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The factorization method in the field of Mikusiński operators

By DJURDJICA TAKAČI (Novi Sad) and ARPAD TAKAČI (Novi Sad)

Abstract. We consider certain second order differential equations with variable coefficients in the field of Mikusiński operators, $\mathcal{F}$. We construct their discrete analogue and, using the factorization method, solve the obtained difference equation and analyze the character of its solution. Then we show that the solution of this difference equation can be treated as the approximate solution of the corresponding initial differential equation in the field $\mathcal{F}$, by estimating the error of approximation.

1. Notations and notions

The elements of the Mikusiński operator field, $\mathcal{F}$, are called *operators*. They are quotients of the form

$$\frac{f}{g}, \quad f \in \mathcal{C}_+, \ 0 \not\equiv g \in \mathcal{C}_+,$$

where the last division is observed in the sense of convolution

$$f(t) * g(t) = \int_0^t f(\tau) g(t-\tau)\, d\tau, \quad t > 0.$$

Any continuous function $a = a(t)$ with support in $[0,\infty)$ can be observed as an operator, which we shall simply denote by a. Then we say that the operator a represents the continuous function $a(t)$ and write $a = \{a(t)\}$. Let us denote by $\mathcal{F}_c$ the subset of $\mathcal{F}$ consisting of the operators representing continuous functions. For example, we have the integral

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operator l (representing the constant function 1 on $[0,\infty)$) and its positive powers l^α:

$$l = \{1\}, \quad l^\alpha = \left\{ \frac{t^{\alpha-1}}{\Gamma(\alpha)} \right\}, \quad \alpha > 0.$$

Also, among the most important operators are the inverse operator to l, the differential operator s, while I is the identity operator. This means that it holds

$$ls = I.$$

By $\mathcal{F}_I$ we denote the subset of $\mathcal{F}$ consisting of the elements of the form αI, for some numerical constant α.

For the theory of differential equations, the following relation, connecting the operator representing the n-th derivative of an n-times differentiable function $x = x(t)$ with the operator x is essential:

$$\{x^{(n)}(t)\} = s^n x - s^{n-1}x(0) - \ldots - x^{(n-1)}(0)I.$$

In this paper, we shall analyze the type I convergence ([2], p. 157). In particular one can show that the infinite series

$$\sum_{i=1}^{\infty} \phi^i,$$

where $\phi \in \mathcal{F}_c$, converges and its sum is an operator from $\mathcal{F}_c$.

The operators can be compared only if they are from $\mathcal{F}_c$. So for two operators $a = \{a(t)\}$ and $b = \{b(t)\}$ from $\mathcal{F}_c$ we define

$$a \le b \quad \text{iff} \quad a(t) \le b(t) \quad \text{for each} \quad t \ge 0$$

(see [2], p. 237). Clearly, $a = b$ iff $a(t) = b(t)$, $t \ge 0$.

Analogously, we shall say for two operator functions that

$$a(x) \le_T b(x), \ \ x \in [c,d],$$

if $a(x)$ and $b(x)$ are representing continuous real valued functions of two variables, $a(x) = \{a(x,t)\}$, $b(x) = \{b(x,t)\}$ and

$$a(x,t) \le b(x,t) \ \text{ for } \ t \in [0,T], \ x \in [c,d].$$

The absolute value of an operator a from $\mathcal{F}_c$, $a = \{a(t)\}$, denoted by $|a|$, is the operator $|a| = \{|a(t)|\}$. Also, we put $|a(x)| = \{|a(x,t)|\}$.

If the operators a and b are from $\mathcal{F}_c$, then it holds

$$|a+b| \le |a| + |b|,$$

$$|ab| = \left| \int_0^t a(\tau)b(t-\tau)d\tau \right| \le |a||b|,$$

and

$$|a| \leq_T \alpha(T)l, \text{ where } \alpha(T) = \max_{t\in[0,T]} |a(t)|.$$

2. Introduction

Let us consider the differential equation

$$s^p u''(x) + B(x)s^q u'(x) + C(x)s^r u(x) = f(x),$$

or

$$u''(x) + B(x)s^{q-p}u'(x) + C(x)s^{r-p}u(x) = l^p f(x), \tag{1}$$

with the conditions

$$u(0) = E, \quad u(1) = F, \tag{2}$$

in the field $\mathcal{F}$. In (1), p, q, r are positive integers, s is the differential operator, $B(x), C(x)$ and $f(x)$ are the given and $u(x)$ the unknown operator functions.

In (1), we assumed that $l^p f(x)$ are operator functions representing continuous function of two variables and in (2), we assumed that E and F are operators which can be written as

$$E = s^\sigma(E_1 I + E_c), \qquad F = s^\sigma(F_1 I + F_c), \tag{3}$$

where $\sigma \in \mathbf{Z}$, E_1, F_1 are numerical constants and E_c, F_c are operators representing continuous functions.

In this paper we construct a discrete analogue in the field $\mathcal{F}$ for the differential equation (1) with (2), present a method for the exact solution of this difference equation with variable coefficients, similarly as it was done for numerical difference equations in the book [1].

As is usual in numerical analysis, for $h > 0$ instead of $u'(x)$ we shall take

$$\frac{u(x+h) - u(x-h)}{2h}$$

and also instead of $u''(x)$ we shall put

$$\frac{u(x+h) - 2u(x) + u(x-h)}{h^2}.$$

So we obtain the difference equation in the field $\mathcal{F}$ corresponding to (1):

$$\frac{u(x+h)-2u(x)+u(x-h)}{h^2}+B(x)s^{q-p}\frac{u(x+h)-u(x-h)}{2h} \\ +C(x)s^{r-p}u(x)=l^p f(x). \tag{4}$$

For $N\in\mathbb{N}$, we define $h=\frac{1}{N}$, and put $x_0=0$, $x_n=x_{n-1}+h$, i.e., $x_n=\frac{n}{N}$, $n=1,2,\ldots,N-1$. Moreover, we define the operators f_n, for $n=1,2,\ldots,N-1$, by $f_n=f(x_n)$, and the numerical constants B_n and C_n, for $n=1,2,\ldots,N-1$, by $B_n=B(x_n)$ and $C_n=C(x_n)$.

Then the equation (4) can be written as the difference equation with variable coefficients

$$a_n u_{n-1}+b_n u_n+c_n u_{n+1}=l^p f_n, \quad n=1,2,\ldots,N-1. \tag{5}$$

The conditions (2) correspond to

$$u_0=E, \qquad u_N=F, \tag{6}$$

where a_n, b_n and c_n, $n=1,2,\ldots,N-1$, are operators from the field $\mathcal{F}$. Putting

$$r_1=q-p \quad \text{and} \quad r_2=r-p,$$

we have

$$a_n=\frac{I}{h^2}\left(I-\frac{s^{q-p}B_n h}{2}\right)=:\alpha I-s^{r_1}\beta_n, \tag{7}$$

$$b_n=-\frac{I}{h^2}\left(2I-s^{r-p}C_n h^2\right)=:-2\alpha I+s^{r_2}\gamma_n, \tag{8}$$

$$c_n=\frac{I}{h^2}\left(I+\frac{s^{q-p}B_n h}{2}\right)=:\alpha I+s^{r_1}\beta_n. \tag{9}$$

In the previous relations $\alpha:=\frac{I}{h^2}$, $\beta_n:=\frac{B_n}{2h}$ and $\gamma_n:=C_n$, $n=1,2,\ldots,N-1$, are assumed to be nonzero numerical constants.

The discrete analogue for the differential equation (1) with conditions (2), in the field $\mathcal{F}$ is the difference equation (5) with the conditions (6).

In Section 3 we construct the exact solution of equation (5) and analyze its character in the field of Mikusiński operators $\mathcal{F}$.

In Section 4 we estimate the error of approximation and show that the approximate solution of difference equation in the field $\mathcal{F}$ can be treated as the solution of differential equation (1) and in special case the solution of the partial differential equation with variable coefficients.

In Section 5 we give an application of the analyzed operator differential equation.

In papers [3], [4] and [6], the general form of the solution of difference equation (5) was constructed in the field $\mathcal{F}$ and its character was analyzed. In [7], a first order difference equation was observed. In all these papers we also estimated the error of approximation.

3. The solution of operator difference equation

It is known that the field of Mikusiński operators has very good algebraic properties, which also means that the usual addition and multiplication with operators can be treated in the same way as with real numbers. Therefore the solutions of the problem (5), (6) can be formed similarly as is done for numerical difference equations (when the coefficients a_n, b_n, c_n and also the solutions u_n are numerical constants). In the book [1], p. 51, the exact solution of difference equation (5) with conditions (6) was constructed.

It can be shown, similarly as was done in [1], that in the field $\mathcal{F}$ the solutions of the difference equation (5) with conditions (6) can be written as

$$\begin{aligned} u_N &= F, \\ u_n &= L_{n+1/2}u_{n+1} + K_{n+1/2}, \quad n = 0, 1, 2, \dots, N-2, N-1, \end{aligned} \tag{10}$$

where $L_{1/2} = 0$ and $K_{1/2} = E$,

$$L_{3/2} := \frac{-c_1}{b_1}, \quad K_{3/2} := \frac{a_1 E - f_1}{-b_1}, \tag{11}$$

and the operators $L_{n+1/2}$ and $K_{n+1/2}$, $1 \le n \le N-1$, have the forms

$$L_{n+1/2} = \frac{-c_n}{b_n + a_n L_{n-1/2}}, \tag{12}$$

$$K_{n+1/2} = \frac{f_n - a_n K_{n-1/2}}{b_n + a_n L_{n-1/2}}. \tag{13}$$

Let us express the operators $L_{n+1/2}$ and $K_{n+1/2}$ in the field $\mathcal{F}$. In order to obtain the character of the operator solution given by relation (10), let us first analyze the dependence of character of the operators $L_{n+1/2}$ and $K_{n+1/2}$ on r_1 and r_2.

Theorem 1. *Assume in equation* (5) *that the coefficients* a_n, b_n *and* c_n, $1 \le n \le N-1$, *are of the form* (7), (8) *and* (9), *respectively, and let the* *operator E be given by (3). If $r_2 > r_1 > 0$, i.e., $r > q$, then the operators $L_{n+1/2}$ for $n = 1, 2, \dots, N-1$, are from $\mathcal{F}_c$ and they can be written as*

$$L_{n+1/2} = -\left(l^{r_2}\frac{\alpha}{\gamma_n} + l^{r_2-r_1}\frac{\beta_n}{\gamma_n}\right)(I+\phi_n), \tag{14}$$

where

$$\phi_1 = \sum_{j=1}^{\infty}\left(2\frac{\alpha}{\gamma_1}l^{r_2}\right)^j, \tag{15}$$

and

$$\begin{aligned}\phi_n = \sum_{j=1}^{\infty}\Bigg(2\frac{\alpha}{\gamma_n}l^{r_2} &+ \left(\frac{\alpha}{\gamma_n}l^{r_2} + l^{r_2-r_1}\frac{\beta_n}{\gamma_n}\right)\\ &\times\left(l^{r_2}\frac{\alpha}{\gamma_{n-1}} + l^{r_2-r_1}\frac{\beta_{n-1}}{\gamma_{n-1}}\right)(I+\phi_{n-1})\Bigg)^j.\end{aligned} \tag{16}$$

Under the upper conditions, the operators $K_{n+1/2}$ can be written as

$$K_{n+1/2} = l^{\kappa_n}\cdot\left(\mathcal{K}_n I + \mathcal{K}_{c,n}\right), \quad n = 1, \dots, N-1, \tag{17}$$

where

$$\begin{aligned}&\kappa_1 = \min\{r_2+p,\ r_2-r_1-\sigma\} \quad \text{and}\\ &\kappa_n = \min\{r_2+p, r_2-r_1+\kappa_{n-1}\}, \quad n = 2, \dots, N-1.\end{aligned} \tag{18}$$

In (17), $\mathcal{K}_n$ are numerical constants, while $\mathcal{K}_{c,n}$ are operators representing continuous functions, $n = 1, 2, \dots, N-1$.

Proof. Using relations (11), (8) and (9) we can write

$$\begin{aligned}L_{3/2} = \frac{-c_1}{b_1} = -\frac{\alpha I + s^{r_1}\beta_1}{-2\alpha I + s^{r_2}\gamma_1} &= -\frac{l^{r_2}\frac{\alpha}{\gamma_1} + l^{r_2-r_1}\frac{\beta_1}{\gamma_1}}{I - 2\frac{\alpha}{\gamma_1}l^{r_2}}\\ &= -\left(l^{r_2}\frac{\alpha}{\gamma_1} + l^{r_2-r_1}\frac{\beta_1}{\gamma_1}\right)\sum_{j=0}^{\infty}\left(2\frac{\alpha}{\gamma_1}l^{r_2}\right)^j\\ &= -\left(l^{r_2}\frac{\alpha}{\gamma_1} + l^{r_2-r_1}\frac{\beta_1}{\gamma_1}\right)(I+\phi_1).\end{aligned}$$

The operator ϕ_1 is from $\mathcal{F}_c$; since by assumption $r_2 > 0$ and $r_2 - r_1 > 0$, the operator $L_{3/2}$ is from $\mathcal{F}_c$, also.

In the same manner, the operator $L_{5/2}$ can be transformed as follows:

$$
\begin{aligned}
L_{5/2} &= \frac{-c_2}{b_2 + a_2 L_{3/2}} \\
&= -\frac{\alpha I + s^{r_1}\beta_2}{-2\alpha I + s^{r_2}\gamma_2 - (\alpha I - s^{r_1}\beta_2)\left(l^{r_2}\frac{\alpha}{\gamma_1} + l^{r_2-r_1}\frac{\beta_1}{\gamma_1}\right)(I+\phi_1)} \\
&= -\left(l^{r_2}\frac{\alpha}{\gamma_2} + l^{r_2-r_1}\frac{\beta_2}{\gamma_2}\right) \\
&\quad \times \sum_{j=0}^{\infty}\left(2\frac{\alpha}{\gamma_2}l^{r_2} + \left(\frac{\alpha}{\gamma_2}l^{r_2} - l^{r_2-r_1}\frac{\beta_2}{\gamma_2}\right)\left(l^{r_2}\frac{\alpha}{\gamma_1} + l^{r_2-r_1}\frac{\beta_1}{\gamma_1}\right)(I+\phi_1)\right)^j.
\end{aligned}
$$

Similar calculations give for $n = 3, \dots, N-1$, the formula (14), with ϕ_{n-1} from (16):

$$
\begin{aligned}
&L_{n+1/2} = \\
&= -\frac{\alpha I + s^{r_1}\beta_n}{-2\alpha I + s^{r_2}\gamma_n + (\alpha I - s^{r_1}\beta_n)\left(-\left(l^{r_2}\frac{\alpha}{\gamma_{n-1}} + l^{r_2-r_1}\frac{\beta_{n-1}}{\gamma_{n-1}}\right)(I+\phi_{n-1})\right)},
\end{aligned}
$$

and thus we obtain the form (14). In particular, it follows that the operators $L_{n+1/2}$ are from $\mathcal{F}_c$.

The operator $K_{3/2}$ can be transformed as

$$
\begin{aligned}
K_{3/2} &= \frac{l^p f_1 - a_1 E}{b_1} = \frac{l^p f_1 - (\alpha I - s^{r_1}\beta_1)s^{\sigma}(E_1 I + E_c)}{-2\alpha I + s^{r_2}\gamma_1} \\
&= \frac{l^{p+r_2} f_1 - (l^{r_2}\alpha - l^{r_2-r_1}\beta_1)s^{\sigma}(E_1 I + E_c)}{\gamma_1}\sum_{j=0}^{\infty}\left(2l^{r_2}\frac{\alpha}{\gamma_1}\right)^j \\
&= \frac{l^{p+r_2} f_1 - (l^{r_2}\alpha - l^{r_2-r_1}\beta_1)s^{\sigma}(E_1 + E_c)}{\gamma_1}(I+\phi_1) \\
&= l^{\kappa_1}\left(K_1 I + \mathcal{K}_{c,1}\right),
\end{aligned} \tag{19}
$$

where $\kappa_1 = \min\{r_2 + p,\ r_2 - r_1 - \sigma\}$.

Similarly we have

$$
\begin{aligned}
K_{5/2} &= \left(\frac{l^{r_2+p} f_2 - (\alpha l^{r_2} - s^{r_1-r_2}\beta_2)K_{3/2}}{\gamma_2}\right)(I+\phi_2) \\
&= l^{\kappa_2}\left(\mathcal{K}_2 I + \mathcal{K}_{c,2}\right),
\end{aligned}
$$

where $\kappa_2 = \min\{r_2 + p, r_2 - r_1 + \kappa_1\}$.

Assume that (17) with κ_{n-1} from (18) for some $n-1 \in \{2, \dots, N-2\}$ holds. Then

$$K_{n+1/2} =$$

$$= \frac{f_n - (\alpha I - s^{r_1}\beta_n) \cdot l^{\kappa_{n-1}}(\mathcal{K}_{n-1}I + \mathcal{K}_{c,n-1})}{(-2\alpha I + s^{r_2}\gamma_n) - (\alpha I + s^{r_2}\beta_2) \cdot \left(l^{r_2}\frac{\alpha}{\gamma_{n-1}} + l^{r_2-r_1}\frac{\beta_{n-1}}{\gamma_{n-1}}\right)(I + \phi_{n-1})}$$

$$= \frac{l^{\kappa_n}(\mathcal{K}'_{n-1}I + \mathcal{K}'_{c,n-1})}{\gamma_n},$$

where κ_n is given by (18). Thus we obtained the representation (17) for $K_{n+1/2}$. □

Corollary 1. *If in relation* (18) *it holds that* $r_2+p \geq (k-1)(r_2-r_1)-\sigma$ *and* $r_2 + p \leq k(r_2 - r_1) - \sigma$, *for some* k, $1 \leq k \leq N-1$, *then*

$$\kappa_i = \begin{cases} i(r_2 - r_1) - \sigma, & i = 1, 2, \dots, k-1, \\ r_2 + p, & i = k, k+1, \dots, N-1. \end{cases} \tag{20}$$

Proof. If $r_2 + p \leq r_2 - r_1 - \sigma$, then $\kappa_1 = \kappa_2 = \ldots = \kappa_{N-1} = r_2 + p$. However, if $r_2 + p > r_2 - r_1 - \sigma$, then $\kappa_1 = r_2 - r_1 - \sigma$, and $\kappa_2 = \min\{r_2 + p, 2(r_2 - r_1) - \sigma)$. It is obvious that $\kappa_i \leq \kappa_{i+1}$, $i = 1, \dots, N-2$. So if $r_2 + p \leq 2(r_2 - r_1) - \sigma$, then $\kappa_2 = \ldots = \kappa_{N-1} = r_2 + p$. If $r_2 + p > 2(r_2 - r_1) - \sigma$, then $\kappa_2 = 2(r_2 - r_1) - \sigma$. Continuing this procedure we obtain relation (20). □

The next theorem will characterize the solutions of equation (5), i.e., operators u_n, $n = 1, \dots, N-1$, given by relation (10).

Theorem 2. *Assume in equation* (5) *that the coefficients* a_n, b_n, c_n, $1 \leq n \leq N$, *are of the form* (7), (8), (9), *respectively, and let* $r_2 > r_1 > 0$. *If* E *and* F *are given by relation* (3), *then the solutions of equation* (5) *can be written as*

$$u_N = s^{\sigma}(F_1 + F_c), \quad u_{N-k} = l^{\omega_{N-k}}(U_{N-k}I + U_{c,N-k}). \tag{21}$$

for numerical constants U_{N-k} *and operators representing continuous functions* $U_{c,N-k}$, $k = 1, \dots, N-1$, *and the powers* ω_n *having the forms*

$$\omega_{N-k} = \begin{cases} k(r_2 - r_1) - \sigma, & k(r_2 - r_1) - \sigma \leq r_2 + p; \\ r_2 + p, & k(r_2 - r_1) - \sigma > r_2 + p. \end{cases} \tag{22}$$

PROOF. The form (21) of the solutions in the field $\mathcal{F}$ follows from relation (10) and Theorem 1, namely from relations (14) and (17). In fact, we have

$$\begin{aligned} u_{N-1} &= L_{N-1/2}u_N + K_{N-1/2} \\ &\quad - \left(l^{r_2}\frac{\alpha}{\gamma_{N-1}} + l^{r_2-r_1}\frac{\beta_{N-1}}{\gamma_{N-1}} \right)(I+\phi_{N-1})s^{\sigma}(F_1+F_c) \\ &\quad + l^{\kappa_{N-1}}\cdot(\mathcal{K}_{N-1}I+\mathcal{K}_{c,N-1}) \\ &= l^{\omega_{N-1}}(U_{N-1}I+U_{c,N-1}), \end{aligned}$$

where $\omega_{N-1} = \min\{r_2-r_1-\sigma,\ \kappa_{N-1}\}$.

- If $\kappa_{N-1} = (N-1)(r_2-r_1)-\sigma)$, then $\omega_{N-1} = r_2-r_1-\sigma$.
- If $\kappa_{N-1} = r_2+p = r$, then
  - either $\omega_{N-1} = r_2-r_1-\sigma$, for $r_2-r_1-\sigma < r_2+p$,
  - or $\omega_{N-1} = r_2+p$ if $r_2-r_1-\sigma > r_2+p$.

Continuing this procedure we obtain

$$\begin{aligned} u_{N-2} &= L_{(N-2)+1/2}u_{N-1} + K_{(N-2)+1/2} \\ &= -\left(l^{r_2}\frac{\alpha}{\gamma_n} + l^{r_2-r_1}\frac{\beta_{N-2}}{\gamma_{N-2}} \right)(I+\phi_{N-2})l^{\omega_{N-1}}(U_{N-1}I+U_{c,N-1}) \\ &\quad + l^{\kappa_{N-2}}\cdot(\mathcal{K}_{N-2}I+\mathcal{K}_{c,N-2}(\phi_{N-2})) \\ &= l^{\omega_{N-2}}(U_{N-2}I+U_{c,N-2}), \end{aligned}$$

where $\omega_{N-2} = \min\{r_2-r_1+\omega_{N-1},\ \kappa_{N-2}\}$. In this case we have the following.

- If $\kappa_{N-2} = (N-2)(r_2-r_1)-\sigma$, meaning $(N-2)(r_2-r_1)-\sigma < r_2+\sigma$, then $\omega_{N-1} = r_2-r_1-\sigma$ and $\omega_{N-2} = 2(r_2-r_1)-\sigma$, if $N-2 \geq 2$.
- If $\kappa_{N-2} = r_2+p$, meaning then $(N-2)(r_2-r_1)-\sigma \geq r_2+p$, then
  - either $2(r_2-r_1)-\sigma < r_2+p$, (then also $r_2-r_1-\sigma < r_2+p$, implying $\omega_{N-1} = r_2-r_1-\sigma$,) and therefore $\omega_{N-2} = 2(r_2-r_1)-\sigma$,
  - or $2(r_2-r_1)-\sigma > r_2+p$. If $r_2-r_1-\sigma < r_2+p$, then $\omega_{N-1} = r_2-r_1-\sigma$, but $\omega_{N-2} = r_2+p$. Otherwise if $r_2-r_1-\sigma > r_2+p$, then $\omega_{N-1} = r_2+p$, then also $\omega_{N-2} = r_2+p$.

Again U_{N-2} is a numerical constant, while $U_{c,N-2}$ is an operator representing continuous function. Continuing the procedure we obtain the form of the solution given by relation (22).

Corollary 2. *If the conditions of Theorem 2 are fulfilled and the solutions have the form* (21),

- *then for $\sigma < 0$ the solutions of the problem* (5), (6) *represent continuous functions;*
- *then for $\sigma \geq 0$ solutions of this problem may represent continuous functions and may not.*

PROOF.

- If $\sigma < 0$, then E and F represent continuous functions, thus $\kappa_n > 0$ and $\omega_n > 0$, $n = 1, \ldots, N-1$. From relation (21) it follows that the operators u_n represent continuous functions.
- Assume $\sigma > 0$.
  - If $r_2 - r_1 - \sigma > 0$, then $\kappa_n > 0$, and $\omega_n > 0$, $n = 1, \ldots, N-1$, and in this case all the operators u_n, $n = 1, \ldots, N-1$ represent continuous functions.
  - However, if $r_2 - r_1 - \sigma < 0$, then $\kappa_1 < 0$ and $\omega_{N-1} < 0$, meaning that u_{N-1} does not represent a continuous function.
    - If $2(r_2 - r_1) - \sigma < 0$, then $\kappa_2 = \omega_{N-2} < 0$, and the solution u_{N-2} does not represent a continuous function.
    - If $2(r_2 - r_1) - \sigma > 0$, then $\kappa_2 > 0$ and $\omega_{N-2} > 0$, and thus the solution u_{N-2} does represent a continuous function.
  - If $(N-1)(r_2-r_1)-\sigma < 0$, then no solution represents a continuous function.
  - If for some k, $3 \leq k < N-1$, $k(r_2 - r_1) - \sigma > 0$, then the operators u_{N-k}, $u_{N-k-1}, \ldots$, $3 \leq k \leq N-1$, represent continuous functions.

Similarly we can prove the following statements, which correspond to the case $r_1 = r_2$.

Theorem 3. *Assume in equation* (5) *that the coefficients a_n, b_n and c_n, $1 \leq n \leq N$, are of the form* (7), (8) *and* (9), *respectively, then put $\delta_1 = \gamma_1 \neq 0$, $\delta_n = \gamma_n + \beta_n \frac{\beta_{n-1}}{\delta_{n-1}}$, $n = 2, \ldots, N-1$, and assume that all δ_n are nonzero.*

If $r_1 = r_2 = m > 0$, then the operators $L_{n+1/2}$ can be written as

$$L_{n+1/2} = -\left(l^m \frac{\alpha}{\delta_n} + \frac{\beta_n}{\delta_n}\right)(I + \tilde{\phi}_n), \tag{23}$$

where the operators $\tilde{\phi}_n$, $n = 1, \ldots, N-1$, are from $\mathcal{F}_c$ and have the forms

$$\tilde{\phi}_1 = \sum_{j=1}^{\infty} \left(2 \frac{\alpha}{\gamma_1} l^m \right)^j, \tag{24}$$

$$\begin{aligned} \tilde{\phi}_n &= \sum_{j=1}^{\infty} \left(\frac{\alpha}{\delta_n} l^m \left(2 + \frac{\beta_{n-1}}{\delta_{n-1}} \right) + \left(\frac{\alpha}{\delta_n} l^m - \frac{\beta_n}{\delta_n} \right) \cdot l^m \frac{\alpha}{\delta_{n-1}} \right. \\ &\qquad \left. + \left(l^m \frac{\alpha}{\delta_n} - \frac{\beta_n}{\delta_n} \right) \left(l^m \frac{\alpha}{\delta_{n-1}} + \frac{\beta_{n-1}}{\delta_{n-1}} \right) \cdot \tilde{\phi}_{n-1} \right)^j \\ &=: \sum_{j=1}^{\infty} \left(\Psi_n + \left(l^m \frac{\alpha}{\delta_n} - \frac{\beta_n}{\delta_n} \right) \left(l^m \frac{\alpha}{\delta_{n-1}} + \frac{\beta_{n-1}}{\delta_{n-1}} \right) \cdot \tilde{\phi}_{n-1} \right)^j . \end{aligned} \tag{25}$$

If the operators E, given by (3), and $l^p f_n$, $n = 1, 2, \ldots, N$, represent continuous functions, then the operators $K_{n+1/2}$, for $n = 1, 2, \ldots, N-1$, are of the forms

$$K_{n+1/2} = l^{\tau} \left(\mathcal{K}_n I + \mathcal{K}_{c,n} \right)), \tag{26}$$

where $\tau = \min\{m + p, -\sigma\}$ and $\mathcal{K}_n$ are numerical constants and $\mathcal{K}_{c,n}$ are operators from $\mathcal{F}_c$.

Theorem 4. *Assume in equation (5) the coefficients a_n, b_n and c_n, $1 \leq n \leq N$, are of the form (7), (8) and (9), respectively, and let $r_2 = r_1 > 0$. If E and F are given by relation (3), then the solutions of equation (5) can be written as*

$$\begin{gathered} u_N = s^{\sigma}(F_1 + F_c), \quad u_{N-k} = l^{\omega}(U_{N-k} I + U_{c,N-k}), \\ k = 1, \ldots, N-1, \end{gathered} \tag{27}$$

for some numerical constants U_{N-k} and operators representing continuous functions $U_{c,N-k}$, $k = 1, \ldots, N-1$, and $\omega = \min\{m + p, -\sigma\}$. Moreover, the following holds.

- *If $\sigma < 0$, then the solutions of the problem (5), (6) represent continuous functions;*
- *If $\sigma \geq 0$, then the solutions of this problem do not represent continuous functions.*

Theorem 5. *Assume in equation (5) that the coefficients a_n, b_n and c_n, $1 \leq n \leq N$, are of the form (7), (8) and (9), respectively, and $r_1 > r_2 > 0$. Let the numerical constants satisfy $g_1 = \gamma_1 \neq 0$, $g_{2k+1} = \gamma_{2k+1} + \beta_{2k+1} \frac{\beta_{2k}}{g_{2k}} \neq 0$, $g_{2k} = \beta_{2k} \frac{\beta_{2k-1}}{g_{2k-1}} \neq 0$.*

1. Then for $n = 2k+1,\ k = 0,1,2,\dots,[(N-1)/2]$, the operators $L_{n+1/2}$ are neither from $\mathcal{F}_c$ nor from $\mathcal{F}_I$; however, they can be written as

$$L_{2k+3/2} = -\left(l^{r_2}\frac{\alpha}{g_{2k+1}} + l^{r_2-r_1}\frac{\beta_{2k+1}}{g_{2k+1}}\right)(I+\psi_{2k+1}),$$

where $g_1 = \gamma_1,\ g_{2k+1} = \gamma_{2k+1} + \beta_{2k+1}\frac{\beta_{2k}}{g_{2k}}$ are numerical constants, while ψ_{2k+1} are from $\mathcal{F}_c$ and have the forms

$$\psi_{2k+1} =: \sum_{j=1}^{\infty}\left(\tilde{\phi}^1_{2k+1} + \left(l^{2r_1-r_2}\frac{\alpha}{g_{2k+1}} - l^{r_1-r_2}\frac{\beta_{2k+1}}{g_{2k+1}}\right)\right.$$
$$\left.\times\left(l^{2r_1-r_2}\frac{\alpha}{g_{2k}} + l^{r_1-r_2}\frac{\beta_{2k}}{g_{2k}}\right)\cdot\psi_{2k}\right)^j,$$

where the form of $\tilde{\phi}^1_{2k+1}$ is ordered similarly as $\tilde{\phi}_{2k+1}$ in relation (25). The operator $K_{3/2}$ is the same as one given in Theorem 3. Further on, we have

$$K_{2k+3/2} = -\left(l^{p+r_2}f_n\frac{I}{g_{2k+1}} - \left(\frac{\alpha}{g_{2k+1}}l^{r_2} - l^{r_2-r_1}\frac{\beta_{2k+1}}{g_{2k+1}}\right)K_{2k+1-1/2}\right)$$
$$\times(I+\psi_{2k+1}).$$

2. If $n = 2k,\ k = 1,2,\dots,[(N-1)/2]$, then the operators $L_{2k+1/2}$ are from $\mathcal{F}_c$ and can be written as

$$L_{2k+1/2} = -\left(l^{2r_1-r_2}\frac{\alpha}{g_{2k}} + l^{r_1-r_2}\frac{\beta_{2k}}{g_{2k}}\right)(I+\psi_{2k})$$

and the operators $K_{2k+1/2}$ are of the forms

$$K_{2k+1/2} = -\left(l^{p+2r_1-r_2}f_n\frac{I}{g_{2k}} - \left(\frac{\alpha}{g_{2k}}l^{2r_1-r_2} - l^{r_1-r_2}\frac{\beta_{2k}}{g_{2k}}\right)K_{2k-1/2}\right)$$
$$\times(I+\psi_{2k}).$$

Again $g_{2k} = \frac{\beta_{2k-1}\beta_{2k}}{g_{2k-1}} \neq 0$ are numerical constants and ψ_n are from $\mathcal{F}_c$ and can be written as

$$\psi_{2k} =: \sum_{j=1}^{\infty}\left(\tilde{\phi}^2_{2k} + \left(l^{r_2}\frac{\alpha}{g_{2k}} - l^{r_2-r_1}\frac{\beta_{2k}}{g_{2k}}\right)\right.$$
$$\left.\times\left(l^{r_2}\frac{\alpha}{g_{2k-1}} + l^{r_2-r_1}\frac{\beta_{2k-1}}{g_{2k-1}}\right)\cdot\ \psi_{2k-1}\right)^j.$$

The solutions in the case when $r_1 > r_2$ can be formed similarly as it was done in Theorems 2 and 4.

4. The error of approximation

Let us suppose that the solution of equation (1) is from $\mathcal{F}_c$ and has a continuous fourth derivative in the field $\mathcal{F}$. Let us denote by $u(x_j)$ the exact solution of equation (1) and by u_j the approximate solution of the same equation (which also belongs to $\mathcal{F}_c$). In fact, u_j is the solution of the difference equation (5).

In order to give the error of approximation, we have to estimate the difference between the equations (1) and (5). So for $j = 1, 2, \ldots, N-1$, we have

$$\left(u''(x_j) - \frac{u_{j+1} - 2u_j + u_{j-1}}{h^2}\right) + B(x_j)s^{q-p}\left(u'(x_j) - \frac{u_{j+1} - u_{j-1}}{2h}\right)$$
$$+C(x_j)s^{r-p}\left(u(x_j) - u_j\right) = 0.$$

From the previous relation we have

$$|u(x_j) - u_j| = \left|\frac{I}{C(x_j)} \cdot \left(l^{r-p}\left(u''(x_j) - \frac{u_{j+1} - 2u_j + u_{j-1}}{h^2}\right)\right.\right.$$
$$\left.\left.+B(x_j)l^{r-q}\left(u'(x_j) - \frac{u_{j+1} - u_{j-1}}{2h}\right)\right)\right|.$$

In this paper we give the error of approximation for $r > q \geq p \geq 0$. Then we have $r_2 = r - p > r_1 = q - p \geq 0$ and therefore the expression

$$\left|\frac{I}{C(x_j)}\left(l^{r-p}\left(u''(x_j) - \frac{u_{j+1} - 2u_j + u_{j-1}}{h^2}\right)\right.\right.$$
$$\left.\left.+B(x_j)l^{r-q}\left(u'(x_j) - \frac{u_{j+1} - u_{j-1}}{2h}\right)\right)\right|$$

represents a continuous function. It can be estimated by

$$\left|\frac{I}{C(x_j)}\left(l^{r-p}\left(u''(x_j) - \frac{u_{j+1} - 2u_j + u_{j-1}}{h^2}\right)\right.\right.$$
$$\left.\left.+B(x_j)l^{r-q}\left(u'(x_j) - \frac{u_{j+1} - u_{j-1}}{2h}\right)\right)\right|$$
$$\leq_T \frac{h^2}{6}\left(R_1\frac{M_4(T)T^{r-p-1}}{2(r-p-1)!} + R_2\frac{M_3(T)T^{r-p-1}}{(r-p-1)!}\right)l^2,$$

where the numerical constants R_1 and R_2 are such that

$$R_1 = \max_{0\le x_j\le 1}\left|\frac{1}{C(x_j)}\right|, \quad R_2 = \max_{0\le x_j\le 1}\left|\frac{B(x_j)}{C(x_j)}\right| \quad \text{and}$$

$$M_i(T) = \max_{x\in[0,1],\ t\in[0,T]}\left|\frac{\partial^i u(x,t)}{\partial x^i}\right|, \quad i=3,4.$$

So the error of approximation can be estimated as

$$|u(x_j)-u_j| \le_T \frac{h^2}{6}\left(R_1\frac{M_4(T)T^{r-p-1}}{2(r-p-1)!} + R_2\frac{M_3(T)T^{r-p-1}}{(r-p-1)!}\right) l^2.$$

Note that the error of approximation is $O(h^2)$, as in the classical case.

5. An application

Let us consider the partial differential equation

$$\frac{\partial^{2+p}u(x,t)}{\partial x^2\partial t^p} + B(x)\frac{\partial^{1+q}u(x,t)}{\partial x\partial t^q} + C(x)\frac{\partial^r u(x,t)}{\partial t^r} = f(x,t), \tag{28}$$

on the set $\{(x,t)|\, 0\le x\le 1,\ t\ge 0\}$, with certain appropriate conditions. Here, we shall assume that

$$\left.\frac{\partial^{\mu+\nu}u(x,t)}{\partial x^\mu\partial t^\nu}\right|_{t=0} = 0, \tag{29}$$

for $\mu=0,\ \nu=0,1,\ldots,r-1;\ \mu=1,\ \nu=0,1,\ldots,q-1;\ \mu=2,\ \nu=0,1,\ldots,p-1$, and

$$u(0,t)=E(t), \quad u(1,t)=F(t). \tag{30}$$

The numbers p,q and r are positive integers, $B(x)$ and $C(x)$ in (28) are continuous functions depending on the variable x, $E(t)$ and $F(t)$ in (30) are continuous functions depending on the variable t, while $f(x,t)$ and $u(x,t)$ are the given and the unknown function of two variables. We assume that for every $x\in[0,1]$ both $B(x)$ and $C(x)$ are different from zero. The problem (28), (29) and (30) corresponds to differential equation (1) with the conditions (2). This means that we can treat the exact solution of equation (5) given by (21) as the approximate solution of the partial differential equation (28).

6. An example

Let us consider the partial differential equation

$$e^x \frac{\partial^2 u(x,t)}{\partial t^2} = \frac{\partial}{\partial x}\left((1+x^2)\frac{\partial u(x,t)}{\partial x}\right) - 1, \tag{31}$$

with the conditions

$$u(x,0) = 0, \quad u_t(x,0) = 0, \tag{32}$$

$$u(0,t) = 1, \quad u(1,t) = \frac{t^2}{2}. \tag{33}$$

In the field $\mathcal{F}$, the equation (31) with the conditions (32) corresponds to the equation

$$u''(x) + \frac{2x}{1+x^2}u'(x) - s^2\frac{e^x}{1+x^2}u(x) = l. \tag{34}$$

The conditions (33) correspond to the conditions

$$u(0) = l, \quad u(1) = l^3. \tag{35}$$

In this case we have $r = 2,\ p = 0,\ q = 0,\ \ r_2 = 2,\ r_1 = 0$, and $r_2 > r_1$, and

$$E = l, \quad F = l^3, \quad B(x) = \frac{2x}{1+x^2} \quad C(x) = -\frac{e^x}{1+x^2}.$$

The difference equation (5) has now the form

$$a_n u_{n-1} + b_n u_n + c_{n+1}u_{n+1} = l.$$

Defining the constants $\alpha,\ \beta_n$ and γ_n as after relation (9), we obtain the solution of the problem (34), (35), in the form

$$\begin{aligned} u_N &= F, \\ u_n &= L_{n+1/2}u_{n+1} + K_{n+1/2}, \quad n = N-1, N-2, \dots, 1. \end{aligned} \tag{36}$$

Hence it follows that

$$L_{3/2} = -l^2\left(\frac{\alpha}{\gamma_1}+\frac{\beta_1}{\gamma_1}\right)\sum_{j=0}^{\infty}\left(2\frac{\alpha}{\gamma_1}l^{r_2}\right)^j,$$

$$L_{n+1/2} = -l^2\left(\frac{\alpha}{\gamma_n}+\frac{\beta_n}{\gamma_n}\right)$$
$$\times\sum_{j=0}^{\infty}l^{2j}\left(2\frac{\alpha}{\gamma_n}+\left(\frac{\alpha}{\gamma_n}+\frac{\beta_n}{\gamma_n}\right)\cdot\left(l^2\frac{\alpha}{\gamma_{n-1}}+l^2\frac{\beta_{n-1}}{\gamma_{n-1}}\right)(I+\phi_{n-1})\right)^j;$$

$$K_{3/2} = \frac{l^3-l^{r_2}(\alpha-\beta_1)E}{\gamma_1}\sum_{j=0}^{\infty}\left(2l^2\frac{\alpha}{\gamma_1}\right)^j,$$

$$K_{n+1/2} = \frac{l^3-l^2(\alpha-\beta_n)K_{n-1/2}}{\gamma_n}$$
$$\times\sum_{j=0}^{\infty}l^{2j}\left(2\frac{\alpha}{\gamma_n}+\left(\frac{\alpha}{\gamma_n}+\frac{\beta_n}{\gamma_n}\right)\cdot\left(l^2\frac{\alpha}{\gamma_{n-1}}+l^2\frac{\beta_{n-1}}{\gamma_{n-1}}\right)(I+\phi_{n-1})\right)^j.$$

Note that the solutions are from $\mathcal{F}_c$, which, of course, is in accordance with Theorem 1.

The error of approximation can be estimated by

$$|u(x_j)-u_j| \leq_T \frac{h^2}{6}\left(\frac{1+e}{e}\cdot\frac{T\cdot M_4(T)}{2}+\frac{2Te}{1+e}\cdot M_3(T)\right)l^2.$$

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DJURDJICA TAKAČI
INSTITUTE OF MATHEMATICS
UNIVERSITY OF NOVI SAD
TRG DOSITEJA OBRADOVIĆA 4
21000 NOVI SAD
YUGOSLAVIA
E-MAIL: takaci@unsim.ns.ac.yu

ARPAD TAKAČI
INSTITUTE OF MATHEMATICS
UNIVERSITY OF NOVI SAD
TRG DOSITEJA OBRADOVIĆA 4
21000 NOVI SAD
YUGOSLAVIA

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