Quadratic statistics in Hilbert space

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§ 1. Introduction

The object of this paper is to generalize some well known finite dimensional statistical results to the infinite dimensional case. Our aim is to give a dimension-free (see GOODMAN and PATHAK [5]) approach to the presented problems. This means that, where it is possible, we use such uniform methods which give the desired results in the same way both in the finite dimensional and in the infinite dimensional cases.

In § 2 we list some basic facts on Hilbert space valued random variables. Our suitable formulations make possible to obtain the properties of the variance operator and the sample variance operator in a very simple way.

§ 3 is devoted to a study of characterizations of the Gaussian distribution in Hilbert space. There are several results on this subject, eg. Goodman and Pathak present characterizations by linear statistics in [5]. In § 3 we give characterizations with the help of quadratic and linear statistics.

In § 4 the operator loss function for a Hilbert space valued parameter is introduced. The idea of the operator loss function is a simple generalization of the notion of the matrix loss function (cf. LINNIK and RUKHIN [9]). An analogue of the Rao—Blackwell theorem is proved.

§ 2. Preliminary remarks and notations

Throughout this paper we deal with Hilbert space valued and operator valued random variables (r. v. 's) defined on a probability space $(\Omega, \mathcal{A}, P)$. Let H denote a real separable Hilbert space with inner product $\langle .,. \rangle$ and norm $|\,.\,|$. The space of bounded linear operators of H is denoted by $L(H)$ and $\|\,.\,\|$ is used for the operator norm. $L_2(H)$ denotes the Hilbert space of Hilbert—Schmidt operators of H (see [8]).

The following definition is valid both for Hilbert valued and for operator valued r.v. 's.

Definition 2.1. Let Y be a Banach valued r.v. The expectation of Y is defined by the Bochner integral and is denoted by EY. The conditional expectation of Y has been defined by Scalora [13] and it is denoted by $E(Y\mid .)$ (see also [2]).

Definition 2.2. Let $x, y \in H$. We define the operator xy' of H as follows:

$$xy'(z) = x\langle y, z\rangle$$

for every $z \in H$.

Proposition 2.3.

(1) $$xy' \in L_2(H) \quad \text{and} \quad \|xy'\| = \|xy'\|_2 = |x|\,|y|,$$

where $\|\,.\,\|_2$ denotes the norm in $L_2(H)$.

(2) If $\{e_i\}_{i=1}^{\infty}$ is an orthonormal basis in H, then $\{e_i e_j'\}_{i,j=1}^{\infty}$ is an orthonormal basis in $L_2(H)$.

One can easily establish the other elementary properties of this operator.

Remark. For $x_i \in H$ $(i=1, \dots, n)$ the n-linear form $\prod_{i=1}^{n}{}' x_i$ of H is defined by the relation

$$\prod_{i=1}^{n}{}' x_i(y_1, \dots, y_n) = \prod_{i=1}^{n} \langle x_i, y_i \rangle,$$

where $y_i \in H$ $(i=1, \dots, n)$. $\prod_{i=1}^{n}{}' x_i$ is a Hilbert—Schmidt type n-linear form (see [8]) and for $n=2$ it can be identified with the operator $x_1 x_2'$.

Definition 2.4. If X and Y are H-valued r.v.'s, then let $E(XY')$ be the following operator of H:

$$\langle E(XY')a, b\rangle = E\{\langle X, b\rangle \langle Y, a\rangle\}$$

for every $a, b \in H$.

The covariance operator of X and Y is defined by the relation $\operatorname{cov}(X, Y) = E\{(X-EX)(Y-EY)'\}$. $\operatorname{cov}(X, X)$ is called the variance operator of X and it is denoted by D^2X.

Proposition. 2.5. *If* $E(|X|\,|Y|) < \infty$, *then* $E(XY')$ *exists and it is equal to the Bochner integral of the measurable operator* XY'.

PROOF. First we recall that a map $A: \Omega \to L(H)$ is called a measurable operator, if the real-valued function $\langle Aa, b\rangle$ is measurable for every $a, b \in H$. If the range of A is in $L_2(H)$, then this notion of measurability and the other ones defined in [6] (p. 74) coincide.

Since $\langle XY'a, b\rangle = \langle X, b\rangle\langle Y, a\rangle$, XY' is a measurable operator. $E(\|XY'\|) = E(|X|\,|Y|) < \infty$ shows that XY' is Bochner integrable. Finally, it is easy to see that the Bochner integral of XY' is equal to $E(XY')$.

Remarks. (1) It is easy to see that $E(XY')$ possesses the same properties as the finite dimensional second moment operator.

(2) If $E|X|^2 < \infty$ and $E|Y|^2 < \infty$, then $\operatorname{trace} E(XY') = E\langle X, Y\rangle < \infty$ (see [8] p. 16.).

By Proposition 2.5. we can immediately prove statistical properties of the sample variance operator.

Proposition 2.6. *Let* $X_1, \dots, X_n$ *be i.i.d. H-valued r.v.'s,* $E|X_1|^2 < \infty$. *Let*

$$(n-1)^{-1} S_n = (n-1)^{-1} \sum_{i=1}^{n} (X_i - \bar{X})(X_i - \bar{X})'$$

be the sample variance, where $\bar{X} = n^{-1} \sum_{i=1}^{n} X_i$ *is the sample mean. Then*

(1) $(n-1)^{-1} S_n$ *is an unbiased estimator of* $D^2 X_1$.

(2) $n^{-1} \sum_{i=1}^{n} X_i X_i'$ *is an unbiased consistent estimator of* $E(X_1 X_1')$.

(3) *If* X_1 *is Gaussian, then* $(n-1)^{-1} S_n$ *is an unbiased consistent estimator of* $D^2 X_1$.

PROOF. (1) It is a simple consequence of Prop. 2.5.

(2) It follows from the law of large numbers (see [2]).

(3) It is an immediate consequence of Lemma 3.6 and (2).

§ 3. Characterizations of the Gaussian distribution

In this paragraph the Gaussian distribution is characterized by the property of constant regression of a quadratic statistic on the sample mean. This is an extension of Theorem 3.1 of CACOULLOS [1]. (The characterization of the Gaussian law in a Hilbert space through the zero regression of two linear statistics was considered by GOODMAN and PATHAK [5].)

For the proof of the main result, Proposition 3.5, we require the following lemmas.

Lemma 3.1. *Let* B *and* F *be Banach spaces. Assume that the function* $f\colon B\times\Omega\to F$ *has the following properties:* $f(x,\omega)$ *is measurable in* ω *and* $f(x,.)$ *is Bochner integrable for every fixed* x. *Suppose that* $\frac{\partial f(x,\omega)}{\partial x}$ *exists and it is a measurable function of* ω *for every* x.

If $\left|\frac{\partial f(x,\omega)}{\partial x}\right| < h(\omega)$ *for* $x\in B$, $\omega\in\Omega$, *where* $h(\omega)$ *is a real-valued integrable function, then* $\frac{d\int_\Omega f(x,\omega)\,P(d\omega)}{dx}$ *exists and it is equal to* $\int_\Omega \frac{\partial f(x,\omega)}{\partial x} P(d\omega)$.

$\int_\Omega \cdot\, P(d\omega)$ denotes Bochner integral and $\frac{df(x)}{dx}$ and $\frac{\partial f(x,\omega)}{\partial x}$ denote Fréchet derivative and Fréchet partial derivative resp.

PROOF. Direct computation (see [3], 8.11.2).

Lemma 3.2. (See [5] Lemma 1.3.) *Let* X *be an* H*-valued r.v. Denote by* φ_X *the characteristic functional of* X. *If* $E|X|^n<\infty$, *then* φ_X *is differentiable* n*-times and*

$$\frac{d^n \varphi_X(t)}{dt^n} = E\{[\prod_{j=1}^{n}{}' (iX)]\, e^{i\langle t, X\rangle}\}, \quad t\in H.$$

PROOF. Use Lemma 3.1 and induction.

Lemma 3.3. *Let* X *be an* H*-valued r.v. and let* Y *be a real-valued r.v. and* $E|Y|<\infty$. Y *has constant regression on* X *if and only if*

(1) $$E\{Ye^{i\langle t, X\rangle}\} = EYE\{e^{i\langle t, X\rangle}\}$$

for each $t\in H$.

PROOF. See [7], 1.1.1 Lemma.

Lemma 3.4. *Let B^* denote the dual of the Banach space B. Assume that there exists a countable norm-determining set F in B^* (in particular let B be separable). Let Y be a B-valued r.v.*

Y has constant regression on X if and only if the relation (1) *holds for all $t\in H$.*

PROOF. By the preceding lemma one can easily prove that the following properties are equivalent.

$E\{Y|X\}=EY$ a.s.

$E\{\langle y^*, Y\rangle|X\}=E\langle y^*, Y\rangle$ a.s. for all $y^*\in F$, where $\langle y^*, y\rangle$ denotes the duality between B and B^*.

$$E\{\langle y^*, Y\rangle e^{i\langle t, X\rangle}\} = E\langle y^*, Y\rangle E\{e^{i\langle t, X\rangle}\} \quad \text{for each} \quad t\in H,\ y^*\in F.$$

$$E\{Ye^{i\langle t, X\rangle}\} = EYE\{e^{i\langle t, X\rangle}\} \quad \text{for each} \quad t\in H.$$

Proposition 3.5. *Let $X_1, \dots, X_n$ be i.i.d. H-valued r.v.'s $(n\geqq 2)$ and let $E|X_1|^2<\infty$. Let A_{jk} $(j, k=1, \dots, n)$ be bounded linear operators and $b_j\in H$ $(j=1, \dots, n)$. Suppose that $\sum_{j=1}^{n} A_{jj}=-\sum_{j\neq k} A_{jk}$ and $A=\sum_{j=1}^{n} A_{jj}$ is invertible, $\sum_{j=1}^{n} b_j=0$.*

Let

$$Q=\sum_{j,k=1}^{n} A_{jk}X_jX_k'+\sum_{j=1}^{n} b_jX_j'$$

and

$$\Lambda=\sum_{j=1}^{n} X_j.$$

Q has constant regression on Λ (that is $E\{Q|\Lambda\}=C$, where C is a bounded linear operator) if and only if X_1 has a Gaussian distribution.

PROOF. First suppose that $E\{Q|\Lambda\}=C$. By Lemma 3.4

$$E\{Qe^{i\langle t, \Lambda\rangle}\} = CE\{e^{i\langle t, \Lambda\rangle}\} \quad \text{for} \quad t\in H. \tag{2}$$

We then have from Lemma 3.2 that

$$-\sum_{j} A_{jj}\frac{d^2f(t)}{dt^2}[f(t)]^{n-1}-\sum_{j\neq k} A_{jk}\frac{df(t)}{dt}\left[\frac{df(t)}{dt}\right]'[f(t)]^{n-2}-$$

$$-i\sum_{j} b_j\left[\frac{df(t)}{dt}\right]'[f(t)]^{n-1}=C[f(t)]^n, \tag{3}$$

where $f(t)$ denotes the characteristic functional of X_1. There exists a neighbourhood K of $0\in H$ such that $f(t)\neq 0$ if $t\in K$. Put $\varphi(t)=\ln[f(t)]$, $t\in K$. (3) implies that

$$-\sum_{j} A_{jj}\frac{d^2\varphi(t)}{dt^2}-\sum_{j,k} A_{jk}\frac{d\varphi(t)}{dt}\left[\frac{d\varphi(t)}{dt}\right]'-$$

$$-i\sum_{j} b_j\left[\frac{d\varphi(t)}{dt}\right]'=C, \quad t\in K.$$

By the assumptions of the theorem we obtain

$$\frac{d^2\varphi(t)}{dt^2} = -A^{-1}C, \quad t\in K.$$

Hence $\frac{d^s\varphi(t)}{dt^s}=0$ if $t\in K$ and $s>2$.

Therefore the Taylor series expansion of $\varphi(t)$ is the following:

$$\varphi(t) = i\langle m, t\rangle - \frac{1}{2}\langle Dt, t\rangle, \quad t\in K, \tag{4}$$

where $m=EX_1$ and $D=D^2X_1$. Hence the characteristic function of $\langle X_1, h\rangle$ $(h\in H)$ is

$$f_{\langle X_1, h\rangle}(\alpha) = \exp\{i\langle m, h\rangle\alpha - \frac{1}{2}\langle Dh, h\rangle\alpha^2\}$$

for some neighbourhood $|\alpha|<\varepsilon$ of the origin. By Marcinkiewicz' theorem ([7], Lemma 2.4.3) $\langle X_1, h\rangle$ is a Gaussian r.v., thus X_1 has a Gaussian distribution.

Conversely, if X_1 has a Gaussian distribution, then $\varphi(t)=\ln[f(t)]$ satisfies (4) (where $f(t)$ is the characteristic functional of X_1). This implies (2), therefore Q has constant regression on Λ.

Finally we prove that the independence of the sample mean and the sample variance is a characteristic property of the Gaussian distribution.

Lemma 3.6. *Let X be an H-valued Gaussian r.v. and let $X_1, \dots, X_n$ $(n\geqq 2)$ be a random sample for X. Then there exist independent Gaussian r.v.'s $Z_1, \dots, Z_n$ for which $EZ_i=0$ $(i=1, \dots, n-1)$ and $EZ_n=\sqrt{n}\,EX$; $D^2Z_i=D^2X$ $(i=1, \dots, n)$ and*

$$\bar{X} = n^{-1/2}Z_n, \quad S_n = \sum_{i=1}^{n-1} Z_i Z_i',$$

where $\bar{X}=n^{-1}\sum_{i=1}^{n} X_i$ is the sample mean and $S_n=\sum_{i=1}^{n}(X_i-\bar{X})(X_i-\bar{X})'$ is the sample variance.

PROOF. It can be proved as the corresponding k-dimensional theorem (see [4]).

Proposition 3.7. *Let $X_1, \dots, X_n$ be a sample for X $(n\geqq 2)$. X has a Gaussian distribution if and only if $\bar{X}$ and S_n are independent.*

PROOF. Suppose that $\bar{X}$ and S_n are independent. Let $\eta_i=\langle X_i, y\rangle$ $(i=1, \dots, n)$, where $y\in H$ is fixed. We have $\langle\bar{X}, y\rangle=\bar{\eta}$ and $\langle S_n y, y\rangle=\sum_{i=1}^{n}(\eta_i-\bar{\eta})^2$. By Theorem 4.2.1 of [7] $\langle X, y\rangle$ is Gaussian, hence X is a Gaussian r.v.

Conversely, if X is a Gaussian r.v., then by Lemma 3.6 $\bar{X}$ and S_n are independent.

§ 4. Operator loss functions

In this paragraph we investigate the estimations of a Hilbert space valued parameter. Instead of real-valued loss functions we use operator loss functions. To study the operator loss function we need the following preliminary lemmas (see [15]).

Lemma 4.1. *Let X be a B-valued Bochner integrable r.v. on the probability space $(\Omega, \mathcal{A}, P)$, where B is a real Banach space with norm $|\,.\,|$.*

Then there exists a sequence of measurable partitions $\mathcal{F}_n=\{A_{in}\}_{i=1}^{i_n}$ of Ω such that the following property is satisfied.

If for all i and n a point ω_{in} is chosen from A_{in} and the simple function X_n is defined as

$$X_n(\omega)=\begin{cases} X(\omega_{in}) & \text{if } \omega\in A_{in} \quad (i=2, \dots, i_n) \\ 0 & \text{if } \omega\in A_{1n}, \end{cases}$$

then $\lim\limits_{n\to\infty} X_n=X$ *a.s. and* $\lim\limits_{n\to\infty} \int\limits_{\Omega} |X_n-X|\,dP=0.$

PROOF. It can be supposed that B is separable. Let B denote the σ-algebra of Borel sets of B. According to [12] Q_X (the distribution of X) is a tight measure on $(B, \mathcal{B})$. Thus for every n there exists a compact set K_n such that $Q_X(B\setminus K_n)<n^{-1}$. We can suppose that $K_n\subset K_{n+1}$ $(n=1, 2, \dots)$. There exists a subdivision of K_n into disjoint measurable sets $\{A'_{in}\}_{i=2}^{i_n}$ such that $|x-y|<n^{-1}$ if $x, y\in A'_{in}$ $(i=2, \dots, i_n;\ n=1, 2, \dots)$. Let $A_{1n}=X^{-1}(B\setminus K_n)$ and $A_{in}=X^{-1}(A'_{in})$ $(i=2, \dots, i_n)$. Let X_n be the function defined in the theorem. Then $|X(\omega)-X_n(\omega)|<n^{-1}$, $\omega\notin A_{1n}$ thus $\lim\limits_{n\to\infty} X_n=X$ a.s. and in L_1.

Definition 4.2. Let B and F be real Banach spaces and let C be a positive closed cone in F. A function $f\colon B\to F$ is called convex if

$$f(\lambda x+(1-\lambda)y)\leqq \lambda f(x)+(1-\lambda)f(y) \quad \text{for} \quad x, y\in B, \quad \lambda\in(0, 1).$$

(For $u, v\in F$ the relation $u\leqq v$ is defined by $v-u\in C$.)

Lemma 4.3. *Let f be the function defined in 4.2. Assume that f is continuous. Let X be a B-valued r.v. on $(\Omega, \mathcal{A}, P)$. If X and $f(X)$ are Bochner integrable, then*

$$f(E\{X|\mathcal{F}\})\leqq E\{f(X)|\mathcal{F}\} \quad a.s., \tag{5}$$

where $\mathcal{F}$ is a σ-subalgebra of $\mathcal{A}$.

PROOF. It is easy to verify that

$$f(\lambda_1 x_1+\dots+\lambda_n x_n)\leqq \lambda_1 f(x_1)+\dots+\lambda_n f(x_n),$$

where $x_i\in B$, $\lambda_i\in[0, 1]$ $(i=1, \dots, n)$, $\sum\limits_{i=1}^{n}\lambda_i=1$.

According to Lemma 4.1 a sequence $X_n=\sum\limits_{i=1}^{i_n} x_{in}\,\chi_{A_{in}}$ of simple functions can be constructed such that $\lim\limits_{n\to\infty} X_n=X$ and $\lim\limits_{n\to\infty} f(X_n)=f(X)$ a.s. and in L_1.

Then

$$f(E\{X_n|\mathscr{F}\}) = f\Big(\sum_{i=1}^{i_n} x_{in} P\{A_{in}|\mathscr{F}\}\Big) \leqq$$

(6)

$$\leqq \sum_{i=1}^{i_n} f(x_{in}) P\{A_{in}|\mathscr{F}\} = E\{f(X_n)|\mathscr{F}\} \quad \text{a.s.},$$

because $\sum_{i=1}^{i_n} P\{A_{in}|\mathscr{F}\}=1$ a.s. and $P\{A_{in}|\mathscr{F}\}\geqq 0$ a.s. Since the conditional expectation is a continuous mapping of L_1 into itself ([10], p. 102), $\lim_{n\to\infty} E\{X_n|\mathscr{F}\}= =E\{X|\mathscr{F}\}$ and $\lim_{n\to\infty} E\{f(X_n)|\mathscr{F}\}=E\{f(X)|\mathscr{F}\}$ in L_1. If a sequence converges in L_1, then one can find a subsequence which converges a.s. Therefore (6) implies (5) because f is continuous.

Definition 4.4. Let $(\Omega, \mathscr{A})$, $(\Theta, \mathscr{T})$ and $(\Delta, \mathscr{D})$ denote the space of observations, the space of parameters and the space of decisions respectively. Let $P(\vartheta, A)$ be a transition probability relative to $(\Theta, \mathscr{T})$ and $(\Omega, \mathscr{A})$. Since we want to investigate the estimation of a parameter belonging to H, we shall assume that $\Theta, \Delta \subset H$.

An estimator is a measurable mapping δ of $(\Omega, \mathscr{A})$ into $(\Delta, \mathscr{D})$.

Definition 4.5. Let C denote the positive closed cone of positive self-adjoint operators in $L(H)$. Let $W: L(H)\to L(H)$ be a continuous, monotone, convex function. Let $W_1: \Theta\times\Delta\to L(H)$ be defined by

$$W_1(\vartheta, \delta) = W[(\delta-\vartheta)(\delta-\vartheta)'],$$

where $\delta\in\Delta$ is the estimator of parameter $\vartheta\in\Theta$. W_1 is called a (monotone, convex) loss function.

Definition 4.6. (1) The risk of an estimator δ is defined by

$$R_\delta(\vartheta) = E_\vartheta[W_1(\vartheta, \delta)] = \int_\Omega W_1(\vartheta, \delta(\omega)) P(\vartheta, d\omega) \quad (\vartheta\in\Theta),$$

where the integral is the Bochner integral in $L(H)$.

(2) We say that an estimator δ_1 is not inferior to δ_2 if $R_{\delta_1}(\vartheta)\leqq R_{\delta_2}(\vartheta)$ for all $\vartheta\in\Theta$.

Proposition 4.7. *Let V be a sufficient statistic for ϑ. Let T be an estimator of ϑ, $E|T|^2<\infty$. Assume that $W_1(\vartheta, T)$ and $W_1(\vartheta, E(T|V))$ are Bochner integrable in $L(H)$. Then the estimator $E(T|V)$ is not inferior to T.*

Proof. It follows from Lemma 4.3 that

$$E\{W[(T-\vartheta)(T-\vartheta)']|V\} \geqq W(E\{(T-\vartheta)(T-\vartheta)'|V\}) \quad \text{a.s.}$$

It is easy to see that $E(XX'|V)\geqq E(X|V)[E(X|V)]'$ a.s. if $|X|^2$ is Bochner integrable. Since W is monotone these inequalities imply

$$E\{W[(T-\vartheta)(T-\vartheta)']|V\} \geqq W([E\{T|V\}-\vartheta][E\{T|V\}-\vartheta]') \quad \text{a.s.}$$

that is

$$E\{W_1(\vartheta, T)|V\} \geqq W_1(\vartheta, E\{T|V\}) \quad \text{a.s.}$$

Taking expectation on both sides we obtain the desired result.

Remarks. It is easy to see that $E\{T|V\}$ is an explicit function of V. If T is unbiased, then $E(T|V)$ is also an unbiased estimator of ϑ and $D^2T \geqq D^2E(T|V)$.

Definition 4.8. A randomized estimator is a transition probability $S=S(\omega, D)$ on $(\Omega, \mathscr{A})$ and $(\Delta, \mathscr{D})$. The risk of S is defined by the relation

$$R_S(\vartheta) = \int_\Omega \int_\Delta W_1(\vartheta, \delta) S(\omega, d\delta) P(\vartheta, d\omega), \quad \vartheta \in \Theta.$$

Remark. To study the risk of a randomized estimator we need an analogue for Banach valued r.v.'s of the Fubini type theorem given in [10] (Prop. III. 2.1). With the help of Theorem 3.7.13 of [6] one can easily establish this result.

Proposition 4.9. *Let W_1 be an operator loss function and let S be a randomized estimator. Then the nonrandom estimator $s(\omega) = \int_\Delta \delta S(\omega, d\delta)$ is not inferior to S (provided that all integrals involved exist).*

PROOF. The Fubini theorem mentioned and the method used in the proof of Prop. 4.7 give the result.

Finally we give a remark in connection with the Stein problem (see Stein [14]).

Proposition 4.10. *Let X be an H-valued Gaussian r.v. with known variance operator $\sum$ and unknown expectation ϑ. Then X is an admissible estimator* ([10], p. 78) *of ϑ with respect to the operator loss function* $W_1(\vartheta, \delta) = (\delta - \vartheta)(\delta - \vartheta)'$.

PROOF. If X is not admissible, then there exists an estimator X_1 better than X. That is, there exists a $h \in H$ such that $\langle R_{X_1}(\vartheta)h, h\rangle \leqq \langle R_X(\vartheta)h, h\rangle$ for $\vartheta \in \Theta$ and $\langle R_{X_1}(\vartheta)h, h\rangle \not\equiv \langle R_X(\vartheta)h, h\rangle$. Therefore

$$E(\langle X_1, h\rangle - \langle \vartheta, h\rangle)^2 \leqq E(\langle X, h\rangle - \langle \vartheta, h\rangle)^2 \quad (\vartheta \in \Theta) \tag{7}$$

and

$$E(\langle X_1, h\rangle - \langle \vartheta, h\rangle)^2 \not\equiv E(\langle X, h\rangle - \langle \vartheta, h\rangle)^2.$$

According to [14] $\langle X, h\rangle$ is an admissible estimator of $\langle \vartheta, h\rangle$ under quadratic loss. Thus in (7) equality is valid for $\vartheta \in \Theta$.

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