Remarks on the convergence of orthogonal series

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In this note we prove some simple results concerning the convergence of orthogonal series with restrictions on the Lebesgue functions and of multiple Fourier series. These solve four earlier posed problems.

I. Let $\lambda=\{\lambda_k\}_{k=1}^{\infty}$ be an increasing sequence of positive numbers tending to the infinity. For an orthonormal system (ONS) $\varphi=\{\varphi_k\}$ defined on $(0, 1)$ let

$$L_n(\varphi;\ x) = \int_0^1 \Big|\sum_{k=1}^{n} \varphi_k(x)\varphi_k(t)\Big|\,dt \quad (0<x<1)$$

be its Lebesgue function, and let $\Omega(\lambda)$ be the set of those φ which satisfy

$$\int_0^1 \sup_n \frac{L_n(\varphi;\ x)}{\lambda_n}\,dx \leqq 1.$$

For a sequence $a=\{a_n\}_1^{\infty}$ we set

$$\|a, \lambda\| = \sup_{\varphi\in\Omega(\lambda)} \int_0^1 \sup_n \Big|\sum_{k=1}^{n} a_k\varphi_k(x)\Big|\,dx$$

and let $M(\lambda)$ be the set of those a for which $\|a;\lambda\|<\infty$ is satisfied. It was proved in [4, Theorem 1] and [5, Theorem 1] that this $M(\lambda)$ with the usual addition and multiplication and with the above norm is a Banach-space and the system $\{e^n\}$, $e^n=\{0, \ldots, 0, \overset{n}{\breve{1}}, 0, \ldots\}$ constitutes a basis in $M(\lambda)$. The importance of $M(\lambda)$ is enlightened by the fact that (see [4, Theorem 2] and [6, Assertion D]) the series $\sum_{k=1}^{\infty} a_k\,\varphi_k(x)$ converges almost everywhere for every ONS $\varphi=\{\varphi_k\}\in\Omega(\lambda)$ if and only if $a=\{a_k\}\in M(\lambda)$.

The second author proved [8, Theorem II] that $\{a_k\}\in M(\lambda)$ implies $a(\{n_k\})\in M(\lambda)$ for every subsequence $\{n_k\}$ of the natural numbers where

$$a(\{n_k\})_i = \begin{cases} a_{n_k} & \text{if} \quad i=n_k, \quad k=1, 2, \ldots, \\ 0 & \text{if} \quad i\neq n_k, \quad k=1, 2, \ldots \end{cases}$$

and raised the question (see [8, p. 298]) if $\{a_k\}\in M(\lambda)$, $|b_k|\leqq|a_k|$ implies $\{b_k\}\in M(\lambda)$, or more generally (see also [7, p. 150]) $\|\{b_k\};\lambda\|\leqq\|\{a_k\};\lambda\|$ holds or not (for related results when there is no restriction on the Lebesgue functions see [9]). Here we prove by Banach-space technique that the answer is positive:

Theorem. *For any sequences* $a=\{a_k\}$, $b=\{b_k\}$, $|b_k|\leqq|a_k|$ *we have* $\|b;\lambda\|\leqq\leqq\|a;\lambda\|$.

PROOF. Since for every sequence $\varepsilon_k=\pm 1$ the system $\{\varepsilon_k\varphi_k\}_1^\infty$ has the same Lebesgue functions as $\{\varphi_k\}_1^\infty$ has, we have for every sequence $\{a_k\}$ $\|\{\varepsilon_k a_k\};\lambda\|==\|\{a_k\};\lambda\|$, and so the theorem follows from the following lemma to be proved below.

Lemma. *If* X *is a Banach space and* $\sum_{n=1}^{\infty} x_n$ *is a convergent series of* X *such that every series* $\sum_{n=1}^{\infty}\pm x_n$ *converges and* $\|\sum_1^\infty \pm x_n\|\leqq\|\sum_1^\infty x_n\|$ *then* $\sum_1^\infty \alpha_n x_n$ *converges and* $\|\sum_1^\infty \alpha_n x_n\|\leqq\|\sum_1^\infty x_n\|$ *for every sequence* $\{\alpha_n\}$, $|\alpha_n|\leqq 1$.

Indeed, the convergence of every series $\sum_1^\infty \alpha_n x_n$, $\alpha_n=O(1)$ follows from a general result on unconditional convergence in Banach spaces (see e.g. [3, Lemma 16.1]). Let $T\colon l^\infty\to X$ be defined by $T(\{\alpha_n\})=\sum_1^\infty \alpha_n x_n$. An application of the uniform boundedness principle yields that T is a bounded linear operator from l^∞ into X. Let U be the unit ball of l^∞ and EU the set of its extremal points. Clearly $\varepsilon=\{\varepsilon_k\}\in EU$ if and only if $\varepsilon_k=\pm 1$ for every k. By the assumption $\|T\varepsilon\|\leqq\|x\|$ where $x=\sum_1^\infty x_n$. If every α_n has the form $\alpha_n=l/N$ with fixed $N\geqq 1$ and $l=-N,-N+2,\dots$ $\dots,N-2,N$ then the sequence $\alpha=\{\alpha_n\}$ can be written as $\alpha=\frac{1}{N}\sum_{j=1}^{N}\varepsilon^j$ where each ε^j belongs to EU and so we can see that in this case $\|T\alpha\|\leqq\frac{1}{N}\sum_{j=1}^{N}\|T\varepsilon^j\|\leqq\|x\|$. Since the α's having the above form constitute a dense subset of U, the boundedness of T yields that $\|T\|\leqq\|x\|$ and this is exactly what we wanted to prove.

Actually, the method above gives much more namely if $\Phi=\{\varphi=\{\varphi_k\}\}$ is a set of function systems from $L^p(0,1)$ $(1\leqq p<\infty)$ such that $\{\varphi_k\}\in\Phi$ implies that $\{\varepsilon_k\varphi_k\}\in\Phi$ for every sequence $\varepsilon_k=\pm 1$ then the norms

$$\|a\|_{\Phi,p}=\sup_{\varphi\in\Phi}\Big\{\int_0^1\big(\sup_n\big|\sum_{k=1}^{n}a_k\varphi_k(x)\big|\big)^p dx\Big\}^{1/p}\quad(1\leqq p<\infty)$$

are monotonous: $|b_k|\leqq|a_k|$ $(k=1,2,\dots)$ implies $\|b\|_{\Phi,p}\leqq\|a\|_{\Phi,p}$. Indeed, for finite sequences a,b (or if $a_k=b_k=0$ for $k\geqq N$) this follows by the method above and then the general case follows by an application of Fatou's lemma. As an application we get for the norm

$$\|a\|=\sup_{\alpha_k}\int_0^1\big(\sup_n\big|\sum_{k=1}^{n}a_k\cos(k\pi(x-\alpha_k))\big|\big)^2 dx$$

that $\|b\|\leqq\|a\|$ whenever $|b_k|\leqq|a_k|$ (compare [10] where the weaker inequality $\|b\|\leqq 6\|a\|$ was proved).

II. Our second theme is connected with multiple Fourier series.

Let $u=\sum_{i,j=0}^{\infty} u_{ij}$ be a double series and $s_{mn}=\sum_{i=0}^{m}\sum_{j=0}^{n} u_{ij}$ its partial sums. There are several convergence definitions concerning double series; the two most important ones are:

(i) u is said to be convergent to s in Pringsheim's sense if for every $\varepsilon>0$ there is an N such that $|s_{mn}-s|<\varepsilon$ whenever $\min(n, m)\geqq N$,

(ii) u is said to be regularly convergent if for every $\varepsilon>0$ there is an N such that $|\sum_{m_1\leqq i\leqq m_2}\sum_{n_1\leqq j\leqq n_2} u_{ij}|<\varepsilon$ whenever $m_2\geqq m_1$, $n_2\geqq n_1$ and $\max(m_1, n_1)\geqq N$.

It is easy to see that regular convergence implies convergence in Pringsheim's sense and convergence in Pringsheim's sense together with the ordinary convergence of the row- and column-series $\sum_{j=0}^{\infty} u_{ij}$ $(i=0, 1, \ldots)$ and $\sum_{i=0}^{\infty} u_{ij}$ $(j=0, 1, \ldots)$ is equivalent to regular convergence.

Let now $I^2=[0, 2\pi]\times[0, 2\pi]$. It is proved in [2] that for a double sequence $\{a_{ij}\}$ the following two statements are equivalent:

(i) for every ONS $\{\varphi_{ij}\}$ defined in I^2, $\sum_{i,j} a_{ij}\varphi_{ij}$ converges almost everywhere in Pringsheim's sense,

(ii) for every ONS $\{\varphi_{ij}\}$ defined on I^2, $\sum_{i,j} a_{ij}\varphi_{ij}$ regularly converges almost everywhere.

The problem if this two kinds of convergence coincide a.e. also for individual double Fourier series was raised by F. Móricz (oral communication). The negative answer is conveyed by the next example. Let $f(x, y)=h(x)g(y)$ where $h, g\in L^1_{2\pi}$, $h(x)=0$ for $0<x<\pi$ and $h(x)>0$ for $\pi<x<2\pi$ and g has almost everywhere boundedly divergent Fourier series (cf. [10. 308]). If $s_{m,n}(f; x, y)$ and $s_m(h; x)$ denote the partial sums of the Fourier series of f and h, respectively, then $s_{m,n}(f; x, y)=s_m(h; x)\cdot s_n(g; y)$. Hence the Fourier series of f converges to 0 for almost every $(x, y)\in(0, \pi)\times(0, 2\pi)$ in Pringsheim's sense (because $s_m(h; x)\to 0$ and $s_n(g; y)=O_y(1)$ as $m, n\to\infty$) but it is not regularly convergent at points (x, y) (and so a.e.) which satisfy the conditions: there is an m with $s_m(h; x)\neq 0$ and $\{s_n(g; x)\}_{n=0}^{\infty}$ diverges.

Another problem of F. Móricz (see [1, p. 215]) was if this "non-coincidence phenomenon" can occur for functions from the classes $L^p(I^2), p>1$. Here the answer is again "NO". In fact, if $f\in L^p(I^2)$, $p>1$ then the function

$$h_{m,x}(y) = c_m\Big(\Big(\int_0^{2\pi} f(\tau, y)\cos m\tau d\tau\Big)\cos mx+\int_0^{2\pi} (f(\tau, y)\sin m\tau\, d\tau)\sin mx\Big),$$

$$c_m = \begin{cases} 1/\pi & \text{if } m>0,\\ 1/2\pi & \text{if } m=0\end{cases}$$

belongs to $L^p_{2\pi}$ for every fixed m and x and the n-th partial sum of the "m-th row" of the Fourier expansion of f at a point (x, y) is $s_n(h_{m,x}; y)$, therefore this row-series converges for every x and almost every y by the Carleson—Hunt-theorem.

Similarly can be proved the a.e. convergence of every column-series and so, according to what has been said after our convergence definitions, the convergence in Pringsheim's sense of the Fourier series of f implies its regular convergence at almost every point (x, y).

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