Lie structure in prime rings with derivations

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Abstract

Recently J. BERGEN, I. N. HERSTEIN and J. W. KERR [2, Theorem 4] proved that if R is a prime ring, char $R\neq 2$, and $U\subseteq Z$ (the centre of R) is a Lie ideal of R; further if δ and d are derivations of R such that $\delta d(U)=0$, then either $\delta=0$ or $d=0$. A remarkable theorem of HERSTEIN [4, Theorem 1] of which we have made several uses states: Let R be a semiprime, 2-torsion free ring and let U be a Lie ideal of R. If $t\in R$ commutes with all tu-ut, $u\in U$, then $[t, U]=0$. Further, AWTAR [1, Theorems 1 and 2] proved an interesting result which can be mentioned as follows Let R be a prime ring, char $R\neq 2$, and let U be a Lie ideal of R. Let d be a nonzero derivation f R: such that $ud(u)-d(u)u\in Z$ for all $u\in U$. Moreover, if char $R\neq 3$ or $u^2\in U$ for all $u\in U$ then $U\subset Z$. The object of this paper is to extend the above mentioned results, to a more general situation.

Throughout this paper by ring R we mean an associative ring. The symbol Z denotes the centre of R. We say that a ring R is semiprime if it contains no nonzero nilpotent ideals, and R is prime if the nonzero elements of Z are not zero divisors in R. For $x, y\in R$, we denote $[x, y]=xy-yx$. An additive subgroup U of R is said to be a Lie ideal of R if $[u, r]\in U$ for all $u\in U$, $r\in R$. For any subset A of R, we denote the centralizer of A by $C_R(A)$, and define as follows: $C_R(A)=\{r\in R\mid [a, r]=0$ for all $a\in A\}$. An additive mapping d from R into itself is said to be a derivation of R if $d(xy)=xd(y)+d(x)y$ for all $x, y\in R$.

We begin this paper with the following theorem which may have some independent interest.

Theorem 1. *Let R be a prime ring,* char $R\neq 2$, *and let $d\neq 0$ be a derivation of R. Suppose that $U\not\subset Z$ is a Lie ideal of R such that $d^2(U)\subset Z$, and if $a\in R$ is such that $d(a)=0$ and $[a, d(U)]\subset Z$. Then $a\in Z$.*

PROOF. By hypothesis, $d^2(u)\in Z$ for all $u\in U$. If $u\in U$, $r\in R$, then $d^2[u, r]\in Z$ and so $[d^2(u), r]+2[d(u), d(r)]+[u, d^2(r)]\in Z$. Thus, since $d^2(u)\in Z$ for $u\in U$, we get

$$(1)\qquad 2[d(u), d(r)]+[u, d^2(r)]\in Z \quad \text{for all} \quad u\in U, \quad r\in R.$$

Replace r by $rd^2(v)$ where $v\in U$ in (1) and expand; then

$$2[d(u), d(r)d^2(v)+rd^3(v)]+[u, d^2(r)d^2(v)+2d(r)d^3(v)+rd^4(v)]\in Z,$$

or,

$$2[d(u), d(r)]d^2(v)+2[d(u), r]d^3(v)+[u, d^2(r)]d^2(v)+2[u, d(r)]d^3(v)+[u, r]d^4(v)\in Z,$$

since $d^2(v), d^3(v)$ and $d^4(v)$ are in Z. From (1) $2[d(u), d(r)]+[u, d^2(r)]\in Z$ and by hypothesis $d^2(v)\in Z$, therefore $2[d(u), d(r)]d^2(v)+[u, d^2(r)]d^2(v)\in Z$. Hence from the last equation, we get that

$$(2) \qquad 2[d(u), r]d^3(v)+2[u, d(r)]d^3(v)+[u, r]d^4(v)\in Z \quad \text{for all} \quad u, v\in U \quad \text{and } r\in R.$$

In (2) replace r by $d(w)$ where $w\in U$ to get $2[d(u), d(w)]d^3(v)+[u, d(w)]d^4(v)\in Z$. Write $r=w$, $w\in U$, in (1) then $2[d(u), d(w)]\in Z$ and by hypothesis $d^3(v)\in Z$, therefore $2[d(u), d(w)]d^3(v)\in Z$. Thus we conclude that $[u, d(w)]d^4(v)\in Z$ for all u, v, $w\in U$. If $d^4(v)\neq 0$ for some $v\in U$, since $d^4(v)\in Z$ and R is prime, then $[u, d(w)]\in Z$ and so $[d(w), [d(w), u]]=0$ for all $u, w\in U$. By Theorem 1 of [4], $[d(w), u]=0$ for all $u, w\in U$; that is $d(U)\subset C_R(U)=Z$ by [2, Lemma 2] and so $U\subset Z$ by [2, Lemma 6]; a contradiction. Hence $d^4(U)=0$. Thus from (2) we get $2[d(u), r]d^3(v)+2[u, d(r)]d^3(v)\in Z$; that is $\{[d(u), r]+[u, d(r)]\}d^3(v)\in Z$ and so $d[u, r]d^3(v)\in Z$ for $u, v\in U$, $r\in R$. If $d^3(v)\neq 0$ for some $v\in U$, then $d[u, r]\in Z$ for $u\in U$, $r\in R$ since $d^3(v)\in Z$ and R is prime. Let $r=rd^2(w)$ where $w\in U$; we get, since $d^2(w)\in Z$, that $d([u, r]d^2(w))\in Z$ and so $d[u, r]d^2(w)+[u, r]d^3(w)\in Z$. Since $d[u, r]\in Z$ and $d^2(w)\in Z$, then $d[u, r]d^2(w)\in Z$. Thus $[u, r]d^3(w)\in Z$ for all $w\in U$, $r\in R$. As $d^3(v)(\neq 0)\in Z$ and R is prime, we force that $[u, r]\in Z$; that is $[u, [u, r]]=0$ for $u\in R$ $r\in R$ and so $U\subset Z$, by lemma 1.1.9 of [3], a contradiction. Hence $d^3(U)=0$.

In (1) write $r=ra$, since $d(a)=0$, we get on expansion

$$2[d(u), d(r)]a+2d(r)[d(u), a]+[u, d^2(r)]a+d^2(r)[u, a]\in Z,$$

or,

$$\{2[d(u), d(r)]+[u, d^2(r)]\}a+2d(r)[d(u), a]+d^2(r)[u, a]\in Z.$$

Commuting the last equation on x where $x\in R$, since by (1) $2[d(u), d(r)]+[u, d^2(r)]\in Z$ and by hypothesis $[a, d(u)]\in Z$, we get

$$\{2[d(u), d(r)]+[u, d^2(r)]\}[a, x]+2[d(r), x][d(u), a]+[d^2(r), x][u, a]$$

$$(3) \qquad +d^2(r)[[u,a],x]=0 \quad \text{for} \quad u\in U; \quad r, x\in R.$$

Replace r by ar in (3) and use (3), since $d(a)=0$, then

$$\{2[d(u), a]d(r)+[u, a]d^2(r)\}[a, x]+2[a, x]d(r)[d(u), a]+[a, x]d^2(r)[u, a] = 0$$

$$(4) \qquad \text{for } u\in U; \quad r, x\in R.$$

Replace r by $d(v)$ in (4) where $v\in U$; since $d^3(U)=0$, $d^2(U)\subset Z$ and $[a, d(U)]\subset Z$ we get $4d^2(v)[d(u), a][a, x]=0$ and so $d^2(v)[d(u), a][a, x]=0$ for $u, v\in U$, $x\in R$. Let $x=xy$ where $y\in R$; we get that $d^2(v)[d(u), a]R[a, x]=0$. Since R is prime, then either $a\in Z$ or $d^2(v)[d(u), a]=0$ for $u, v\in U$. If $d^2(U)=0$, since $d\neq 0$, then by Theorem 1 of [2] $U\subset Z$; a contradiction. So $d^2(U)\neq 0$ therefore the above relation gives us $[d(u), a]=0$ for all $u\in U$. By Theorem 2 of [2], since $U\not\subset Z$ and $d\neq 0$, $a\in Z$. Hence $a\in Z$. This proves the Theorem.

An immediate consequence of Theorem 1 is the following

Theorem 2. *Let R be a prime ring, char $R\neq 2$, and let U a Lie ideal of R. Suppose that $a\in R$ is such that $[a, [a, u]]\in Z$ for all $u\in U$. Then $[a, U]=0$. Moreover, if $U\not\subset Z$ then $a\in Z$.*

PROOF. For $x\in R$, let $d(x)=[a, x]$ be an inner derivation of R. Then $d(a)=0$. By hypothesis $[a, d(u)]\in Z$ and $d^2(u)\in Z$ for all $u\in U$. If for some $u\in U$, $[a, u]\neq 0$ then $d\neq 0$ and $U\not\subset Z$. Hence, by Theorem 1 $a\in Z$, a contradiction. Thus $[a, U]=(0)$. If $U\not\subset Z$, since $[a, U]=0$; that is $a\in C_R(U)$, then by lemma 2 of [2] $C_R(U)=Z$ and so $a\in Z$.

We are in position to prove the following theorem which extends some due to HERSTEIN [4, Theorem 1].

Theorem 3. *Let R be a semiprime, 2-torsion free ring and let U be a Lie ideal of R. Suppose that $a\in R$ is such that $[a,[a,u]]\in Z$ for all $u\in U$. Then $[a, U]=0$.*

PROOF. Since R is semiprime, then $\bigcap_{i\in I} P_i=(0)$ where I is an indexed set and P_i is a prime ideal of R. Thus $R_i=R/P_i$ is a prime ring of characteristic not equal to 2, since R is 2-torsion free; and $U_i=U/P_i$ is a Lie ideal of R_i. Since $[a,[a,u]]\in Z$ for all $u\in U$, then $[\bar{a},[\bar{a},u_i]]\in Z_i$ (the centre of R_i) for all $u_i\in U_i$ where $\bar{a}=a+P_i\in R_i$. By Theorem 2, $[\bar{a}, u_i]=0$ for all $u_i\in U_i$ and so $[a, U]\subset P_i$. Hence $[a, U]\subset\bigcap_{i\in I} P_i=(0)$. With this theorem 3 is proved.

We are now proving the key theorem of this paper which generalizes Theorem 2 of [2].

Theorem 4. *Let R be a prime ring,* char $R\neq 2$, *and let $U\not\subset Z$ be a Lie ideal of R. If $a\in R$ is such that $[a, d(u)]\in Z$ for all $u\in U$, where $d\neq 0$ is a derivation of R, then $a\in Z$.*

PROOF. Suppose on the contrary that $a\notin Z$. We claim that $d(a)\in Z$. This we prove in the same way as in [2]. Let $V=[U, U]\subset U$ be a Lie ideal of R. It is clear that, if $v\in V$ then $d(v)\in U$. By thypothesis $[a, d(v)]\in Z$ for all $v\in V$. Then, $d[a, d(v)]\in Z$ and so $[d(a), d(v)]+[a, d^2(v)]\in Z$. But $[a, d^2(v)]\in Z$ for all $v\in V$, since $d(v)\in U$; therefore $[d(a), d(v)]\in Z$ for all $v\in V$. Replace v by $[a, v]$, then $[d(a), d[a, v]]\in Z$ and so $[d(a),[d(a), v]+[a, d(v)]]\in Z$. Since $[a, d(v)]\in Z$, we conclude that $[d(a),[d(a), v]]\in Z$ for all $v\in V$. Since $U\not\subset Z$, $V\not\subset Z$ by lemma 1 of [4]. By Theorem 2, $d(a)\in Z$.

For $r\in R$, let $\alpha(r)=[a, r]$ be an inner derivation of R. Then by hypothesis $\alpha d(u)\in Z$ for $u\in U$. For $u\in U$, $r\in R$ it follows that $[u, r]\in U$ and so $\alpha d[u, r]\in Z$ which yields on expansion

$$[\alpha d(u), r]+[\alpha(u), d(r)]+[d(u), \alpha(r)]+[u, \alpha d(r)]\in Z,$$

or,

$$[\alpha(u), d(r)]+[d(u), \alpha(r)]+[u, \alpha d(r)]\in Z \quad \text{for all} \quad u\in U, \quad r\in R. \tag{1}$$

Replace r by xy where $x, y\in R$ in (1) to get

$$\begin{aligned}&[\alpha(u), d(x)y+xd(y)]+[d(u), \alpha(x)y+x\alpha(y)]+[u, \alpha d(x)y+\alpha(x)d(y)+\\ &\quad +d(x)\alpha(y)+x\alpha d(y)]\in Z \quad \text{for} \quad x, y\in R, \ u\in U.\end{aligned} \tag{2}$$

In (2) write $x=y=a$, since $\alpha(a)=0$ and $d(a)\in Z$ so $\alpha d(a)=0$, then $2[\alpha(u), ad(a)]\in Z$; that is $[\alpha(u), a]d(a)\in Z$. Thus $[a,[a, u]]d(a)\in Z$ for all $u\in U$.

If $d(a)\neq 0$, since R is prime and $d(a)\in Z$, we conclude that $[a,[a,u]]\in Z$ for all $u\in U$. By Theorem 2 $a\in Z$; a contradiction. Hence $d(a)=0$.

Replace y by a in (2), since $d(a)=\alpha(a)=0$, we get

$$[\alpha(u), d(x)a]+[d(u), \alpha(x)a]+[u, \alpha d(x)a]\in Z,$$

or,

$$\{[\alpha(u), d(x)]+[d(u), \alpha(x)]+[u, \alpha d(x)]\}a+d(x)[\alpha(u), a]+\alpha(x)[d(u), a]+\alpha d(x)[u,a]\in Z,$$

or,

$$\{[\alpha(u), d(x)]+[d(u), \alpha(x)]+[u, \alpha d(x)]\}a-d(x)\alpha^2(u)-\alpha(x)\alpha d(u)-\alpha d(x)\alpha(u)\in Z \tag{3}$$

$$\text{for all} \quad u\in U, \quad x\in R.$$

Commute (3) with a, as $\alpha d(u)\in Z$ and by (1) $[\alpha(u), d(x)]+[d(u), \alpha(x)]+[u, \alpha d(x)]\in Z$, to get

$$[d(x), a]\alpha^2(u)+d(x)[\alpha^2(u), a]+[\alpha(x), a]\alpha d(u)+[\alpha d(x), a]\alpha(u)+\alpha d(x)[\alpha(u), a]=0$$

or,

$$\alpha d(x)\alpha^2(u)+d(x)\alpha^3(u)+\alpha^2(x)\alpha d(u)+\alpha^2 d(x)\alpha(u)+\alpha d(x)\alpha^2(u)=0 \tag{4}$$

$$\text{for all} \quad u\in U, \; x\in R.$$

Replace u by $d(v)$ in (4) where $v\in V$, as $\alpha d(v)\in Z$ so $\alpha^2 d(v)=0$, then

$$\alpha^2(x)\alpha d^2(v)+\alpha^2 d(x)\alpha d(v)=0 \quad \text{for all} \quad x\in R, \; v\in V. \tag{5}$$

In (5) replace x by w where $w\in U$, then $\alpha^2(w)\alpha d^2(v)=0$ since $\alpha^2 d(w)=0$. As $d(v)\in U$, $\alpha d^2(v)\in Z$ for all $v\in V$. If $\alpha d^2(v)\neq 0$ for some $v\in V$, then $\alpha^2(U)=0$ since R is prime. Thus $[a,[a,u]]=0$ for all $u\in U$ and so by Theorem 2 we force that $a\in Z$; a contradiction. Hence $\alpha d^2(v)=0$ for all $v\in V$. Therefore, by (5), $\alpha^2 d(x)\alpha d(v)=0$ for all $x\in R$, $v\in V$. If $\alpha d(v)=[a, d(v)]=0$ for all $v\in V$, since $V\not\subset Z$ and $d\neq 0$, by Theorem 2 of [2] we conclude that $a\in Z$, a contradiction. Thus $\alpha d(v)\neq 0$ and so from the above we get $\alpha^2 d(x)=0$ for $x\in R$, since R is prime and $\alpha d(v)\in Z$. Replace x by $d(x)d(v)$ where $v\in V$, then

$$0=\alpha^2 d\big(d(x)d(v)\big)=\alpha^2\{d^2(x)d(v)+d(x)d^2(v)\}=$$

$$=\alpha^2 d^2(x)d(v)+2\alpha d^2(x)\alpha d(v)+d^2(x)\alpha^2 d(v)+\alpha^2 d(x)d^2(v)+2\alpha d(x)\alpha d^2(v)+d(x)\alpha^2 d^2(v).$$

As we have seen above that $\alpha^2 d(R)=0$ and $\alpha d^2(v)=0$, therefore from the last equation we get $2\alpha d^2(x)\alpha d(v)=0$ and so $\alpha d^2(x)\alpha d(v)=0$ for all $x\in R$, $v\in V$. We have seen above that $\alpha d(v)\neq 0$, and as it is in the center Z, so $\alpha d^2(R)=0$. Now, for $x\in R$, $\alpha d(x)=[a, d(x)]=d[a,x]=d\alpha(x)$. Thus $\alpha d=d\alpha$. Therefore, $\alpha d\alpha=d\alpha^2=$ $=\alpha^2 d=0$ and $d\alpha d=d^2\alpha=\alpha d^2=0$.

Since $\alpha^2 d=0$, therefore from (4) we get

$$2\alpha d(x)\alpha^2(u)+d(x)\alpha^3(u)+\alpha^2(x)\alpha d(u)=0 \quad \text{for} \quad u\in U, \quad x\in R. \tag{6}$$

Replace u by $\alpha(u)=[a,u]$ and x by v, $v\in U$ in (3), since $\alpha d\alpha=0$ and $\alpha d(U)=$ $=d\alpha(U)\subset Z$, we get

$$[\alpha^2(u), d(v)]a-d(v)\alpha^3(u)-\alpha d(v)\alpha^2(u)\in Z \quad \text{for all} \quad u, v\in U.$$

In (6) replace x by v, where $v\in U$ and then adding to the last equation to get

(7) $$[\alpha^2(u), d(v)]a+\alpha d(v)\alpha^2(u)+\alpha^2(v)\alpha d(u)\in Z \quad \text{for} \quad u, \quad v\in U.$$

Replace x by $\alpha(v)$ in (3) where $v\in U$, as $\alpha d\alpha=0$ and $d\alpha(v)=\alpha d(v)\in Z$,

(8) $$[d(u), \alpha^2(v)]a-d\alpha(v)\alpha^2(u)-\alpha^2(v)\alpha d(u)\in Z \quad \text{for} \quad u,v\in U.$$

Adding (7) and (8) to get

(9) $$\{[\alpha^2(u), d(v)]+[d(u), \alpha^2(v)]\}a\in Z \quad \text{for all} \quad u, v\in U.$$

Let $r=\alpha(v)$, $v\in U$ in (1); we get, since $\alpha d\alpha=0$ and $d\alpha(v)=\alpha d(v)\in Z$, that $[d(u), \alpha^2(v)]\in Z$. Replace r by v, $v\in U$ and u by $\alpha(u)=[a, u]$ in (1), then $[\alpha^2(u), d(v)]\in$ $\in Z$. After adding these, we have $\beta=[\alpha^2(u), d(v)]+[d(u), \alpha^2(v)]\in Z$. If $\beta\neq 0$, then in view of (9) we get $a\in Z$; a contradiction. Thus $\beta=0$; so

(10) $$[\alpha^2(u), d(v)]+[d(u), \alpha^2(v)]=0 \quad \text{for all} \quad u, v\in U.$$

Replace u by $\alpha(u)$ in (10), then we get $[\alpha^3(u), d(v)]=0$ for all u, $v\in U$. By Theorem 2 of [2], $\alpha^3(U)\subset Z$. Let $x=d(x)$ in (6); we get, since $\alpha d^2=\alpha^2 d=0$, that $d^2(x)\alpha^3(u)=0$ for $x\in R$, $u\in U$. If $\alpha^3(U)\neq 0$, as it is in the centre Z, $d^2=0$. However, as proof of Lemma 1.1.9 of [3] shows, if R is a semiprime, 2-torsion free ring and d is a derivation of R such that $d^2=0$ then $d=0$. Hence $\alpha^3(U)=0$.

Putting $u=d(w)$ in (10) where $w\in V$, since $\alpha^2 d=0$, then

(11) $$[d^2(w), \alpha^2(v)]=0 \quad \text{for} \quad v\in U, \quad w\in V.$$

Replace x by $\alpha^2(v)$, $v\in U$ and u by $d(w)$, $w\in V$ in (2), since $d\alpha^2=0$, $\alpha^3(U)=0$ and $\alpha d(w)\in Z$, we have

$$[d^2(w), \alpha^2(v)\alpha(y)]+[d(w), \alpha^2(v)\alpha d(y)]\in Z,$$

or,

$$[d^2(w), \alpha^2(v)]\alpha(y)+\alpha^2(v)[d^2(w), \alpha(y)]+[d(w), \alpha^2(v)]\alpha d(y)+$$
$$+\alpha^2(v)[d(w), \alpha d(y)]\in Z \quad \text{for} \quad v\in U, w\in V, y\in R.$$

In view of (11) the last equation reduces to

(12) $$\gamma=\alpha^2(v)[d^2(w), \alpha(y)]+[d(w), \alpha^2(v)]\alpha d(y)+\alpha^2(v)[d(w), \alpha d(y)]\in Z$$

for all $v\in U, w\in V$ and $y\in R$.
Let $y=ya$ in (12); we get, since $d(a)=\alpha(a)=0$ and $\alpha d^2=0$,

$$\gamma a+\alpha^2(v)\alpha d(y)[d(w), a]\in Z,$$

or

$$\gamma a-\alpha^2(v)\alpha d(y)\alpha d(w)\in Z.$$

Commuting the last equation with $d(u)$, $u\in U$, as $\gamma\in Z$, to get $\gamma[a, d(u)]=$ $[\alpha^2(v)\alpha d(y)\alpha d(w), d(u)]=[\alpha^2(v)\alpha d(y), d(u)]\alpha d(w)$. Since γ and $[a, d(u)]$ are in Z, then $[\alpha^2(v)\alpha d(y), d(u)]\alpha d(w)\in Z$ for all $u, v\in U$, $w\in V$ and $y\in R$. As we have seen above that $\alpha d(V)\neq 0$, and as it is in the center Z, we conclude that $[\alpha^2(v)\alpha d(y), d(u)]\in$ $\in Z$ for $u, v\in U$ and $y\in R$. In particular, $[d(w), \alpha^2(v)]\,\alpha d(y)+\alpha^2(v)[d(w), \alpha d(y)]\in Z$ for $v\in U$, $w\in V$ and $y\in R$. Thus, in view of (12) and the last equation, we conclude that $\alpha^2(v)[d^2(w), \alpha(y)]\in Z$ for $v\in U$, $w\in V$ and $y\in R$. In particular, $\alpha^2(v)[d^2(w), \alpha(u)]\in Z$ for $u, v\in U$ and $w\in V$. In (1) replace u by $d(w)$, $w\in V$ and r by u, then $[d^2(w), \alpha(u)]\in Z$ for $w\in V$, $u\in U$. If $[d^2(w), \alpha(u)]\neq 0$, as it is in the center Z, $\alpha^2(v)\in Z$ for all $v\in U$ since R is prime. That is $[a, [a, v]]\in Z$ for all $v\in U$. By Theorem 2 $a\in Z$; a contradiction. Hence $[d^2(w), \alpha(u)]=0$ for all $u\in U$, $w\in V$. Since $a\notin Z$, then $\alpha\neq 0$. By Theorem 2 of [2], $d^2(V)\subset Z$. Thus we have $a\in R$ is such that $d(a)=0$, $[a, d(V)]\subset Z$ and $d^2(V)\subset Z$ where $V\not\subset Z$ is a Lie ideal of R and $d\neq 0$ is a derivation of R. So, by Theorem 1, $a\in Z$. This proves the Theorem.

An immediate consequence of Theorem 4 is the following theorem which extends Theorem 1 of [2].

Theorem 5. *Let R be a prime ring,* char $R\neq 2$, *and let U be a Lie ideal of R. If $d\neq 0$ is a derivation of R such that $d^2(U)\subset Z$ then $U\subset Z$.*

PROOF. Suppose on the contrary that $U\not\subset Z$. By hypothesis, $d^2(u)\in Z$ for all $u\in U$. If $u, v\in U$ then $d^2(u)$, $d^2(v)\in Z$ and $d^2[u, v]\in Z$; that is, $[d^2(u), v)]+ +2[d(u), d(v)]+[u, d^2(v)]\in Z$ and so $2[d(u), d(v)]\in Z$, in consequence of which we get $[d(u), d(v)]\in Z$ for all u, $v\in U$. By Theorem 4 $d(U)\subset Z$ and so $U\subset Z$ by lemma 6 of [2], a contradiction. Hence $U\subset Z$.

Now we are in position to prove a result which generalizes simultaneously those of Theorems 2, 4 and 1 of [2].

Theorem 6. *Let R be a prime ring,* char $R\neq 2$, *and let $U\not\subset Z$ be a Lie ideal of R. Suppose that δ and d are derivations of R such that $\delta d(U)\subset Z$. Then either $\delta=0$ or $d=0$.*

PROOF. Suppose that $d\neq 0$ and $\delta\neq 0$. By hypothesis, $\delta d(u)\in Z$ for all $u\in U$. If $u\in U$, $r\in R$ then $\delta d[u, r]\in Z$ and $\delta d(u)\in Z$. Therefore, $[\delta d(u), r]+[d(u), \delta(r)]+ +[\delta(u), d(r)]+[u, \delta d(r)]\in Z$; that is $[d(u), \delta(r)]+[\delta(u), d(r)]+[u, \delta d(r)]\in Z$ for all $u\in U$, $r\in R$. Replace r by $d(v)$ where $v\in V$, as $d(v)\in U$ and $\delta d(U)\subset Z$, we get $[\delta(u), d^2(v)]\in Z$ for all $u\in U$, $v\in V$. By theorem 4, $d^2(V)\subset Z$ since $\delta\neq 0$. Since V is a Lie ideal of R and $d\neq 0$, then by Theorem 5 $V\subset Z$ and so $U\subset Z$, a contradiction. Hence either $d=0$ or $\delta=0$. This completes the proof of Theorem 6.

We are closing this paper by proving the following theorem which extends some due to Awtar [1, Theorems 1 and 2].

THEOREM 7. *Let R be a prime ring of characteristic different from 2. Let d be a nonzero derivation of R, and U a Lie ideal of R with $[u, d(u)]\in Z$ for all $u\in U$. Then $U\subset Z$.*

PROOF. By Lemma 2 of [1], $[[d(r), u], u]\in Z$ for all $u\in U$, $r\in R$. Its linearization on $u=u+d(v)$ where $v\in V$ yields on expansion $[[d(r), d(v)], u]+ +[[d(r), u], d(v)]\in Z$ for all $u\in U$, $v\in V$ and $r\in R$. In particular, $[d(v), [d(v), u]]\in Z$ for all $u\in U$, $v\in V$. If $U\not\subset Z$, then by Theorem 2 $d(V)\subset Z$ and so $V\subset Z$ by Lemma 6 of [2]. Thus by lemma 1 of [4] we conclude that $U\subset Z$, a contradiction. Hence $U\subset Z$.

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