On the laxity of rooted trees with vertices of degree three

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§ 1. Introduction

The present paper is devoted to studying rooted trees in which each vertex (except the end vertices) is of degree three. The basic notion of the article is laxity, this is a number which characterizes the dispersion of the distance of the end vertices from the root of the tree. (Roughly speaking, the laxity is small if all the end vertices lie almost at the same distance, it is large when the distances of the end vertices are strongly different from each other.)

In § 2, mainly the simplest notions concerning the considered trees are introduced. Certain numerical functions (occurring in the later results) are discussed in the next section. The laxity is defined in § 4 in a formal manner. After studying the trees whose laxity is minimal, § 6 presents a perspicuous characterization of laxity. In § 7, the trees with maximal laxity are constructed, it is shown that such a tree is determined uniquely for each (odd) number k of edges, and a formula is obtained which gives the maximal laxity among the trees having k edges. The question of enumerating the trivalent trees with respect to their laxity is touched in the last section.

It seems probable that the present investigations admit a natural extension to the class of all rooted trees which are regular in the sense that the degree of every inner vertex of a tree is the same (depending on the tree only).

§ 2. Basic notions

A tree (i.e., a connected finite non-directed graph having no circuit) is called *trivalent* if its each vertex is of degree one or of degree three. A vertex P is called an *end vertex* if its degree $d(P)$ equals 1, P is an *inner vertex* otherwise (i.e., if $d(P)=3$).

It is well-known that when the number of the edges of a tree T is k, then T contains $k+1$ vertices. For trivalent trees, some additional statements are also true:

Proposition 1. *Let T be a trivalent tree, denote by k the number of edges of T. Then*

k is an odd number,
T has $(k+3)/2$ end vertices, and
T has $(k-1)/2$ inner vertices.

PROOF. Denote the number of end vertices by x. Then the number of inner vertices is $k+1-x$. The equality

$$2k = x+3(k+1-x)$$

holds because the left-hand side is the double number of edges and the right-hand side is the sum of the degrees of the vertices. Hence we get $2x=k+3$ and

$$k+1-x = k+1-\frac{k+3}{2} = \frac{k-1}{2}.$$

$k(=2x-3)$ is odd because x is an integer. Q.E.D.

Let P and Q be two vertices of a tree. Their *distance* $\delta(P, Q)$ is defined as the length (meant as the number of edges) of the (single) chain between P and Q.

A pair (T, R) is called a *rooted tree* if R is a vertex of the tree T. The distinguished vertex R is called the *root*; we write simply T (instead of (T, R)) if it is clear that a root has been selected in the tree.

If P is an arbitrary vertex of a rooted tree (T, R), then the distance $\delta(P, R)$ is called the *height* of P and denoted by $\nu(P)$. Let us consider an edge $e=PQ$ of T; it is obvious that

$$|\nu(P)-\nu(Q)| = 1.$$

The greater of $\nu(P)$, $\nu(Q)$ is called the *height* $\nu(e)$ of e. The following assertions are evident:

If m is positive, then the number of vertices of height m equals the number of edges of height m.

There is no edge of height zero.

The root R is the only vertex of height zero.

Let T be a rooted tree. The maximal height occurring in T is denoted by $\mu(T)$. It is obvious that *each vertex P fulfilling $\nu(P)=\mu(T)$ is an end vertex.* (T, R) is called a *solid tree*[1]) if $\nu(P)\geqq\mu(T)-1$ for every end vertex P of (T, R). Fig. 1 shows two solid trees.

We denote by $\sigma(n)$ the number of edges e which satisfy $\nu(e)=n$. Let $\sum_{j=1}^{n}\sigma(j)$ be denoted by $\tau(n)$.

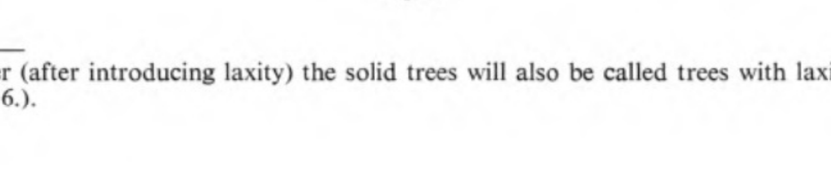

Fig. 1

[1]) Later (after introducing laxity) the solid trees will also be called trees with laxity zero (cf. Proposition 6.).

§ 3. Some numerical functions

For a positive integer n, let $\alpha(n)$, $\beta(n)$ be defined by $\alpha(n)=3\cdot 2^{n-1}$ and $\beta(n)=$ $=3\cdot(2^n-1)$. Let $\gamma(n)$ be the integer i which satisfies[2])

$$2^{i-1} < \frac{n}{3} \leqq 2^i.$$

It is obvious that $\alpha(n+1)=2\alpha(n)$.

Proposition 2. *We have*

$$\beta(n) = \sum_{j=1}^{n} \alpha(j).$$

Proof. We use induction for n. In case $n=1$, $\alpha(1)=3=\beta(1)$. Suppose that the assertion holds for $n-1$. The deduction

$$\beta(n)-\sum_{j=1}^{n-1}\alpha(j) = \beta(n)-\beta(n-1) = 3(2^n-1)-3(2^{n-1}-1) = 3\cdot 2^{n-1} = \alpha(n)$$

shows that it is valid for n, too. Q.E.D.

Lemma 1. *We have* $\gamma(n)+1=\gamma(2n)$ *for each* n.

Proof. We have defined $\gamma(n)$ by postulating

$$2^{\gamma(n)-1} < \frac{n}{3} \leqq 2^{\gamma(n)}.$$

Multiply these quantities by two. We get

$$2^{(\gamma)n} < \frac{2n}{3} \leqq 2^{\gamma(n)+1},$$

this means the validity of the lemma. Q.E.D.

Remark. It is easy to see that $\gamma(n)=\gamma(n+1)$ if $n(\geqq 5)$ is odd.

Proposition 3. *We have* $\gamma(\alpha(n))=n-1$ *for each* n. *If* $n\geqq 4$, *then* $\alpha(\gamma(n))$ *satisfies the following statements:*
(i) $\alpha(\gamma(n))$ *is of form* $3\cdot 2^i$,
(ii) $n/2\leqq\alpha(\gamma(n))<n$.

Remark. The statements (i), (ii) determine $\alpha(\gamma(n))$ uniquely (for a fixed n). To see this, let m_1 and m_2 fulfil (i), (ii) (as $\alpha(\gamma(n))$ where $m_1>m_2$. It follows from (i) that $m_1/m_2\geqq 2$. (ii) implies $m_1/m_2<2$.

Proof of Proposition 3. The first statement is true since

$$\gamma(\alpha(n)) = \gamma(3\cdot 2^{n-1}) = n-1$$

follows at once from the definitions of α, γ.

[2]) This definition can be written in the from $\gamma(n)=[\log_2(n/3)]^*$ where $[x]^*$ is the integer satisfying $x\leqq[x]^*<x+1$.

Suppose $n \geqq 4$. Then $n/3 > 1$, thus $\gamma(n) \geqq 1$ and $\alpha(\gamma(n))$ is defined. The validity of (ii) will be proved inductively. In case $n=4$, it is clear that $\alpha(\gamma(4)) = \alpha(1) = 3$.

Consider an arbitrary $n(\geqq 5)$, suppose that (ii) holds for each of $4, 5, 6, \ldots, n-1$ (instead of n). We separate two cases depending on the parity of n.

Case 1. n is even. The induction hypothesis (applied for $n/2$) implies

$$\frac{n}{2} \leqq 2\alpha\left(\gamma\left(\frac{n}{2}\right)\right) < n.$$

Since the deduction

$$2\alpha\left(\gamma\left(\frac{n}{2}\right)\right) = \alpha\left(\gamma\left(\frac{n}{2}\right)+1\right) = \alpha(\gamma(n))$$

is true (the second equality follows from Lemma 1), (ii) is valid for n.

Case 2. n is odd. Analogously to the preceding case, we have

$$\frac{n+1}{2} \leqq 2\alpha\left(\gamma\left(\frac{n+1}{2}\right)\right) < n+1$$

and

$$2\alpha\left(\gamma\left(\frac{n+1}{2}\right)\right) = \alpha\left(\gamma\left(\frac{n+1}{2}\right)+1\right) = \alpha(\gamma(n+1)) = \alpha(\gamma(n)).$$

(cf. the Remark before Proposition 3). Q.E.D.

Let the function η be defined by $\eta(3)=0$ and

$$\eta(n) = \sum_{j=0}^{\gamma(n)-1} (n - 3 \cdot 2^j) \quad (n \geqq 4).$$

Lemma 2. *We have*

$$\eta(n) - \eta(n-1) = \gamma(n)$$

for each $n(\geqq 4)$.

PROOF. The validity of the lemma is obvious when $n=4$. If $n(>4)$ is of form $3 \cdot 2^i + 1$, then $\gamma(n-1) = \gamma(n) - 1$, thus

$$\eta(n) - \eta(n-1) = \sum_{j=0}^{\gamma(n-1)} (n - 3 \cdot 2^j) - \sum_{j=0}^{\gamma(n-1)-1} (n-1-3 \cdot 2^j) =$$

$$= n - 3 \cdot 2^{\gamma(n-1)} + \sum_{j=0}^{\gamma(n-1)-1} \left((n-3 \cdot 2^j) - (n-1-3 \cdot 2^j)\right) = n-(n-1)+\gamma(n-1) = \gamma(n).$$

If n is not of form $3 \cdot 2^i + 1$, then $\gamma(n) = \gamma(n-1)$, hence

$$\eta(n) - \eta(n-1) = \sum_{j=0}^{\gamma(n)-1} \left((n-3 \cdot 2^j) - (n-1-3 \cdot 2^j)\right) = \gamma(n).$$

Q.E.D.

Proposition 4. *We have*

$$\eta(n) = \sum_{j=3}^{n} \gamma(j)$$

for each $n(\geqq 3)$.

PROOF. The statement is verified by induction. $\eta(3)=0=\gamma(3)$ holds clearly. Suppose that the assertion is true for $n-1$. Then

$$\sum_{j=3}^{n} \gamma(j) = \gamma(n) + \sum_{j=3}^{n-1} \gamma(j) = \gamma(n) + \eta(n-1) = \eta(n)$$

by Lemma 2. Q.E.D.

§ 4. The notion of laxity

Proposition 5. *Let* T *be a trivalent rooted tree and* n *be a positive integer. Then* $\sigma(n) \leqq \alpha(n)$ *and* $\tau(n) \leqq \beta(n)$.

Remark. The example of a solid tree with $\beta(n)$ edges shows that $\alpha(n)$ and $\beta(n)$ are the best upper bounds.

PROOF OF PROPOSITION 5. It is obvious that $\sigma(1)$ equals either 1 or $3(=\alpha(1))$. The number $\sigma(n+1)$ is clearly not greater than $2\sigma(n)$. Hence the first inequality follows by induction (on n). The second inequality is a consequence of the first one and of Proposition 2. Q.E.D.

Now we introduce the most important notion of this paper. Let T be a trivalent rooted tree, and denote the number of edges T by k. The number

$$\frac{\sum_{j=1}^{\mu(T)-1} \left(\min(\beta(j), k) - \tau(j)\right)}{2} \tag{4.1}$$

is called the *laxity* of T and is denoted by $\lambda(T)$.

Remarks. Each term occurring in the numerator of (4.1) is non-negative in consequence of the trivial formula $\tau(j) \leqq k$ and of the second assertion of Proposition 5. The upper bound for the summation could also be written as ∞, because

$$\min(\beta(j), k) = k = \tau(j)$$

if $j \geqq \mu(T)$.

We shall see later (Corollary 1) that the laxity is always an integer.

§ 5. Trees with laxity zero

Proposition 6. *A trivalent rooted tree* T *is solid if and only if* $\lambda(T)=0$.

PROOF. Let T be solid. It is clear that $\sigma(j)=\alpha(j)$ and $\tau(j)=\beta(j)(<k)$ hold when $1 \leqq j < \mu(T)$. Hence the laxity of T is zero.

Suppose that T is not solid. Let P be an end vertex of T such that $\nu(P)$ is minimal (among the end vertices). By the supposition, $\nu(P)<\mu(T)-1$. It is easy to see that $\sigma(j)=\alpha(j)$ if $1\leqq j\leqq \nu(P)$ but $\sigma(\nu(P)+1)<\alpha(\nu(P)+1)$. Hence (by Proposition 2 and the definition of τ) $\tau(\nu(P)+1)<\beta(\nu(P)+1)$. This and the inequality

$$\tau(\nu(P)+1) < \tau(\mu(T)) = k$$

(which follows from $\nu(P)+1<\mu(T)$) implies $\lambda(T)>0$ (cf. the remarks after (4.1)). Q.E.D.

Proposition 7. *Let T be a solid tree, denote the number of edges of T by k. Then $\mu(T)=\gamma(k+3)$.*

PROOF. We use induction on the odd number k. Obviously $\gamma(6)=1$ and $\mu(T)$ equals 1 if T is the solid tree with 3 edges.

It is evident that

$$\gamma(k+3)=\begin{cases}\gamma(k+1)+1 & \text{if } k \text{ is of form } 3\cdot 2^i-1,\\ \gamma(k+1) & \text{otherwise.}\end{cases} \tag{5.1}$$

Consider a solid tree T with $k+2$ edges ($k\geqq 3$). Choose the vertices A, B, C so that $\nu(A)=\nu(B)=\mu(T)$ and C is adjacent to both A, B. (This choice is always possible.) Delete A, B and the edges AC, BC. We get a solid tree T' with k vertices. $\mu(T')=\gamma(k+3)$ by the induction hypothesis. It can be seen easily that $\mu(T)=\mu(T')$ unless k is of form $3\cdot 2^i-3$; in the latter case, $\mu(T)=\mu(T')+1$. Comparing these inequalities with (5.1), we obtain $\mu(T)=\gamma(k+5)$. Q.E.D.

Proposition 8. *Let T, k be as in Proposition 7. Among the end vertices of T, there are*

$$k+3-\alpha(\gamma(k+3)) \tag{5.2}$$

vertices of height $\mu(T)$, and

$$\alpha(\gamma(k+3))-\frac{k+3}{2} \tag{5.3}$$

vertices of height $\mu(T)-1$.

PROOF. First we verify (5.2). If $k=3$ or k is of form $3\cdot 2^i-1$, then we show its validity immediately; in the other cases, we use induction on k.

The tree T with $k=3$ has 3 end vertices (each of height $1=\mu(T)$) and

$$3+3-\alpha(\gamma(6)) = 6-\alpha(1) = 6-3 = 3.$$

Assume that $k(\geqq 5)$ is of form $3\cdot 2^i-1$. On the one hand, $\alpha(\gamma(k+3))=k+1$ by the second sentence of Proposition 3, hence

$$k+3-\alpha(\gamma(k+3)) = 2.$$

On the other hand, if we form the tree T' as in the proof of Proposition 7 (now T' has $k-2$ edges!), then it is easy to see that

(i) every end vertex P of T' satisfies $\nu(P)=\mu(T')$,

(ii) $\mu(T)=\mu(T')+1$,

(iii) T has exactly two vertices (namely A and B) whose height is $\mu(T)$.

Consider now a tree with $k+2$ edges where $k(\geqq 5)$ is not of form $3\cdot 2^i-3$. Suppose that (5.2) holds for the solid trees with k edges. Form again T' as in the preceding proof. It is clear that $\mu(T)=\mu(T')$. If P is an end vertex of T' whose height equals $\mu(T')$, then P satisfies these statements in T, too. In addition, A and B are also end vertices of height $\mu(T)$ in T. Thus the number of the end vertices in question is

$$2+k+3-\alpha(\gamma(k+3)) = k+5-\alpha(\gamma(k+3)) = k+5-\alpha(\gamma(k+5))$$

where the second equality is true by (5.1).

We have shown (5.2). The formula (5.3) is an immediate consequence of (5.2) and the second statement of Proposition 1. Q.E.D.

Example. Figure 2 contains two solid trees with 13 edges which are not isomorphic. (Indeed, the distance $\delta(P, Q)$ is 6 in the first tree; no distance in the second tree exceeds 5.)

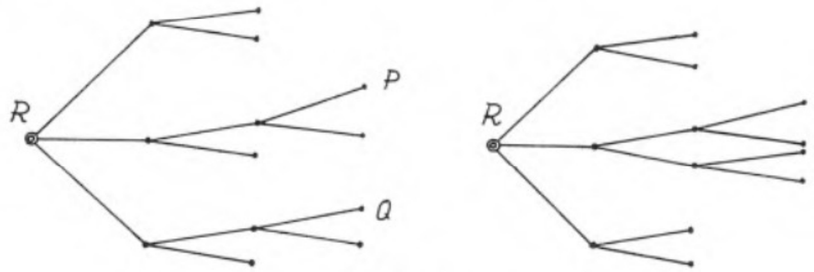

Fig. 2

§ 6. Characterization of the laxity by transformations

Let T be a trivalent rooted tree with k edges. Choose in it four vertices A, B, C, D such that A, B, D are pairwise different end vertices and the edges AC, BC exist in T. The following procedure is called an *elementary transformation:*

(i) we delete the vertices A, B and the edges AC, BC;

(ii) we add the new vertices A', B' and the new edges $A'D$, $B'D$ to the tree.

The difference $v(C)-v(D)$ is called the *weight* of the elementary transformation. It is easy to see that a *tree T admits an elementary transformation with positive weight if and only if T is not solid.*

Let

$$T_1,\ T_2, \ldots, T_q \quad (q \geqq 1) \tag{6.1}$$

be a sequence of trees such that T_i is obtained by an elementary transformation from T_{i-1} $(2\leqq i\leqq q)$. The sequence (6.1) is then called a *transformation* and the *weight* of this transformation is defined as the sum of the weights of the $q-1$ elementary transformations.[3]) (6.1) is called a *positive transformation* if each elementary transformation in it has a positive weight. Starting with an arbitrary tree T, there is

[3]) If $q=1$, then the weight of (6.1) is zero.

at least one transformation which terminates with a solid tree. (Take, for example, a positive transformation which cannot be continued further.)

Proposition 9. *Consider an arbitrary trivalent rooted tree T. Construct a transformation which starts with T and terminates with a solid tree. Then the weight of this transformation is $\lambda(T)$.*

PROOF. We verify the statement by induction on q. If $q=1$, then T itself is solid and the assertion is trivial (by Proposition 6).

If $q \geqq 2$, then let us denote T by T_1 and consider the sequence (6.1). The second member T_2 exists and $\lambda(T_2)$ equals the weight of the transformation

$$T_2, T_3, \ldots, T_q$$

by the induction hypothesis.

Compare the numerator of (4.1) when it is formed for T_1 and for T_2. The number k and the values $\beta(i)$ do not change. Among the values $\tau(j)$, certain ones, however, will be modified. If $v(C) > v(D)$, then[4])

$$\tau(v(D)+1), \quad \tau(v(D)+2), \ldots, \tau(v(C))$$

are greater by two for T_2 than for T_1. If $v(C) < v(D)$, then

$$\tau(v(C)+1), \quad \tau(v(C)+2), \ldots, \tau(v(D))$$

are greater by two for T_1 than for T_2. (The other $\tau(j)$'s are unchanged.) We have in both cases

$$\lambda(T_1) - \lambda(T_2) = v(C) - v(D),$$

thus

$$\lambda(T_1) = \lambda(T_2) + (v(C) - v(D));$$

the right-hand side here is clearly the weight of the transformation (6.1). Q.E.D.

Corollary 1. *The laxity of an arbitrary tree is an integer.*

PROOF. The statement is an immediate consequence of Proposition 9 and of the fact that the weight of a transformation is an integer. Q.E.D.

§ 7. Trees with maximal laxity

Let an odd number k be fixed. We construct a rooted tree $(T^{(k)}, R)$ in the following manner:

the vertices of $T^{(k)}$ are $R, A_1, A_2, \ldots, A_{(k+1)/2}, B_2, B_3, \ldots, B_{(k+1)/2}$,

there exist the edges $RA_1, A_{i-1}A_i, A_{i-1}B_i$ where $2 \leqq i \leqq (k+1)/2$ (but no other edge exists).

$T^{(k)}$ has k edges and is trivalent (see Fig. 3). It is evident that $\mu(T^{(k)}) = (k+1)/2$.

[4]) We use the notations C and D in the same sense, as at the beginning of the section.

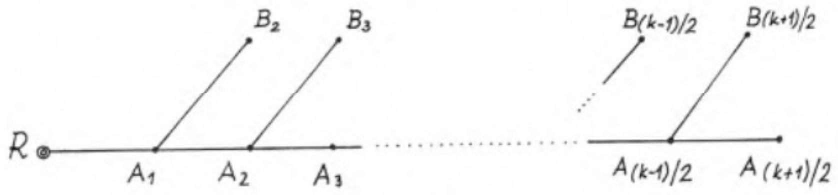

Fig. 3

Lemma 3. *We have*

$$\lambda(T^{(k+2)})-\lambda(T^{(k)})=\frac{k+3}{2}-\gamma(k+5)$$

for each odd $k(\geqq 3)$.

PROOF. Define the quantities $\Omega(j,k)$ and $\Omega(j,k+2)$ by

$$\Omega(j,k)=\min\left(\beta(j),k\right)-\tau^{(k)}(j)$$

and

$$\Omega(j,k+2)=\min\left(\beta(j),k+2\right)-\tau^{(k+2)}(j)$$

where (k is fixed and) j runs through the integers

$$1,2,\dots,(k+1)/2\left(=\mu(T^{(k+2)})-1\right)$$

and the superscripts in $\tau^{(k)}$ or $\tau^{(k+2)}$ denote that τ is meant in the tree $T^{(k)}$ of $T^{(k+2)}$, respectively. It is clear that

$$\tau^{(k)}(j)=2j-1=\tau^{(k+2)}(j)$$

for every considered value of j. Thus

$$\Omega(j,k+2)-\Omega(j,k)=\begin{cases}0 & \text{if}\quad \beta(j)\leqq k,\\ 2 & \text{if}\quad \beta(j)\geqq k+2.\end{cases}\tag{7.1}$$

($\beta(j)$ is either zero or an odd number, therefore it cannot equal $k+1$.) The falsity of the condition $\beta(j)\geqq k+2$ can be expressed in the form $3\cdot(2^j-1)<k+2$, this inequality being equivalent to $2^j<(k+5)/3$, i.e., to $j<\gamma(k+5)$. Hence there are

$$\frac{k+1}{2}-\left(\gamma(k+5)-1\right)=\frac{k+3}{2}-\gamma(k+5)\tag{7.2}$$

values of j which satisfy the inequalities

$$\beta(j)\geqq k+2,\quad j\leqq\frac{k+1}{2}.$$

Consequently,

$$\lambda(T^{(k+2)})-\lambda(T^{(k)})=\frac{\sum\limits_{j=1}^{(k+1)/2}\Omega(j,k+2)}{2}-\frac{\sum\limits_{j=1}^{(k-1)/2}\Omega(j,k)}{2}=$$

$$=\frac{1}{2}\sum_{j=1}^{(k+1)/2}\left(\Omega(j,k+2)-\Omega(j,k)\right)=\frac{1}{2}\left(2\left(\frac{k+3}{2}-\gamma(k+5)\right)\right).$$

(Indeed, the first equality follows from the definitions of λ and Ω, the second one is true because of

$$\Omega((k+1)/2,\, k) = k-k = 0,$$

the third one is a consequence of (7.1) and (7.2).) Q.E.D.

Proposition 10. *We have*

$$\lambda(T^{(k)}) = \frac{k^2-1}{8} - \eta\left(\frac{k+3}{2}\right) \tag{7.3}$$

for each odd $k(\geqq 3)$.

Proof. We use induction. $\lambda(T^{(3)})=1$ and

$$\frac{3^2-1}{8} - \eta\left(\frac{3+3}{2}\right) = 1-\eta(3) = 1$$

show that the assertion is valid for the smallest possible k.

Assume that the statement holds for k; the deduction

$$\lambda(T^{(k+2)}) = \lambda(T^{(k)}) + \frac{k+3}{2} - \gamma(k+5) = \frac{k^2-1}{8} - \eta\left(\frac{k+3}{2}\right) + \frac{k+3}{2} - \gamma(k+5) =$$

$$= \frac{k^2-1+4k+12}{8} - \left[\eta\left(\frac{k+3}{2}\right) + \gamma\left(\frac{k+5}{2}\right) + 1\right] = \frac{(k+2)^2+7}{8} - \left[\eta\left(\frac{k+5}{2}\right) + 1\right] =$$

$$= \frac{(k+2)^2-1}{8} - \eta\left(\frac{k+5}{2}\right)$$

shows its validity for $k+2$ (where the induction hypothesis and the three lemmas have been utilized). Q.E.D.

Proposition 11. *If T is an arbitrary trivalent rooted tree with k edges, then $\lambda(T) \leqq \lambda(T^{(k)})$. Equality holds in this formula exactly when T and $T^{(k)}$ are isomorphic as rooted trees.*

Proof. Consider the definition formula (4.1). Since the quantities $k, \beta(i)$ are common for T and $T^{(k)}$,

$$\lambda(T^{(k)}) - \lambda(T) = \frac{\sum_{j=1}^{\infty} \left(\tau_2(j) - \tau_1(j)\right)}{2}$$

is valid where τ_1 is meant for $T^{(k)}$ and τ_2 is meant for T.

The evident formulae

$$\tau_1(1) = 1 \leqq \tau_2(1),$$

$$\tau_1(j+1) = \min\left(k,\, \tau_1(j)+2\right)$$

and

$$\tau_2(j+1) \geqq \min\left(k,\, \tau_2(j)+2\right)$$

imply that the differences $\tau_2(j) - \tau_1(j)$ are non-negative, therefore $\lambda(T^{(k)}) \geqq \lambda(T)$.

Suppose that T and $T^{(k)}$ are not isomorphic.

Case 1. The degree of the root of T is 1. There exists a smallest $m(\geqq 1)$ such that the number of vertices of T whose height is m and whose degree is 3 exceeds one. It is easy to see that T contains exactly two vertices having degree 3 and height m. Hence

$$\sigma_2(m+1) = 4 = \sigma_1(m+1)+2$$

(where the subscripts serve again for the distinction of the two trees), thus

$$\tau_2(m+1) = \tau_1(m+1)+2 > \tau_1(m+1)$$

(since $\tau_1(m)=\tau_2(m)$). Consequently, $\lambda(T^{(k)})>\lambda(T)$.

Case 2. The degree of the root of T is 3. Then

$$\tau_2(1) = 3 > 1 = \tau_1(1).$$

It follows that $\lambda(T^{(k)})>\lambda(T)$ in this case, too. Q.E.D.

§ 8. An enumeration question

In this paper, I did not treat questions of enumerating the trivalent trees. The enumeration problems have an extensive literature; difficult but powerful techniques have been elaborated for them (see e.g. [1], [3]). It can be expected that most ot (or all) the problems in question can be solved by specializing some known methods. However, it would be of interest to discuss (if possible) this particular topic in an easier manner, than by applying the (somewhat complicated) general theory.

Now I expose the central enumeration question about the trees considered in this article. Let $k(\geqq 3)$ be an odd integer and λ be a non-negative integer such that λ does not exceed the right-hand side of the formula (7.3). Take a partition $m_0+m_1+m_2+\ldots+m_\mu$ of $(k+3)/2$ such that $m_0, m_1, m_2, \ldots, m_{\mu-1}$ are non-negative and m_μ is positive. *Determine the number* $\varphi(m_0, m_1, m_2, \ldots, m_\mu; \lambda)$ *of the rooted trivalent trees* T (up to isomorphism as rooted trees) *for which the following assertions hold:*

T has k edges,

T contains (exactly) m_i end vertices of height i $(0\leqq i\leqq\mu)$,

the laxity of T is λ.

Concerning the particular case $\lambda=0$, it follows from Propositions 7, 8 that $\varphi(m_1, m_2, \ldots, m_\mu; 0)$ takes a positive value if and only if

$$\mu = \gamma(k+3),$$

each of $m_0, m_1, m_2, \ldots, m_{\mu-2}$ is zero,

$m_{\mu-1}$ equals the quantity (5.3), and

m_μ equals the quantity (5.2).

Moreover, it is easy to see that φ takes the value 1 if k is of form either $3\cdot 2^i-5$ or $3\cdot 2^i-3$ or $3\cdot 2^i-1$ (and $\mu, m_0, \ldots, m_\mu$ are as above). φ can be greater than one for other (odd) values of k.

References

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