A remark on lattices satisfying the maximum condition

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In this note we are going to prove the following [1])

Proposition. *Let* $a_0 \prec a_1 \prec \ldots \prec a_r$ *and* $b_0 \prec b_1 \prec \ldots \prec b_s$ *be two chains of a lattice satisfying the maximum condition, and let* $a_1, \ldots, a_r, b_1, \ldots, b_s$ *be join-irreducible elements. Let moreover be* $a_0 \| b_0$. *Then* $a_j \cap b_k = a_0 \cap b_0$ *for any pair of indices* (j, k) $(j=0, 1, \ldots, r;\ k=0, 1, \ldots, s)$.

First we establish the following

Lemma.

$$\left.\begin{array}{l} a \prec c \\ b \prec c \\ a \neq b \end{array}\right\} \Longrightarrow a \cup b = c.$$

PROOF. We have $a \| b$, because, say, $a < b (\prec c)$ would contradict $a \prec c$. Now, $a, b < a \cup b$ because, say, $a = a \cup b \Longleftrightarrow a \geqq b$. At the same time $a \cup b \leqq c$, and here $a \cup b < c$ cannot hold, because

$$a, b < a \cup b < c$$

would contradict our hypothesis. Thus $a \cup b = c$.

The Proposition is now capable of the following

PROOF.

$$a_j \| b_0 \quad (j = 0, 1, \ldots, r). \tag{1}$$

Indeed, let $j \in [1, r]$. Then $a_j \neq b_0$, since a_j is comparable with a_0, while b_0 is not.

We cannot have $a_j < b_0$, since $a_j < b_0 \Longrightarrow a_0 < b_0$, in contradiction to $a_0 \| b_0$.

Suppose now $b_0 < a_j$ where $j > 0$ is the smallest index for which this inequality holds. By the maximum condition there exists an element $h \in L$ satisfying $b_0 \leqq h \prec a_j$. Clearly $h \neq a_{j-1}$, since $b_0 \neq a_{j-1}$ and $b_0 < h = a_{j-1}$ would contradict the choice of j. The Lemma now yields

$$a_j = h \cup a_{j-1},$$

in contradiction to the join-irreducibility of a_j. This establishes (1).

[1]) This proposition was suggested by Exercise 16. on p. 57 of the book [1], of which we are also using the terminology and notations. It seemed however necessary to add the condition $a_0 \| b_0$, while finiteness has been replaced by the maximum condition, and join-irreducibility has been postulated only for $a_j (j > 0)$ and b_k $(k > 0)$.

Let us remark that in deriving (1) we have used the incomparability of b_0 with a_0, but nothing else about b_0. Accordingly we can replace b_0 by any element h of L provided $a_0 \| h$, thus obtaining $a_j \| h$ $(j=0, 1, \ldots, r)$.

Having established (1), we see that by symmetry

(2) $$a_0 \| b_k \quad (k = 0, 1, \ldots, s)$$

also holds.

Consider now the chains $a_1 \prec \ldots \prec a_r$ and $b_0 \prec b_1 \prec \ldots \prec b_s$. Since $a_1 \| b_0$ by (1), we can replace a_0 by a_1 in (2):

$$a_1 \| b_k \quad (k = 0, 1, \ldots, s).$$

Again, consider $a_0 \prec a_1 \prec \ldots \prec a_r$ and $b_1 \prec \ldots \prec b_s$. Since $a_0 \| b_1$ by (2), we can replace b_0 by b_1 in (1):

$$a_j \| b_1 \quad (j = 0, 1, \ldots, r).$$

Given $a_2 \prec \ldots \prec a_r$ and $b_2 \prec \ldots \prec b_s$, we infer with the help of $a_2 \| b_0$ that

$$a_2 \| b_k \quad (k = 0, 1, \ldots, s).$$

Starting with $a_0 \prec a_1 \prec \ldots \prec a_r$ and $b_2 \prec \ldots \prec b_s$, and taking into account $a_0 \| b_2$, we obtain

$$a_j \| b_2 \quad (j = 0, 1, \ldots, r).$$

Continuing this process, we finally reach a_r and $b_0 \prec b_1 \prec \ldots \prec b_s$ as well as $a_0 \prec a_1 \prec \ldots \prec a_r$ and b_s, and we see that

(3) $$a_j \| b_k \quad (j = 0, 1, \ldots, r;\ k = 0, 1, \ldots, s).$$

Let us now show that

(4) $$a_j \cap b_k \leqq a_0 \quad (j = 0, 1, \ldots, r;\ k = 0, 1, \ldots, s).$$

We cannot have $a_0 < a_j \cap b_k$, since this would imply $a_0 < b_k$, thus contradicting $a_0 \| b_k$.

Suppose now $a_0 \| (a_j \cap b_k)$. Replacing b_0 by $a_j \cap b_k$ in (1), we obtain $a_j \| (a_j \cap b_k)$ in contradiction to $a_j \cap b_k \leqq a_j$. This establishes (4), and by symmetry

(5) $$a_j \cap b_k \leqq b_0.$$

(4) and (5) together yield

$$a_j \cap b_k \leqq a_0 \cap b_0.$$

The reverse inequality being trivial, this completes the proof of the proposition.

Reference

[1] G. Szász, Introduction to Lattice Theory. *New York and London,* 1963.

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