On the coincidence of two kinds of ellipses in Minkowskian spaces and in Finsler planes

Dedicated to Professor András Rapcsák on the occasion of his seventieth birthday

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1. On the euclidean plane R_2 an ellipse consists of those points P, for which the distances from two given fixed points F and F^* (from the foci) yield a constant sum 2a. We denote such an ellipse by ε_1:

$$\varepsilon_1(F, F^*, 2a) := \{P \mid \overline{PF} + \overline{PF}^* = 2a\} \quad (2a > \overline{FF}^* \equiv 2c), \tag{1}$$

where $\overline{PF}$ stands for the euclidean distance of the points P and F. At the same time every ellipse ε_1 is also the locus of those points whose distances from a fixed point (from a focus) and from an appropriate line (from the directrix) has a constant ratio $\lambda < 1$. Let us denote such a curve by ε_2:

$$\varepsilon_2(F, g, \lambda) := \left\{P \,\middle|\, \frac{\overline{PF}}{\overline{Pg}} = \lambda\right\}. \tag{2}$$

Let us call these curves ellipse of first and of the second kind respectively. It is well-known that the two classes of these curves coincide: $\{\varepsilon_1\} = \{\varepsilon_2\}$ and $\lambda = \frac{c}{a}$. The two classes $\{\eta_1\}$ and $\{\eta_2\}$ of the hyperbolas of first and second kind can be defined similarly. On the R_2 also $\{\eta_1\} = \{\eta_2\}$.

These curves can be defined on a Minkowski plane M_2, or on a Finsler plane F_2, or on a Riemannian plane V_2 (M_2, F_2, V_2 being homeomorphic to a euclidean plane R_2) in the same way as above provided that straight lines are replaced by geodesics. Prof A. MOÓR put the question: in which F_2 do the two classes $\{\varepsilon_1\}$ and $\{\varepsilon_2\}$, or $\{\eta_1\}$ and $\{\eta_2\}$ coincide, as they do in an R_2? An attempt to determine by analytic method the fundamental functions of these F_2 led to complicated calculations. In what follows we present a solution by synthetic method in an M_n and in F_2.

2. As known, an indicatrix curve I makes the euclidean plane into a Minkowski-plane: $M_2(I)$. We suppose the indicatrix I to be a bounded, differentiable, central-symmetric, closed and strictly convex curve. Then I is a Minkowskian unit circle.

Proposition. *If the two classes of the ellipses of first and second kind coincide on a Minkowski plane, then it is a euclidean plane.*

First we make some preparatory statements and prove some lemmas. A line r is called perpendicular to another line $s: r \perp s$, if the tangents at the intersections of I and of that straight line which runs parallel to r and goes through the center 0 of I are parallel to s.

Let us consider the line $|F, F^*|$ through the foci F, F^* of the ellipse ε of an M_2. Because of the presupposed equality $\{\varepsilon_1\}=\{\varepsilon_2\}$ it is irrelevant whether ε is of the first or of the second kind. On $|F, F^*|$ there lie two points A and B of ε. Let A be the one nearer to F.

Lemma 1. *The only nearest point of* ε *to* F *is* A.

Let k_F be a Minkowski circle with centre F and radius $\|F, A\|=a-c$, where $\| \, , \, \|$ means the Minkowski distance of the two points occurring in it, and a and c denote the quantities appearing in (1). Let P be a point in the inside or on the periphery of k_F differring from A. Then, because of the triangle inequality valid in the M_2, and of the position of P we have $\|P, F^*\| \leqq \|P, F\| + \|F, F^*\| \leqq a-c+2c=a+c$. However, from the two signs $\leqq$ we have at least once the sign $<$ for every allowed P. Thus we get $\|P, F^*\| < a+c$. But according to our assumption $\|P, F\| \leqq a-c$. The sum of these relations yields $\|P, F\| + \|P, F^*\| < 2a$. Therefore no point not farther from F than $a-c$ lies on ε except A.

Lemma 2. *The only nearest point of* ε *to the directrix* g *is* A.

Indeed, let assume that a point P of ε different from A is not farther from g than A. Then because of $A, P \in \varepsilon$

$$\frac{\|A, F\|}{\|A, g\|} = \lambda = \frac{\|P, F\|}{\|P, g\|}. \tag{3}$$

But $\|A, F\| < \|P, F\|$ according to Lemma 1, which together with our assumption $\|A, g\| \geqq \|P, g\|$ contradicts (3).

The foot of a point on a line is the nearest point of the line to the given point. This point is unique. Let us denote the line $|F, F^*|$ by t, the intersection point of t and the indicatrix by R.

Lemma 3. *The foot of* A *on* g *is* R.

Let us assume that the foot g_A of A on g is not R, and consider the Minkowski circle k_{g_A} with center g_A and radius $\|g_A, A\|$. The tangent $\bar{e}_A$ of this circle at A coincides with the tangent e_A of ε at A. For if these lines were intersecting, then ε would have a point nearer to g than A in contradiction to Lemma 2. Thus $|g_A, A| \perp \bar{e}_A = e_A$.

However, we know that the circle k_F used in Lemma 1 is in the interior of ε except the point A. In consequence of this the tangents at A of ε and k_F coincide. So $|F, A| \equiv |R, A| \equiv t \perp e_A$. This, together with $|g_A, A| \perp e_A$, means that both points g_A and R of g lie on t. Thus $g_A = R$.

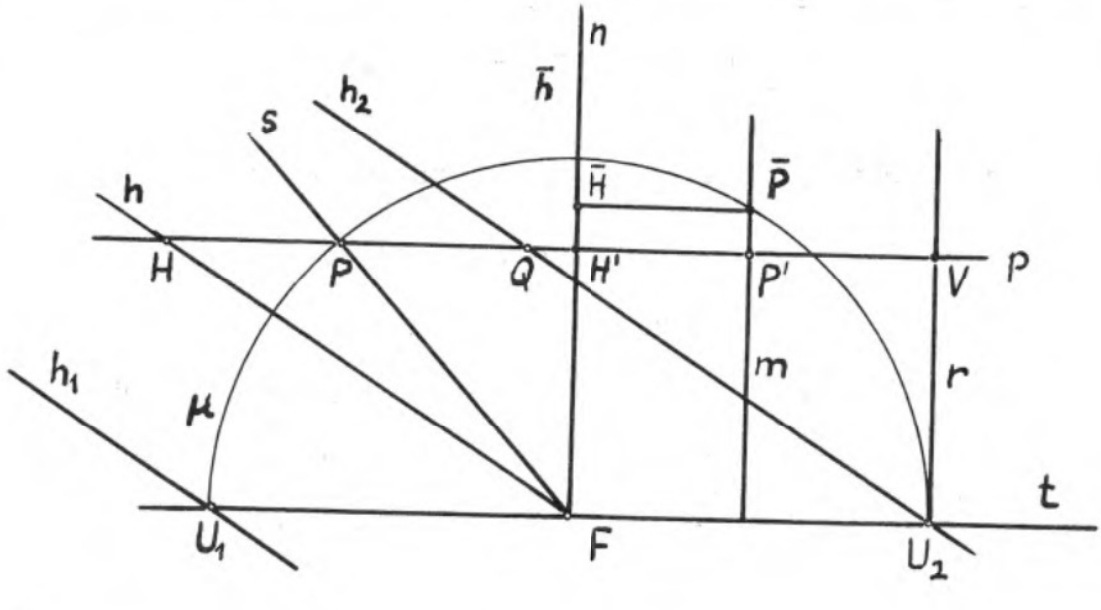
Lemma 4. $t \perp g$.

This immediately follows from the fact that the line through a point and through its foot on a given line is perpendicular to the given line ([1], (17.24)). If this point is A, and the given line is g, then according to Lemma 3, R is the foot of A on g, and therefore $|A, R| = t \perp g$.

3. We know the existence of such euclidean metrics on the set of points of the Minkowski plane for which Minkowski geodesics and euclidean straight lines coincide. So called associated euclidean metrics ([1], § 17) are such ones. If M_2 is defined as in the first paragraph of section **2**, then the euclidean metric used there is associated, and all the other associated euclidean metrics are obtained by affine transformations from the first one. In case of associated euclidean metrics Minkowskian and euclidean parallelity also coincide. We are going to show that in an associated euclidean metric the indicatrix I is a euclidean ellipse, and thus $M_2 = R_2$ ([1], (17.11)).

Let us consider on the set of points of the M_2 an associated euclidean metric, an arbitrary point F and an indicatrix I centered at F. If I is a euclidean circle then our theorem is true. If it is not, then I has a euclidean nearest and a euclidean fartest point to F. Let these euclidean distances be d_1 and d_2 $(d_1 < d_2)$. Consider a euclidean circle μ around F with radius $d: d_1 < d < d_2$. Let us go over by a euclidean homothety to a euclidean metric where μ appears as a unit circle. μ has two intersection points with I not diametrally opposite. Denote the lines joining these points and F by t and by s respectively. Thus in the directions of t and s the euclidean and Minkowskian metric coincide.

Let us denote the tangents to I at the intersection points U_1, U_2 of I and t by h_1 and h_2, and the line parallel to these through F by h. Perform now in our plane an affine transformation $\mathfrak{a}$ such that: 1) the points of t are fixpoints of $\mathfrak{a}$ (i.e. t is an axis), 2) the image $\bar{h} = \mathfrak{a}h$ of h is euclidean perpendicular to t, 3) the euclidean measures of corresponding segments on s and on $\bar{s} = \mathfrak{a}s$ are equal. We show the existence of such an affinity. It is well-known that an affinity (on the plane) is uniquely determined by its axis t and by a pair of corresponding points $P \notin t$ and $\bar{P} = \mathfrak{a}P$. Let P be the common point of s, μ and I, and p the line through P parallel to t, and n the euclidean perpendicular line to t through F. We denote the intersection point



of h and p by H: $h \cap p = H$, and similarly $n \cap p = H'$. Let us measure the euclidean distance $\overline{HH'}$ from P on p, the resulting point being P', so that the orientation of $\overrightarrow{HH'}$ and of $\overrightarrow{PP'}$ should coincide. Draw through P' a line m euclidean perpedicular to t. If $\bar{P} = m \cap \mu$ exists, then the axis t and the corresponding points P and $\bar{P}$ already determine an affinity $\mathfrak{a}$ with the required property. (In this affinity $\mathfrak{a}H = \bar{H}$ lies on n, and thus $\bar{h} = n$.)

We still show that $\bar{P}$ really exists. I lies in the strip between h_1 and h_2. Thus in the case $\sphericalangle(t, s) > \sphericalangle(t, h)$ (as on the figure) P must be between H and $Q = p \cap h_2$. Therefore $\overline{HP} < \overline{HQ} = 1$. In consequence of this P' lies between H' and $V = r \cap p$, where r is the line perpendicular to t through U_2. Because of such a position of P' m intersects μ. — If $\sphericalangle(t, s) \leqq \sphericalangle(t, h)$ then it is easy to see the truth of the statement.

4. We show that, in the associated euclidean metric introduced at the beginning of the previous section **3**, $\bar{I} = \mathfrak{a}I$ is already a euclidean circle, and therefore I is an ellipse in the associated euclidean metric.

Let us consider a Minkowski ellipse ε on M_2 whose main axis and one of its foci are the line t, resp. the point F of the previous section. We put $\varepsilon \cap t = A, B$; $\varepsilon \cap s = R, S$. The directrix g of ε is Minkowski perpendicular to $t: t \perp g$ according to Lemma 4. Since h_1 is a tangent at U_1 of the indicatrix centered at F, we have $g \| h_1$. $\mathfrak{a}$ transforms the indicatrix I into an indicatrix $\bar{I} = \mathfrak{a}I$ and it takes the ellipse ε of the Minkowski plane $M_2(I)$ into the ellipse $\bar{\varepsilon} = \mathfrak{a}\varepsilon$ of $M_2(\bar{I})$, for the Minkowski distance with respect to I of two arbitrary points K and L equals the Minkowski distance with respect to $\bar{I}$ of the images of these points: $\|K, L\|_I = \|\mathfrak{a}K, \mathfrak{a}L\|_{\mathfrak{a}I}$. Denoting the images of points and lines in the affinity $\mathfrak{a}$ by the same letter provided with a bar, we obviously have $A = \bar{A}$, $B = \bar{B}$, $\bar{R}, \bar{S} \in \bar{\varepsilon}$ and we find that $\bar{g}$, which is perpendicular to t according to section **3**, is a directrix of $\bar{\varepsilon}$. Since in the direction of t and $\bar{s}$ the Minkowskian and euclidean units coincide and $\bar{g}$ is euclidean perpendicular to t, the points $A, B, \bar{R}, \bar{S}$ are elements of a euclidean ellipse ε_2^{eu} of second kind determined by F, $\bar{g}$ and a λ, and also of a euclidean ellipse ε_1^{eu} of the first kind determined by F, F^* and $2a = \overline{AB} = \|A, B\|_I$ coinciding with ε_2^{eu}. We remark that in view of

$$\lambda = \frac{\|A, F\|_I}{\|A, g\|_I} = \frac{\|\bar{A}, \bar{F}\|_{\bar{I}}}{\|\bar{A}, \bar{g}\|_{\bar{I}}} = \bar{\lambda} = \frac{\overline{AF}}{\overline{A\bar{g}}}$$

the parameter λ of ε_2 equals the parameter $\bar{\lambda}$ of $\bar{\varepsilon}_2$ and the parameter of ε_2^{eu}.

From $\bar{R} \in \varepsilon_2^{eu}, \bar{\varepsilon}_1$ we have

$$\overline{\bar{R}F} + \overline{\bar{R}F^*} = \overline{AB}$$

$$\|\bar{R}, F\|_{\bar{I}} + \|\bar{R}, F^*\|_{\bar{I}} = \|A, B\|_{\bar{I}}.$$

Now, on t and $\bar{s}$ euclidean and Minkowskian units are the same. Therefore the right hand sides are equal and so are the first terms of the left hand sides too. From these we obtain $\overline{\bar{R}F^*} = \|\bar{R}, F^*\|_{\bar{I}}$. This means the coincidence of euclidean and Minkowskian units also on the line $|\bar{R}, F^*|$. The same consideration can be performed for $\bar{s}$ too. This yields the equivalence of the euclidean and Minkowskian units also on the line $|\bar{S}, F^*|$.

$\bar{\varepsilon}_1$ is determined by F, $\bar{R}$ and F^*. If we fix F and $\bar{R}$, and let F^* vary on the open interval (F, ∞) of t not containing A, then we obtain on $M_2(\bar{I})$ a Minkowskian ellipse $\bar{\varepsilon}_1$ for any position of F^*, the directrix of $\bar{\varepsilon}_1$ being always parallel to $\bar{g}$. Therefore we can perform our above consideration for every $\bar{\varepsilon}_1$ so obtained. Accordingly the Minkowskian unit is the euclidean unit in the direction of all lines $|F^*, \bar{R}|$ and $|F^*, \bar{S}|$, where $F^* \in (F, \infty)$. However in this way all directions are obtained. Therefore $\bar{I}$ is a euclidean circle, and $I = \alpha^{-1}\bar{I}$ is a euclidean ellipse in the starting associated euclidean metric. Consequently $M_2 = R_2$.

5. Now we immediately get

Theorem 1. *If the two classes of the ellipses of first and second kind coincide on every plane of a Minkowski space, then it is a euclidean space.*

Let L_2 be an arbitrary plane through the center 0 of I. We know from our Proposition that in an associated euclidean metric every $L_2 \cap I$ is an ellipse. But in this case I is an ellipsoid, as shown in [1], (16.12), and hence $M_n = R_n$.

6. The constructions leading to the result of section **4** can be performed in an arbitrary small domain of the plane, for example in a given neighbourhood of F, since the image of any ellipse occurring in section **4** subjected to an appropriate homothetic transformation with center F already lies in the given neighbourhood of F together with the image of the portion of its directrix used in the construction.

Let us consider a Finsler space F_2. This can be approximated with an arbitrary accuracy by an M_2 in an appropriately small neighbourhood of any of its points. If $\{\varepsilon_1\} = \{\varepsilon_2\}$ holds in F_2, then it holds in the above neighbourhood of each of its points and in this neighbourhood the considerations and constructions of section **4** can already be performed. Consequently the indicatrix of such an F_2 is an ellipse at every point. Therefore the space is a Riemannian space. Thus we have.

Theorem 2. *If the ellipses of the first and second kind of a Finsler plane F_2 coincide, then it is a Riemannian plane V_2.*

The case of Riemannian V_2 where $\{\varepsilon_1\} = \{\varepsilon_2\}$ will be discussed in a forthcoming paper.

References

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