On a combinatorial identity

B. GYIRES (Debrecen)

In this paper there are proved two theorems. The first one is a combinatorial identity, which is suitable to calculate the Bernoulli numbers by a recursive formula too. It is known that there are more such recursive formulas to the calculation of the Bernoulli numbers. After the examination of the attainable special literature, it seems that this identity is new. — In the second theorem we give an asymtotic formula for the Bernoulli numbers by the first theorem.

Theorem 1. *The difference equation*

$$\Phi[2(\nu+1)]+\sum_{k=1}^{\nu}\binom{2\nu}{2k-1}\Phi(2k)\Phi[2(\nu-k+1)]=0 \quad (\nu=1,2,\dots) \tag{1}$$

with initial condition $\Phi(2)=\frac{1}{4}$ *has the only solution*

$$\Phi(2k)=\frac{2^{2k}-1}{2k}B_{2k} \quad (k=1,2,\dots), \tag{2}$$

where $\{B_{2k}\}_{k=1}$ *is the sequence of the Bernoulli numbers with even indices.*

PROOF. Suppose that the function $\varphi(t)$ of the real variable t is differentiable twice in a neighbourhood of the origin satisfying conditions

$$\varphi(0)=0, \quad \varphi'(0)=0, \quad \varphi''(0)=-1.$$

It is not difficult to show that the only solution of the differential equation

$$\varphi''(t)+[\varphi'(t)]^2+1=0$$

is given by

$$\varphi(t)=\lg\cos t, \quad t\in R_1, \quad t\neq(2k+1)\frac{\pi}{2} \quad (k=0,\pm1,\pm2,\dots).$$

Therefore the power series

$$\varphi(z)=\sum_{\nu=2}^{\infty}\frac{\varphi_{(0)}^{(\nu)}}{\nu!}z^{\nu} \tag{3}$$

is convergent inside the circle with radius $\frac{\pi}{2}$. We get from the formula (3)

$$\varphi''(t)=-[\varphi'(t)]^2-1,$$

thus by the Leibnitz rule we have

$$\varphi_{(0)}^{(\nu+2)} = -\sum_{k=0}^{\nu} \binom{\nu}{k} \varphi_{(0)}^{(k+1)} \varphi_{(0)}^{(\nu-k+1)} \quad (\nu = 0, 1, \ldots). \tag{4}$$

Since $\varphi'(0)=0$, we obtain $\varphi_{(0)}^{(3)}=0$ by (4). If ν is even and $k+1$ is oven, then $\nu-k+1$ is odd, and conversely. Therefore it is very easily to prove by induction that

$$\varphi_{(0)}^{(2\nu+1)} = 0 \quad (\nu = 0, 1, 2, \ldots)$$

Thus

$$\varphi_{(0)}^{(2\nu+2)} = -\sum_{k=1}^{\nu} \binom{2\nu}{2k-1} \varphi_{(0)}^{(2k)} \varphi_{(0)}^{(2(\nu-k+1))} \quad (\nu = 0, 1, 2, \ldots), \tag{5}$$

by (4). Accordingly

$$\varphi(z) = \sum_{\nu=0}^{\infty} \frac{\varphi_{(0)}^{(2\nu+2)}}{(2\nu+2)!} z^{2\nu+2} = \lg \cos z \tag{6}$$

on the circle with radius $\frac{\pi}{2}$, and

$$\varphi'(z) = \sum_{\nu=0}^{\infty} \frac{(2\nu+2)\varphi_{(0)}^{(2\nu+2)}}{(2\nu+2)!} z^{2\nu+1} = -\operatorname{tg} z \tag{7}$$

on the same domain, respectively. But ([1], p. 259)

$$\operatorname{tg} z = \sum_{\nu=0}^{\infty} (-1)^{\nu} 2^{2(\nu+1)} (2^{2(\nu+1)} - 1) \frac{B_{2(\nu+1)}}{(2\nu+2)!} z^{2\nu-1} \tag{8}$$

on the circle with radius $\frac{\pi}{2}$. Thus

$$-(2\nu+2)\varphi_{(0)}^{(2\nu+2)} = (2\nu+2) \sum_{k=1}^{\nu} \binom{2\nu}{2k-1} \varphi_{(0)}^{(2k)} \varphi_{(0)}^{(2(\nu-k+1))} = \tag{9}$$
$$= (-1)^{\nu} 2^{2(\nu+1)} (2^{2(\nu+1)} - 1) B_{2(\nu+1)}$$

fram the formulas (5), (7) and (8), consequently

$$\varphi_{(0)}^{(2\nu+2)} = (-1)^{\nu+1} 2^{2(\nu+1)} \Phi(2\nu+2) \quad (\nu = 0, 1, 2, \ldots), \tag{10}$$

where Φ is the function defined by (2). Substituting (10) in the expression (9), we get that functions (2) satisfy the difference equation (1) indeed. Solution (2) is the only one, if the initial condition $\Phi(2)=\frac{1}{4}$ is given.

Theorem 2. *The Bernoulli numbers with even indices satisfy the following asymptotic formula:*

$$\lim_{\nu\to\infty} \frac{1}{\nu} |B_{2\nu}|^{\frac{1}{2\nu}} = \frac{1}{\pi e}.$$

PROOF. Since the convergence radius of the power series (6) is equal to $\frac{\pi}{2}$, we have

$$\limsup_{\nu\to\infty}\left(\frac{2^{4(\nu+1)}\left[1-\frac{1}{2^{2(\nu+1)}}\right]|B_{2\nu+2}|}{(2\nu+2)(2\nu+2)!}\right)^{\frac{1}{2\nu+2}}=\frac{2}{\pi}$$

using (10). By the application of the Stirling formula

$$2^2\limsup_{\nu\to\infty}\left(\frac{|B_{2\nu+2}|}{\left[\frac{2(\nu+1)}{e}\right]^{2\nu+2}[2\pi(2\nu+2)]^{\frac{1}{2}}}\right)^{\frac{1}{2\nu+2}}=\frac{2}{\pi},$$

and from here the relation

$$\limsup_{\nu\to\infty}\frac{1}{\nu+1}|B_{2\nu+2}|^{\frac{1}{2\nu+2}}=\frac{1}{\pi e}$$

holds.

Still remains to be showed that sequence

$$\left\{\frac{1}{\nu}|B_{2\nu}|^{\frac{1}{2\nu}}\right\}_{\nu=1}^{\infty} \tag{11}$$

is convergent. Since the first member of the expression (1) has the sign $(-1)^\nu$, and the other ones have the sign $(-1)^{\nu-1}$, the relation

$$|\Phi(2\nu+2)|=\sum_{k=1}^{\nu}\binom{2\nu}{2k-1}|\Phi(2k)|\,\big|\Phi\big(2(\nu-k+1)\big)\big|.$$

holds, which is a recursive formula to calculate successantly the Bernoulli numbers with even indices. From this formula we get the inequality

$$\frac{\nu+1}{\nu}\left(\frac{2}{\nu+1}\frac{2^{2(\nu+1)}-1}{2^{2\nu}-1}\right)^{\frac{1}{2\nu}}\frac{\left(|B_{2\nu+2}|^{\frac{1}{2\nu+2}}\right)^{\frac{2\nu+1}{\nu}}}{\nu+1}>\frac{|B_{2\nu}|^{\frac{1}{2\nu}}}{\nu}> \tag{12}$$

$$>\frac{\nu-1}{\nu}\left(\frac{\nu}{2}\frac{2^{2(\nu-1)}-1}{2^{\nu}-1}\right)^{\frac{1}{2\nu}}\frac{\left(|B_{2(\nu-1)}|^{\frac{1}{2(\nu-1)}}\right)^{\frac{\nu-1}{\nu}}}{\nu-1}\quad(\nu=2,3,\ldots).$$

Since the limit inferior of the sequence with elements of the left side of (12), and the limit superior of the sequence with elements of the right side of (12) is equal to the limit inferior, and to the limit superior of sequence (11), respectively, we obtain that sequence (11) is convergent. Thus the proof of Theorem 2 is finished.

We can get an other proof of Theorem 2 starting from the konwn inequality ([1], p. 245)

$$\frac{2(2n)!}{(2\pi)^{2n}}<|B_{2n}|\leqq\frac{(2n)!}{12(2\pi)^{2n-2}}.$$

References

[1] Jordan, Ch. Calculus of finite differences. *Budapest.* 1939.

Kossuth L. University of Debrecen (Hungary)

(Received July 23, 1982.)