Multilinear relations

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In [4], following ARENS [1] and BERGE [2, p. 133], we defined a relation S from a vector space X into another Y over the same field K to be linear if

$$S(x)+S(y)\subset S(x+y) \text{ and } \lambda S(x)\subset S(\lambda x)$$

for all $x, y\in X$ and $\lambda\in K$.

Essentially the same definition was used in our former paper [3], where we proved that a relation S from one vector space X into another Y over the same field is linear iff $S(o)$ is a subspace of Y and there exists a linear function f from X into Y such that $S(x)=f(x)+S(o)$ for all $x\in X$.

In the present note, we will only show that the above basic characterization theorem of linear relations can easily be extended to multilinear relations. For this, we need the following straightforward

Definition. Let X_i $(i=1, \dots, n)$ and Y be vector spaces over the same field K and $X=\bigtimes_{i=1}^{n} X_i$. Then a relation S from X into Y will be called multilinear if it is linear in each of its variables separately in the sense that for each $x=(x_i)\in X$ and $i=1, \dots, n$ the relation S_{xi} from X_i into Y defined by

$$S_{xi}(t) = S(x_1 \dots, x_{i-1}, t, x_{i+1}, \dots, x_n)$$

is linear.

Note that the fact that X has also a vector space structure is quite immaterial here, and thus the following characterization theorem of multilinear relations cannot be derived from that of linear relations.

Theorem. *Let X_i $(i=1, \dots, n)$ and Y be vector spaces over the same field K, and S be a relation from $X=\bigtimes_{i=1}^{n} X_i$ into Y. Then the following properties are equivalent:*

(i) *S is multilinear;*

(ii) *$S(o)$ is a subspace of Y and there exists a multilinear function f from X into Y such that*

$$S(x) = f(x)+S(o)$$

for all $x\in X$.

PROOF. Suppose that (i) holds. For each $i=1, \dots, n$ let E_i be a basis for X_i, and for each $x_i \in X_i$, denote $\hat{x}_i$ the unique function from E_i into K such that

$$x_i = \sum_{e_i \in E_i} \hat{x}_i(e_i)e_i.$$

Define $E = \overset{n}{\underset{i=1}{\times}} E_i$, and let φ be a function from E into Y such that $\varphi(e) \in S(e)$ for all $e \in E$. Moreover, define the function f from X into Y by $f(x) = \sum_{e \in E} \hat{x}(e)\,\varphi(e)$, where $\hat{x}(e) = \prod_{i=1}^{n} \hat{x}_i(e_i)$. Then, it is clear that f is multilinear. On the other hand, we clearly have

$$f(x) \in \sum_{e \in E} \hat{x}(e)\, S(e) \subset S(x)$$

for all $x \in X$. Thus, by [4, Theorem 3.3],

$$S(x) = f(x) + S_{x1}(o)$$

for all $x \in X$. Moreover, by [4, Theorem 2.2],

$$S_{x1}(o) = S_{x1}(o) + S_{x1}(o) \text{ and } S_{x1}(o) = S(o, -x_2, x_3, \dots, x_n),$$

and hence

$$S_{x1}(o) = S(o, o, x_3, \dots, x_n)$$

for all $x \in X$. From this, by induction, it is clear that we also have $S_{x1}(o) = S(o)$ for all $x \in X$. Finally, it is also clear that $S(o)$ is a subspace of Y.

The proof of the converse implication is quite straightforward, and may therefore be completely omitted.

Remark. A two dimensional version of this paper was accepted for publication in The Mathematics Student by M. K. Singal in 1976.

Moreover, this paper was presented at the Confence on Functional Equations in Retzhof (Austria) in 1978. (See *Aequationes Math.* **19** (1979), 267.)

References

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