot\partial_k F-2(\dot\partial_k\mu)\,\dot\partial_i\dot\partial_j F].$$

Since Φ_i^ϱ are covariant vectors, the left hand side of (2.12) is a tensor, and this is true also for the coefficient of $\ddot x^k$ on the right hand side. But $\ddot x^k$ itself is not a vector, and hence (2.12) can be invariant iff this coefficient vanishes (see [2] page 24), so that

$$(2.13)\qquad 2(\dot\partial_k\mu)\,\dot\partial_i\dot\partial_j F\equiv(\dot\partial_i\mu)\,\dot\partial_j\dot\partial_k F+(\dot\partial_j\mu)\,\dot\partial_i\dot\partial_k F.$$

It is well-known from the Finsler-Geometry that

$$(2.14)\qquad g_{ij}=l_i l_j+h_{ij},$$

where

$$(2.14a)\qquad l_i=\dot\partial_i F;\quad h_{ij}=F\,\dot\partial_i\dot\partial_j F.$$

We can write (2.13) after multiplication by $F(x,\dot x^i)$ in the following form

$$(2.15)\qquad 2\,\dot\partial_k\mu\,h_{ij}\equiv\dot\partial_i\mu\,h_{jk}+\dot\partial_j\mu\,h_{ik}.$$

By contraction of (2.14) by g^{ij} with respect to (2.14a) we get

$$(2.16)\qquad h_j^h\equiv Fg^{ih}\,\dot\partial_i\partial_j F=\delta_j^h-l^h l_j.$$

In particular, contracting over h and j, we find

$$(2.17)\qquad h_j^j\equiv Fg^{ij}\,\dot\partial_i\dot\partial_j F=\delta_j^j-l^j l_j=n-1.$$

We now multiply (2.15) by g^{ij} and since μ is homogeneous of degree zero in $\dot x^i$, using (2.17), it is found that

$$(2.18)\qquad 2(n-1)\,\dot\partial_k\mu=\dot\partial_i\mu(\delta_k^i-l^i l_k)+\dot\partial_j\mu(\delta_k^j-l^j l_k)=2\,\dot\partial_k\mu.$$

This gives

$$(2.18a)\qquad 2(n-2)\,\dot\partial_k\mu=0.$$

Excluding the special case $n=2$, we therefore infer that

$$(2.19)\qquad \dot\partial_k\mu=0.$$

Thus μ is independent of $\dot x^k$.

We can summarize the result in

Theorem 1. *If* (1.3) *and* (1.1) *hold, then for* $n>2$, μ *is necessarily independent of* $\dot{x}^k$.

3§. The explicit form of F and F^*

Since according to Theorem 1 μ is independent of $\dot{x}^i$, it follows from (2.3a) on account of the homogenity of ψ in $\dot{x}^i$, that $\psi(x, \dot{x})$ must be linear in $\dot{x}^k$:

$$\psi(x, \dot{x}) = S_k(x)\dot{x}^k. \tag{3.1}$$

Substituting this into (2.3) we find

$$F^*(x, \dot{x}) = \mu(x)F(x, \dot{x}) + S_k(x)\dot{x}^k. \tag{3.2}$$

Furthermore, let us substitute (3.1) and (3.2) in (2.4). Thus we have

$$F\partial_i\mu - \frac{d\mu}{dt}\dot{\partial}_i F + (\partial_i S_k - \partial_k S_i)\dot{x}^k + \\ +(\lambda^{*\varrho} - \mu(x)\lambda^\varrho)(\partial_i \underset{(\varrho)}{A}_k - \partial_k \underset{(\varrho)}{A}_i)\dot{x}^k \equiv \Phi^\varrho_i(x, \dot{x})\underset{(\varrho)}{A}_k(x)\dot{x}^k. \tag{3.3}$$

From this we get the following theorem:

Theorem 2. *If* $\Phi^\varrho_i = \dot{\partial}_i\Phi^\varrho(x, \dot{x})$, $\underset{(\varrho)}{A}_i = \partial_i \underset{(\varrho)}{A}(x)$ *and* $S_i = \partial_i S(x)$ *then the fundamental function* $F(x, \dot{x})$ *has the form:*

$$F(x, \dot{x}) = \left(\frac{d\mu}{dt}\right)^{-1}\left(2N(x, \dot{x}) - \frac{1}{2}\Phi^\varrho(x, \dot{x})\frac{d\underset{(\varrho)}{A}(x)}{dt}\right), \tag{3.4}$$

where $\mu = \mu(x)$, *and* $\Phi^\varrho(x, \dot{x})$, $N(x, \dot{x})$ *are positively homogeneous functions of first and second degree respectively in* $\dot{x}^i$.

Proof. First we transform the relation (3.3). Differentiating (3.3) with respect to $\dot{x}^j$ we have

$$(\partial_i\mu)\dot{\partial}_j F - (\partial_j\mu)\dot{\partial}_i F - \frac{d\mu}{dt}\dot{\partial}_i\dot{\partial}_j F + \partial_i S_j - \partial_j S_i + \\ +(\lambda^{*\varrho} - \mu(x)\lambda^\varrho)(\partial_i \underset{(\varrho)}{A}_j - \partial_j \underset{(\varrho)}{A}_i) - \Phi^\varrho_i \underset{(\varrho)}{A}_j - \dot{\partial}_j\Phi^\varrho_i \underset{(\varrho)}{A}_k \dot{x}^k \equiv 0. \tag{3.5}$$

Now we take the skew-symmetric part in (i, j) of (3.5)

$$(\partial_i\mu)\dot{\partial}_j F - (\partial_j\mu)\dot{\partial}_i F + \partial_i S_j - \partial_j S_i + (\lambda^{*\varrho} - \mu(x)\lambda^\varrho)(\partial_i \underset{(\varrho)}{A}_j - \partial_j \underset{(\varrho)}{A}_i) - \\ -\frac{1}{2}(\Phi^\varrho_i \underset{(\varrho)}{A}_j - \Phi^\varrho_j \underset{(\varrho)}{A}_i) - \frac{1}{2}(\dot{\partial}_j\Phi^\varrho_i - \dot{\partial}_i\Phi^\varrho_j)\underset{(\varrho)}{A}_k(x)\dot{x}^k \equiv 0. \tag{3.6}$$

By the assumptions of Theorem 2 this reduces to

$$(\partial_i\mu)\dot{\partial}_j F - (\partial_j\mu)\dot{\partial}_i F - \frac{1}{2}(\dot{\partial}_i\Phi^\varrho\partial_j \underset{(\varrho)}{A} - \dot{\partial}_j\Phi^\varrho\partial_i \underset{(\varrho)}{A}) \equiv 0. \tag{3.7}$$

This can be written in the following form

$$\partial^{\cdot}_j \left(F \partial_i \mu + \frac{1}{2} \Phi^{\varrho} \partial_i \underset{(\varrho)}{A} \right) - \partial^{\cdot}_i \left(F \partial_j \mu + \frac{1}{2} \Phi^{\varrho} \partial_j \underset{(\varrho)}{A} \right) \equiv 0. \tag{3.8}$$

From this it follows that $F \partial_j \mu + \frac{1}{2} \Phi^{\varrho} \partial_j \underset{(\varrho)}{A}$ has the form

$$F \partial_j \mu + \frac{1}{2} \Phi^{\varrho} \partial_j \underset{(\varrho)}{A} \equiv \partial^{\cdot}_j N(x, \dot{x}). \tag{3.9}$$

Multiplying (3.9) by $\dot{x}^j$, we obtain with respect to the homogenity of the functions Φ^{ϱ} and N in $\dot{x}^i$:

$$F \frac{d\mu}{dt} + \frac{1}{2} \Phi^{\varrho} \frac{d \underset{(\varrho)}{A}}{dt} = 2N(x, \dot{x}). \tag{3.10}$$

Since $\frac{d\mu}{dt} \neq 0$, so from (3.10) we get the statement of Theorem 2.

We still calculate the form of F^* in case if Theorem 2 holds. Substituting (3.4) in (3.2) and using $S_i = \partial_i S(x)$ we have

$$F^*(x, \dot{x}) = \mu(x) \left(\frac{d\mu}{dt} \right)^{-1} \left(2N(x, \dot{x}) - \frac{1}{2} \Phi^{\varrho}(x, \dot{x}) \frac{d \underset{(\varrho)}{A}(x)}{dt} \right) + \frac{dS(x)}{dt}. \tag{3.11}$$

4 §. Geometrical remarks

Proposition 1. *If* $S_i = \partial_i S(x)$, $\mu = \mu(x)$ *and no constraints exist, then the extremals of* $\int F^*(x, \dot{x})\, dt$ *and* $\int \mu(x) F(x, \dot{x})\, dt$ *are identical.*

PROOF. Because of (3.2)

$$F^*(x, \dot{x}) = \mu(x) F(x, \dot{x}) + \frac{dS(x)}{dt}. \tag{4.1}$$

F^* and μF differ by a total differential. The addition of a total differential to the integrand evidently cannot affect any extremals, which completes the proof.

Remark: In [1] A. MOÓR has shown that if the relations $\mathscr{E}_i(F^*(x, \dot{x})) - \mu \mathscr{E}_i(F(x, \dot{x}) \equiv 0$ hold, where μ=const., then $S_i = \partial_i S(x)$ and $F^* = \mu F + \frac{dS}{dt}$ is also satisfied. So in this case $\int F^*\, dt$ and $\int F\, dt$ have identical extremals.

Proposition 2. *If along the curve* $x^i = x^i(t)$ $\underset{(\varrho)}{A}_k \dot{x}^k = 0$ *and* $\Phi^{\varrho}_i(x, \dot{x}) \underset{(\varrho)}{A}^i(x) = 0$ *then* $\mu(x(t)) = const.$

PROOF. The symmetric-part in (i, j) of (3.5) always vanishes

$$\frac{d\mu}{dt} \partial^{\cdot}_i \partial^{\cdot}_j F + \frac{1}{2} (\Phi^{\varrho}_i \underset{(\varrho)}{A}_j + \Phi^{\varrho}_j \underset{(\varrho)}{A}_i) + \frac{1}{2} (\partial^{\cdot}_j \Phi^{\varrho}_i + \partial^{\cdot}_i \Phi^{\varrho}_j) \underset{(\varrho)}{A}_k \dot{x}^k \equiv 0. \tag{4.2}$$

Let us multiply this by g^{ij}. Using (2.17) we obtain

$$\frac{1}{F}(n-1)\frac{d\mu}{dt}+\Phi_i^{\varrho}\underset{(\varrho)}{A^i}+\dot{\partial}_j\Phi_i^{\varrho}g^{ij}\underset{(\varrho)}{A_k}\dot{x}\equiv 0. \tag{4.3}$$

From this, and from the conditions of Proposition 2 it follows

$$\frac{1}{F}(n-1)\frac{d\mu}{dt}=0. \tag{4.4}$$

So

$$\frac{d\mu}{dt}\equiv 0, \tag{4.5}$$

and thus $\mu(x(t))=const.$

References

[1] A. Moór, Über äquivalente Variationsprobleme erster und zweiter Ordnung. *J. Reine Angew. Math.* **223** (1966), 131—137.

[2] H. Rund, Quasi-equivalent problems in the calculus of variations. *Colloquium on the calculus of variation. University of South Africa,* (1966) 1—34.

[3] H. Rund, The Hamilton—Jacobi theory in the calculus of variations. *London & New York* (1966).

[4] A. Moór, Über nicht-holonome allgemeine metrische Linienelementräume. *Acta Math. Acad. Sci. Hungar.* **101** (1959), 201—233.

[5] M. Sz. Kirkovits, Equivalent problems in the calculus of variations whose fundamental functions involve second-order derivatives. *Acta Math. Acad. Sci. Hungar.* **43** (1984),

ERDÉSZETI ÉS FAIPARI EGYETEM
MATEMATIKA TANSZÉK
H—9401 SOPRON
PF.: 132.
HUNGARY

(Received January 23, 1983.)