Characterization of integervalued R-additive functions

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1. Let $\mathfrak{N}$ be the set of natural numbers. For an arbitrary subset $S \subseteqq \mathfrak{N}$ let $\bar{S} = S \cup \{0\}$, and $|S|$ be the cardinality of S. We shall write $\mathfrak{N}_0 = \mathfrak{N} \cup \{0\}$ $(= \bar{\mathfrak{N}})$.

Let $R_i \subseteqq \mathfrak{N}$ $(i=0, 1, \ldots)$. We shall say that $\{\bar{R}_0, \bar{R}_1, \ldots\}$ is a finite direct decomposition (*FDD*) of $\mathfrak{N}_0$ if every $n \in \mathfrak{N}$ can be stated uniquely as

$$n = r_{i_1} + \ldots + r_{i_s}, \quad r_{i_k} \in R_{i_k} \quad (k = 1, \ldots, s). \tag{1.1}$$

We shall say that this *FDD* is an R-system, if:

a) R_i is finite and nonempty,

b) the smallest element of R_i is smaller than the smallest element of R_j for every $i<j$.

We shall say that an R-system is monotonic if the smallest element of R_j is larger than the greatest element of R_i for every $i<j$.

The sets $R_i = \{q^i, 2q^i, \ldots, (q-1)q^i\}$ $(i=0, 1, \ldots)$ generate the q-ary number-system, this is a monotonic R-system.

Let $f(n)$ be an arbitrary complex-valued function defined on $\mathfrak{N}_0$. We shall say that f is an R-additive function if $f(0)=0$ and for $n \in \mathfrak{N}$ written in the form (1.1) we get

$$f(n) = f(r_{i_1}) + \ldots + f(r_{i_s}). \tag{1.2}$$

This is a straightforward generalization of q-additivity that has been introduced by A. O. GELFOND [1].

For the sake of brevity let

$$R = \bigcup_{i=0}^{\infty} R_i, \tag{1}$$

$$A_i = \{\lambda_0 + \ldots + \lambda_{i-1} | \lambda_j \in \bar{R}_j\} \setminus \{0\}, \tag{2}$$

$$T_i = \{\lambda_i + \lambda_{i+1} + \ldots + \lambda_h | h = i+1, i+2, \ldots;\ \lambda_j \in \bar{R}_j\} \setminus \{0\}$$

The following assertion has been proved in [2] and [3].

Theorem A. *Let us be given an R-system and let f be an integervalued R-additive function. Assume that*

(a) *the R-system is monotonic, or there exists a suitable prime p for which $p \nmid m$ holds for infinitely many $m \in R$;*

(b) *there exists an index i_0 such that $n \mid f(n)$ for every $n \in T_{i_0}$.*

Then $f(n) = c \cdot n$ holds for every $n \in T_{i_0}$, where c is a suitable integer.

2. We shall assume that $f, g, f_1, \ldots, f_s, g_1, \ldots, g_s, h_1, \ldots, h_s$ are integer-valued functions, furthermore that $M_1 < M_2 < \ldots < M_s$ are fixed nonnegative integers.

Let us consider the conditions

$$(A_s) \qquad \sum_{i=1}^{s} f_i(n+M_i) \equiv \sum_{i=1}^{s} f_i(M_i) \pmod{n} \quad (\forall n \in \mathfrak{N});$$

$$(B_s) \qquad \sum_{i=1}^{s} g_i(n+M_i) \equiv 0 \pmod{n} \quad (\forall n \in \mathfrak{N}).$$

It is clear that A_1 with $M_1 = 0$ is the same as condition (b) in Theorem A.

We shall say that $(f_1, \ldots, f_s)$ is a trivial solution of (A_s) if $f_i(n) = c_i n$ $(i = 1, \ldots, s)$ with integer constans c_i. Similarly $(g_1, \ldots, g_s)$ is a trivial solution of (B_s) if $g_i(n) = a_i n$ $(i = 1, \ldots, s)$ with integer constants a_i. It is obvious that in the latter case $\sum_{i=1}^{s} a_i M_i = 0$.

Theorem 1. *If* (A_s) *has a nontrivial solution then so has* (B_s), *and vice versa.*

First we shall prove

Lemma 1. *Let* f *be an integer-valued* R*-additive function. If*

$$(2.1) \qquad f(n) \equiv A \pmod{n}$$

holds for every $n \in T_{i_0}$, *then* $A = 0$.

Proof. Let P be a large integer, $P > |A|$. There exists an infinite sequence $r_{i_1}, r_{i_2}, \ldots$ of integers, $r_{i_k} \in R_{i_k}$ $(i = 1, 2, \ldots)$ such that $i_0 \leqq i_1 < i_2 <, \ldots,$ and

$$r_{i_k} \equiv z_{i_k} \pmod{P}, \quad f(r_{i_k}) \equiv w \pmod{P} \quad (k = 1, 2, \ldots),$$

where z and w are suitable residues mod P. Then the integer

$$N = \sum_{k=1}^{P} r_{i_k}$$

belongs to T_{i_0}, $N \equiv Pz \equiv 0 \pmod{P}$, $f(N) \equiv 0 \pmod{P}$, consequently $P | A$, which involves that $A = 0$. □

Proof of Theorem 1. Let $(f_1, \ldots, f_s)$ and $(h_1, \ldots, h_s)$ be some solutions of (A_s); $H = \sum_{i=1}^{s} h_i(M_i)$, $F = \sum_{i=1}^{s} f_i(M_i)$. Then $(g_1, \ldots, g_s)$ defined by

$$(2.2) \qquad g_i(n) = Hf_i(n) - Fh_i(n) \quad (i = 1, \ldots, s)$$

is a solution of (B_s). If $(f_1, \ldots, f_s)$ is a nontrivial solution and $(h_1, \ldots, h_s)$ is a trivial solution such that $H \neq 0$ (there exists such a solution), then $(g_1, \ldots, g_s)$ is a nontrivial solution.

To prove the second part of the assertion it is enough to show that for every solution $(g_1, \ldots, g_s)$ satisfying (B_s) the relation

$$(2.3) \qquad \sum_{i=1}^{s} g_i(M_i) = 0$$

holds. Indeed, under (2.3) the functions $(g_1, \dots, g_s)$ give a solution satisfying (A_s).

Let i_0 be such an index for which $M_i \in A_{i_0}$ $(i=1, \dots, s)$. From (B_s) we get that

$$\sum_{i=1}^{s} g_i(n) \equiv -\sum_{i=1}^{s} g_i(M_i) \pmod n$$

holds for every $n \in T_{i_0}$, and so by Lemma 1 applied for $f=\sum g_i$ we get that (2.3) holds. □

If $(g_1, \dots, g_s)$ is a solution of (B_s), then $\big(g_1(n)+c_1 n, \dots, g_s(n)+c_s n\big)$ with arbitrary integer constants $c_1, \dots, c_s$ is a solution of (A_s). This is a straightforward consequence of (2.3). Furthermore, if $(f_1, \dots, f_s)$ is a solution of (A_s) and the integer constants $c_1, \dots, c_s$ are so chosen that $\sum_{i=1}^{s} c_i M_i = 0$, then $\big(f_1(n)-c_1 n, \dots, f_s(n)-c_s n\big)$ is a solution of (A_s). Consequently it is enough to characterize the solutions of (B_s) only.

Theorem 2. *Let $F, g_1, \dots, g_s$ be integer-valued R-additive functions, $0<M_1<<M_2<\dots<M_s$ be fixed integers. Assume that*

(α) the R-system is monotonic, or there exists a suitable prime p such that $p \nmid m$ holds for infinitely many $m \in R$, and

(β) that the relation

$$\sum_{i=1}^{s} g_i(n+M_i) \equiv F(n) \pmod n \tag{2.4}$$

holds for every positive integer n.

Then

$$\sum_{i=1}^{s} g_i(n+M_i) = F(n)+cn$$

identically, with a suitable integer c.

Proof. Let i_0 be so large that $M_i \in A_{i_0}$ holds for every i. Then for $n \in T_{i_0}$ we get that $g_i(n+M_i)=g_i(n)+g_i(M_i)$, consequently from (2.4) we deduce that

$$\sum_{i=1}^{s} g_i(M_i) + \sum_{i=1}^{s} g_i(n) \equiv F(n) \pmod n.$$

Taking into account Lemma 1 with $f=F-g_1-\dots-g_s$, we get that

$$\sum_{i=1}^{s} g_i(M_i) = 0,$$

and so from Theorem A that $f(n)=cn$, i.e.

$$\sum_{i=1}^{s} g_i(N) = F(N)+cN \quad (\forall N \in T_{i_0}). \tag{2.5}$$

Let now $b \in \mathfrak{N}_0$ be an arbitrary integer. Let i_1 be so large that $i_1 \geqq i_0$, $b \in A_{i_1}$, $b+M_i \in A_{i_1}$ $(i=1, \dots, s)$. Let us consider now (2.4) with $n=N+b$, $N \in T_{i_1}$. Taking into account (2.5) we get immediately that

$$\sum_{i=1}^{s} g_i(b+M_i) \equiv F(b)-cN \pmod{(N+b)},$$

i.e. that

$$\sum_{i=1}^{s} g_i(b+M_i) \equiv F(b)+cb \pmod{(N+b)} \quad (\forall N \in T_{i_1}). \tag{2.6}$$

Since T_{i_1} contains arbitrary large elements, we have that

$$\sum_{i=1}^{s} g_i(b+M_i) = F(b)+cb.$$

The proof is finished. □

Corollary 1. *Let f be an integervalued R-additive function, $M \geqq 0$ be a fixed integer. Assume that condition* (a) *in* Theorem A *holds, and that*

$$f(n+M) \equiv f(M) \pmod{n} \tag{2.7}$$

is satisfied for every $n \in \mathfrak{N}$.

Then $f(n)=cn$ for every $n \in \mathfrak{N}_0$.

Proof. Let the integers c and d be so chosen that $df(M)=cM$. Let $g(n)=df(n)-cn$. From (2.7) we get that $g(n+M)\equiv 0 \pmod{n}$ holds identically, and so from Theorem 2 ($s=1$, $F=0$) that $g(n)=0$. Consequently $f(n)=\frac{c}{d}n$ c/d is an integer, since f is an integervalued function. □

Remark. Theorem 2 contains the following special case. If f and g are integervalued R-additive functions, $M \geqq 0$ being an integer, furthermore condition (a) in Theorem A and the relation

$$g(n+M) \equiv f(n) \pmod{n} \quad (\forall n \in \mathfrak{N})$$

hold, then

$$g(n+M) = f(n)+cn. \tag{2.8}$$

We could give a complete description of the solutions of (2.8) in the case $R_0 = \{1, 2, \ldots, k\}$ [4].

3.

Theorem 3. *Let the R-system be arbitrary, f_i $(i=1, \ldots, s)$ be arbitrary integervalued R-additive functions and φ_i $(i=1, \ldots, s)$ be arbitrary integervalued functions defined on $\mathfrak{N}_0$. Assume that*

$$\sum_{i=1}^{s} f_i(n+M_i) \equiv \sum_{i=1}^{s} \varphi_i(M_i) \pmod{n} \tag{C_s}$$

holds for every $n, M_1, \ldots, M_s \in \mathfrak{N}$.

Then $f_i(n)=c_i n$, $\varphi_i(n)=f_i(n)+a_i$, *where* c_i, a_i *are integers and* $\sum_{i=1}^{s} a_i=0$.

Lemma 2. *Let* f *be an integervalued R-additive function.*
Assume that

$$(3.1) \qquad f(n+K)-f(n+M) \equiv f(K)-f(M) \pmod n$$

holds for every $n, K, M \in \mathfrak{N}$.
Then $f(n)=cn$, c *being an integer.*

Proof of Lemma 2. Let $g(n)=f(n)-f(1)\cdot n$. Then $g(1)=0$, and the requirements stated for f in Lemma 2 hold for g too. Let $n\in\mathfrak{N}$ be arbitrary. Let i_0 and $i_1>i_0$ be so chosen, that $n+1\in A_{i_0}$, N_1 and N_2 be so that $N_1\in T_{i_0}\cap A_{i_1}$ and $N_2\in T_{i_1}$. Let us consider (3.1) with $M=N_2$ and $K=N_2+1$. Then we have

$$(3.2) \qquad g(n+1)-g(n) \equiv 0 \pmod{(n+N_1)}.$$

Observing that N_1 can be an arbitrary large number we get that $g(n+1)=g(n)$. From $g(1)=0$ we get that $g(n)=0$ identically. □

Proof of Theorem 3. Let $M_1, \ldots, M_s$ be fixed. Let i_0 be so large that $M_i\in A_{i_0}$ $(i=1, \ldots, s)$. Then (C_s) involves that

$$\sum_{i=1}^{s} f_i(n) \equiv \sum_{i=1}^{s} \varphi_i(M_i)-\sum_{i=1}^{s} f_i(M_i) \pmod n \quad \forall n\in T_{i_0}.$$

Lemma 1 gives that the right hand side is zero, i.e.

$$(3.3) \qquad \sum_{i=1}^{s} \varphi_i(M_i) = \sum_{i=1}^{s} f_i(M_i).$$

(3.3) holds for every choice of $M_1, \ldots, M_s$, therefore $\varphi_i(M)-f_i(M)=\varphi_i(1)-f_i(1)$ is a constant, i.e.

$$(3.4) \qquad \varphi_i(n) = f_i(n)+a_i, \quad a_i \text{ integer}.$$

After substituting (3.4) into (C_s) we get that

$$(3.5) \qquad \sum_{i=1}^{s} f_i(n+M_i) \equiv \sum_{i=1}^{s} f_i(M_i)+\sum_{i=1}^{s} a_i \pmod n$$

holds for every choice of $n, M_1, \ldots, M_s\in\mathfrak{N}$. Hence we get immediately that

$$f_i(n+K)-f_i(n+M) \equiv f_i(K)-f_i(M) \pmod n$$

holds for every $n, K, M\in\mathfrak{N}$, i.e. the condition (3.1) holds for $f_i=f$. From Lemma 2 we get that $f_i(n)=c_i n$. The assertion $\sum a_i=0$ is an immediate consequence of (3.5). □

The following assertion is a straightforward consequence of Lemma 2.

Corollary 2. *Let f be an arbitrary integervalued R-additive function. Assume that*

$$f(n+M) \equiv f(n)+f(M) \pmod n$$

holds for every $n, M \in \mathfrak{N}$.

Then $f(n)=cn$, c *being an integer constant.*

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