*Publ. Math. Debrecen*
**50 / 3-4** (1997), 305–316

# The topological structure of the set of $P$-sums of a sequence

By J. E. NYMANN (El Paso) and RICARDO A. SÁENZ (El Paso)

**Abstract.** Let $P = \{p_0, p_1, \dots, p_N\}$ where $p_{i-1} < p_i$ for $i = 1, 2, \dots, N$. Let $\lambda = \langle \lambda_n \rangle$ be a sequence of real numbers for which $\sum |\lambda_n|$ converges and $|\lambda_n| \geq |\lambda_{n+1}| > 0$ for all $n$. Let

$$S(P, \lambda) = \left\{ \sum_{n=1}^{\infty} \varepsilon_n \lambda_n : \varepsilon_n \in P \right\}.$$

$S(P, \lambda)$ is called the set of $P$-sums for the sequence $\lambda$. The topological structure of these sets is studied in this paper.

## 1. Introduction

Let $P = \{p_0, p_1, \dots, p_N\}$ where $p_{i-1} < p_i$ for $i = 1, 2, \dots, N$. Let $\lambda = \langle \lambda_n \rangle$ be a sequence of real numbers for which $\sum |\lambda_n|$ converges and $|\lambda_n| \geq |\lambda_{n+1}| > 0$ for all $n$. Let

$$S(P, \lambda) = \left\{ \sum_{n=1}^{\infty} \varepsilon_n \lambda_n : \varepsilon_n \in P \right\}.$$

We will call $S(P, \lambda)$ the set of $P$-sums for the sequence $\lambda$. In the case $P = \{0, 1\}$ we will write $S(P, \lambda) = S(\lambda)$ and call this set the set of subsums of $\lambda$. In [2] Boros studied conditions under which $S(P, \lambda)$ is an interval. (He also required a strict inequality between $|\lambda_n|$ and $|\lambda_{n+1}|$.) In this paper we will study the topological structure of the set $S(P, \lambda)$.

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*Mathematics Subject Classification*: Primary, 11B05; Secondary, 11K31, 40A05.
*Key words and phrases*: Sequence, subsum, Cantor set.

In [3] the following theorem was proved.

**Theorem 1.1.** *If $\lambda$ is a positive term sequence, then $S(\lambda)$ is one of the following:*

*(i)* *a finite union of intervals;*

*(ii)* *homeomorphic to the Cantor set $C$;*

*(iii)* *homeomorphic to $S(\beta)$ where $\beta = \langle \beta_n \rangle$ with $\beta_{2n-1} = 3/4^n$ and $\beta_{2n} = 2/4^n$ $(n = 1, 2, \dots)$.*

*Remark 1.1.* In order to visualize the set $S(\beta)$ above we give the following example of a set which is homeomorphic to $S(\beta)$ (see [3]). Let $S_n$ denote the union of the $2^{n-1}$ open middle thirds which are removed from $[0,1]$ at the $n$th step in the geometric construction of $C$. Then

$$C = [0,1] \setminus \bigcup_{n=1}^{\infty} S_n,$$

and $S(\beta)$ is homeomorphic to $C \cup \bigcup_{n=1}^{\infty} S_{2n-1}$.

In Section 2 we will see that the hypothesis in Theorem 1.1 that $\lambda$ have only positive terms can be deleted. In Section 3 of this paper we will give an example of a set $S(P,\lambda)$ which is not one of the above three types. In Section 4 will give some conditions on $P$ under which $S(P,\lambda) = S(\alpha)$ for an appropiate sequence $\alpha$. In those cases the known results on sets of subsums can be used to determine the topological structure of $S(P,\lambda)$.

At this point we introduce some notation and terminology which will be used later in the paper. As usual, if $E$ is any set, $aE + b = \{ax + b : x \in E\}$. We will say $E$ is symmetric (about $a$) if the condition

$$a + x \in E \iff a - x \in E$$

is satisfied. (It is easy to see that this is equivalent to $E = 2a - E$.)

With $\lambda$ and $P$ as given above, $|\lambda|$ will denote the sequence $\langle |\lambda_n| \rangle$, $s$ will denote $\sum \lambda_n$ and $\bar{s}$ will denote $\sum |\lambda_n|$. Also set

$$r_n = \sum_{k=n+1}^{\infty} \lambda_k$$

and

$$\bar{r}_n = \sum_{k=n+1}^{\infty} |\lambda_k|.$$

For $k$ a positive integer, $S_k(P,\lambda)$ will denote the set

$$\left\{ \sum_{n=k+1}^{\infty} \varepsilon_n \lambda_n : \varepsilon_n \in P \right\}$$

and will be called the set of $P$-sums of the $k$-tail of $\lambda$ and $F_k(P,\lambda)$ will denote

$$\left\{ \sum_{n=1}^{k} \varepsilon_n \lambda_n : \varepsilon_n \in P \right\}$$

and will be called the set of $k$-finite $P$-sums of $\lambda$.

*Remark 1.2.* Using the above notation, the following decomposition for $S(P,\lambda)$ is easy to see.

$$S(P,\lambda) = \bigcup_{f \in F_k(P,\lambda)} (f + S_k(P,\lambda)).$$

The following three facts were discovered in 1914 by Kakeya [5], and rediscovered by Hornich [4] in 1941 (see also the papers by Barone [1], Koshi and Lai [6] and Menon [8] ):

A. $S(\lambda)$ is a perfect set.

B. $S(\lambda)$ is a finite union of intervals if and only if $|\lambda_n| \le \bar{r}_n$ for $n$ sufficiently large. (Also, $S(\lambda)$ is an interval if and only if $|\lambda_n| \le \bar{r}_n$ for all $n$.)

C. If $|\lambda_n| > \bar{r}_n$ for $n$ sufficiently large, then $S(\lambda)$ is homeomorphic to the Cantor set.

(Some of the papers referenced above only consider the case where $\lambda$ is a positive term sequence, but from Proposition 2.2 in the next section it is easy to see that this condition can be removed.)

In Section 2 we will prove that $S(P,\lambda)$ is a perfect set and, for $P$ symmetric, we will prove results analogous to B and C above.

## 2. Basic results on $S(P,\lambda)$.

The following results are known or easy to prove, so the proofs will be omitted.

**Proposition 2.1.** *$S(aP+b,\lambda) = aS(P,\lambda) + bs$ for any real numbers $a$ and $b$.*

The above proposition tells us that if $P$ is shifted horizontally and/or expanded or contracted, then essentially the same thing happens to $S(P,\lambda)$.

**Proposition 2.2.** *If $P$ is symmetric about $a$, then $S(P,\lambda)=S(P,|\lambda|)+(s-\bar{s})a$.*

*Remark 2.1.* $P=\{0,1\}$ is clearly symmetric about $1/2$. Hence $S(|\lambda|)$ is just a horizontal shift of $S(\lambda)$. Therefore the hypothesis of "positive terms" in Theorem 1.1 can be deleted.

**Theorem 2.3.** *$S(P,\lambda)$ is a perfect set.*

Proof. First we show $S(P,\lambda)$ has no isolated points. Let $x\in S(P,\lambda)$. It is sufficient to show that for any $\delta>0$ there is a $y\in S(P,\lambda)$ such that $0<|y-x|<\delta$. Now $x=\sum\varepsilon_n\lambda_n$ where $\varepsilon_n\in P$ for all $n$. For a given $\delta>0$, choose $K$ so large that $|\lambda_K|<\delta/(p_N-p_0)$. Now let $y=\sum\varepsilon_n'\lambda_n$ where $\varepsilon_n'=\varepsilon_n$ if $n\neq K$ and $\varepsilon_K'$ is any element of $P$ except $\varepsilon_K$. Then, since $|y-x|=|\varepsilon_K'-\varepsilon_K||\lambda_K|$, $0<|y-x|<\delta$.

Secondly we show that $S(P,\lambda)$ is closed. Set $b=N+2$ and let $E=\left\{\sum_{n=1}^{\infty}\varepsilon_n/b^n:\varepsilon_n=0,1,2,\ldots,b-2=N\right\}$. Note that $E$ is the set of numbers in $[0,1]$ which do not have a $b-1$ in their base $b$ expansion. Hence the representation as given in the definition of $E$ is unique. Also $E$ is a closed set since $E$ can be obtained from $[0,1]$ by removing countably many open intervals. Now consider the mapping $f$ from $E$ onto $S(P,\lambda)$ given by

$$f\Big(\sum_{n=1}^{\infty}\varepsilon_n/b^n\Big)=\sum_{n=1}^{\infty}p_{\varepsilon_n}\lambda_n.$$

Since $E$ is compact, it is sufficient to show that $f$ is continuous. Let $x=\sum\varepsilon_n/b^n\in E$ and let $\varepsilon>0$ be given. Choose $K$ such that

$$\sum_{n=K}^{\infty}(p_N-p_0)|\lambda_n|<\varepsilon$$

and choose $\delta$ so that $0<\delta<\sum_{n=K}^{\infty}N/b^n$. Then if $y\in E$, $|y-x|<\delta$ and $y=\sum_{n=1}^{\infty}\varepsilon_n'/b^n$, then $\varepsilon_n=\varepsilon_n'$ for $n<K$. Hence

$$|f(y)-f(x)|=\big|\sum p_{\varepsilon_n'}\lambda_n-\sum p_{\varepsilon_n}\lambda_n\big|\leq\sum_{n=K}^{\infty}|p_N-p_0||\lambda_n|<\varepsilon.$$

Hence $f$ is continuous at $x\in E$ and since $x$ was an arbitrary element of $E$, $f$ is continuous on $E$.

The following theorem generalizes statements B and C in the cases where $\lambda$ has positive terms or $P$ is symmetric.

**Theorem 2.4.** *Assume $\lambda$ is a positive term sequence.*

*B'. $S(P,\lambda)$ is a finite union of intervals if*

$$\lambda_n \leq \frac{(p_N - p_0)}{\max(p_i - p_{i-1})} r_n \tag{1}$$

*for $n$ sufficiently large. (Also $S(P,\lambda)$ is an interval if (1) holds for all $n$.) Conversely, if*

$$\frac{p_1 - p_0}{p_N - p_0} \geq \frac{\lambda_n}{\lambda_{n-1}} \tag{2}$$

*for $n$ sufficiently large and $S(P,\lambda)$ is a finite union of intervals, then (1) holds for $n$ sufficiently large. (Also if (2) holds for all $n$ and $S(P,\lambda)$ is an interval, then (1) holds for all $n$.)*

*C'. $S(P,\lambda)$ is homeomorphic to the Cantor set $C$ if*

$$\lambda_n > \frac{p_N - p_0}{\min(p_i - p_{i-1})} r_n \tag{3}$$

*for $n$ sufficiently large.*

*Furthermore, if $P$ is symmetric the requirement that $\lambda$ have only positive terms can be deleted if $\lambda_n$ and $r_n$ are replaced by $|\lambda_n|$ and $\bar{r}_n$ in the above inequalities.*

Proof. By replacing $P$ by $(1/(p_N - p_0))(P - p_0)$ we may assume $p_0 = 0$ and $p_N = 1$. Then, of course, (1), (2) and (3) become

$$\max(p_i - p_{i-1})\lambda_n \leq r_n \tag{1'}$$

$$p_1 \lambda_{n-1} \geq \lambda_n \tag{2'}$$

$$\min(p_i - p_{i-1})\lambda_n > r_n. \tag{3'}$$

By Remark 1.2, to prove the first part of B', it is sufficient to prove that $S(P,\lambda) = [0,s]$ if (1') holds for all $n$. Let $x \in (0,s]$. We will define a sequence $\langle \varepsilon_n \rangle$, $\varepsilon_n \in P$, inductively as follows: $\varepsilon_1 = \max\{p_i : p_i\lambda_1 < x\}$. Once $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n-1}$ have been selected, let

$$\varepsilon_n = \max\Big\{p_i : \Big(\sum_{k=1}^{n-1} \varepsilon_k \lambda_k\Big) + p_i \lambda_n < x\Big\}.$$

Set $x_n = \sum_{k=1}^{n} \varepsilon_k \lambda_k$. Then $\langle x_n \rangle$ is an increasing sequence which is bounded above by $x$. Set $x_0 = \lim x_n$. Then $x_0 \leq x$. If $x_0 = x$, we are done. We will show that if $d = x - x_0 > 0$, then (1') does not hold for all $n$ which will complete the proof of $S(P,\lambda)=[0,s]$. Now if $\varepsilon_n \neq 1 = p_N$, then $x_{n-1}+p_N\lambda_n \geq x$ so that $p_N\lambda_n \geq x - x_{n-1} \geq x - x_0 = d$. Hence $\varepsilon_n \neq 1$ for only finitely many $n$. Since $\sum \varepsilon_n\lambda_n = x_0 < x \leq \sum \lambda_n$, there are $n$ for which $\varepsilon_n \neq 1$. Let $t = \max\{n : \varepsilon_n \neq 1\}$. Suppose $\varepsilon_t = p_j$. Then $x_{t-1} + p_j\lambda_t + r_t = x_0 < x \leq x_{t-1} + p_{j+1}\lambda_t$. Hence $r_t < (p_{j+1}-p_j)\lambda_t \leq \max(p_i - p_{i-1})\lambda_t$ and we see that (1') does not hold for $n = t$.

Now suppose (2') holds for $n \geq N_0$ and that $S(P,\lambda)$ is a finite union of intervals. Assume, with the goal of obtaining a contradiction, that $\max(p_i - p_{i-1})\lambda_n > r_n$ for infinitely many $n$. Then there is a sequence $\langle n_k \rangle$ such that $n_k \geq N_0$ and $\max(p_i - p_{i-1})\lambda_{n_k} > r_{n_k}$. Let $p_v - p_{v-1} = \max(p_i - p_{i-1})$. We will show that

$$(p_{v-1}\lambda_{n_k} + r_{n_k}, p_v\lambda_{n_k}) \cap S(P,\lambda) = \emptyset$$

and hence the complement of $S(P,\lambda)$ has infinitely many components which completes the contradiction. Now the inequality (2'), i.e. $p_1\lambda_{n_k-1} \geq \lambda_{n_k}$ assures us that $0 < p_1\lambda_{n_k} < \ldots < p_{v-1}\lambda_{n_k} < p_v\lambda_{n_k} < \ldots < \lambda_{n_k}$ are the $N+1$ smallest elements of $F_{n_k}(P,\lambda)$. Also $p_{v-1}\lambda_{n_k} + r_{n_k}$ is the largest element of $p_{v-1}\lambda_{n_k} + S_{n_k}(P,\lambda)$ and $p_v\lambda_{n_k}$ is the smallest element of $p_v\lambda_{n_k} + S_{n_k}(P,\lambda)$. Hence, by Remark 1.2, $S(P,\lambda)$ can have no elements between $p_{v-1}\lambda_{n_k} + r_{n_k}$ and $p_v\lambda_{n_k}$.

Now by Remark 1.2, in order to prove C', it is sufficient to prove $S(P,\lambda)$ is homeomorphic to the Cantor set $C$ if (3') holds for all $n$. So let us assume $\max(p_i - p_{i-1})\lambda_n > \lambda_n$ for all $n$. First we show that the representation of every element in $S(P,\lambda)$ is unique. Suppose $\sum \varepsilon_n\lambda_n = \sum \varepsilon'_n\lambda_n$ where $\varepsilon_n, \varepsilon'_n \in P = \{0 = p_0 < p_1 < \ldots < p_N = 1\}$ and let $t = \min\{n : \varepsilon_n \neq \varepsilon'_n\}$. We can, without loss of generality, assume that $\varepsilon_t > \varepsilon'_t$. Then

$$(\varepsilon_t - \varepsilon'_t)\lambda_t + \sum_{n=t+1}^{\infty} \varepsilon_n\lambda_n = \sum_{n=t+1}^{\infty} \varepsilon'_n\lambda_n.$$

Hence $r_t \geq (\varepsilon_t - \varepsilon'_t)\lambda_t \geq \min(p_i - p_{i-1})\lambda_t$. This contradiction shows that the set $\{n : \varepsilon_n \neq \varepsilon'_n\}$ must be empty.

Now to complete the proof of statement C', consider the continuous one-to-one mapping $f$ of $E$ onto $S(P,\lambda)$ defined in the proof of Theorem 2.3. Since the representations of the elements in $S(P,\lambda)$ are unique, $f^{-1}$ can be shown to be continuous in the same way as $f$ was shown to be continuous in the proof of Theorem 2.3. Hence $S(P,\lambda)$ and $E$ are homeomorphic and since $E$ is homeomorphic to $C$, the proof of statement C' is complete.

The final statement of the theorem follows easily from Proposition 2.2.

## 3. Examples

Originally we had hoped to prove that $S(P,\lambda)$ was always a set of one of the three types given in Theorem 1.1. The examples given in this section show that this is not the case.

In this section we set $\lambda_n = 1/3^n$ and $P = \{0,1,2,9\}$. Clearly $[0,1] \subset S(P,\lambda) \subset [0,9/2]$. We will show the following three facts about $S(P,\lambda)$:

I. $[0,9/8] \subset S(P,\lambda)$.

II. For every $\varepsilon > 0$, $(9/8, 9/8+\varepsilon) \setminus S(P,\lambda) \neq \emptyset$.

III. For every $\varepsilon > 0$, $(9/8, 9/8+\varepsilon) \cap S(P,\lambda)$ contains a closed interval.

II and III guarantee that $S(P,\lambda)$ is not a finite union of intervals nor homeomorphic to the Cantor set. I guarantees that $S(P,\lambda)$ is not homeomorphic to the set $S(\beta)$ of Remark 1.1. (If $f$ were a homeomorphism of $S(P,\lambda)$ onto $S(\beta)$, $[0,9/8]$ would be mapped onto an interval $J$ of $S(\beta)$. Now $f(0)$ would be an endpoint of $J$ and since in $S(\beta)$ there are intervals of $S(\beta)$ arbitrarily close to $f(0)$, $f^{-1}$ would have to map these intervals to intervals "close" to 0, but that is impossible since $f^{-1}[J] = [0,9/8]$.)

In verifying I, II, and III, we will use two forms for base 3 expansions, e.g.

$$\frac{9}{8} = 1.0101\ldots = 1.\overline{01} = \sum_{n=0}^{\infty} \frac{1}{3^{2n}}.$$

*Verification of I.* Since $9/8 = \sum_{n=1}^{\infty} 9/3^{2n}$, it only remains to show that $(1,9/8) \subset S(P,\lambda)$. Let $x \in (1,9/8)$ and write $x$ in its base 3 expansion, say $x = \sum_{k=0}^{\infty} a_k/3^k$. Since $1 < x < 9/8$, $a_0 = 1$ and $a_1 = 0$. If $a_2 = 1$, then $a_3 = 0$; if also $a_4 = 1$, then $a_5 = 0$, etc. Let $K = \min\{k : a_{2k} = 0\}$. Then $x = 1.01\ldots0100a_{2K+1}a_{2K+2}\ldots$. Then we can write $x = \sum \varepsilon_n \lambda_n$, where

$$\varepsilon_n = \begin{cases} 0 & \text{if } n \text{ is odd and } n < 2K \\ 9 & \text{if } n \text{ is even and } n \le 2K \\ a_n & \text{if } n > 2K. \end{cases}$$

Hence $x \in S(P, \lambda)$.

*Verification of II.* Let $\varepsilon > 0$ be given. Choose $K$ such that $3^{2K} > 1/\varepsilon$. Set

$$x_0 = \sum_{k=0}^{2K+2} \frac{a_k}{3^k}$$

where

$$a_k = \begin{cases} 1 & \text{if } k \text{ is even and } k \le 2K \\ 0 & \text{if } k \text{ is odd and } k < 2K \\ 2 & \text{if } k = 2K+2. \end{cases}$$

Then $x_0 = 1.01\ldots0102 = 1.01\ldots0101\overline{2}$. Clearly $x_0 > 9/8$ and

$$\begin{aligned} x_0 - \frac{9}{8} &= \frac{2}{3^{2K+3}} + \frac{1}{3^{2K+4}} + \frac{2}{3^{2K+5}} + \frac{1}{3^{2K+6}} + \ldots \\ &= \frac{7}{8 \cdot 3^{2K+2}} < \frac{1}{3^{2K}} < \varepsilon, \end{aligned}$$

so $x \in (9/8, 9/8 + \varepsilon)$. It only remains to show that $x_0 \notin S(P, \lambda)$. Set

$$x^* = \sum_{k=0}^{2K} \frac{b_k}{3^k}$$

where

$$b_k = \begin{cases} 1 & \text{if } k \text{ is even} \\ 0 & \text{if } k \text{ is odd.} \end{cases}$$

Then $x^* = 1.01\ldots01$. Now $x^* \in S(P, \lambda)$ since

$$x^* = \sum_{k=1}^{\infty} \frac{\varepsilon_k}{3^k}$$

where

$$\varepsilon_k = \begin{cases} 0 & \text{if } k \text{ is odd or } k > 2K+2 \\ 9 & \text{if } k \text{ is even and } k \le 2K+2. \end{cases}$$

Note also that this representation of $x^*$ as an element of $S(P,\lambda)$ is unique. Now, $x_0 - x^* = 2/3^{2K+2}$ and can be expressed as an element in $S(P,\lambda)$ in two ways.

$$\frac{2}{3^{2K+2}} = \sum_{k=1}^{\infty} \frac{\varepsilon'_k}{3^k}$$

where

$$\varepsilon'_k = \begin{cases} 2 & \text{if } k = 2K+2 \\ 0 & \text{otherwise} \end{cases}$$

or

$$\varepsilon'_k = \begin{cases} 0 & \text{if } k < 2K+2 \\ 1 & \text{if } k = 2K+2 \\ 2 & \text{if } k > 2K+2. \end{cases}$$

Hence $x_0 = x^* + 2/3^{2K+2}$ is not in $S(P,\lambda)$, since $\varepsilon_{2K+2} = 9$ in the representation of $x^*$ as an element of $S(P,\lambda)$.

*Verification of III.* Let $\varepsilon > 0$ be given. Choose $K$ such that $3^{2K} > 1/\varepsilon$. Set

$$a = \sum_{k=0}^{2K+1} \frac{a_k}{3^k}$$

where

$$a_k = \begin{cases} 1 & \text{if } k \text{ is even or } k = 2K+1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$b = a + \frac{1}{3^{2K+2}}.$$

Then $a = 1.0101\ldots011$ and $b = 1.0101\ldots0111$ and it is easy to check that $a, b \in S(P,\lambda)$. It remains to show that $[a,b] \subset S(P,\lambda)$. Let $x \in [a,b]$. Then if $x = \sum x_k/3^k$, $x_k \in \{0,1,2\}$,

$$x = 1.0101\ldots0110x_{2K+3}x_{2K+4}\ldots.$$

Now since

$$x = \sum_{k=1}^{\infty} \frac{\varepsilon_k}{3^k}$$

where

$$\varepsilon_k = \begin{cases} 9 & \text{if } k \text{ is even and } 2 \le k \le 2K+2 \\ 0 & \text{if } k \text{ is odd and } 1 \le k \le 2K-1 \\ 1 & \text{if } k = 2K+1 \\ x_k & \text{if } k \ge 2K+3, \end{cases}$$

$x \in S(P, \lambda)$.

Now, if we set $P' = 9 - P = \{0, 7, 8, 9\}$, Proposition 2.1 tells us that $S(P', \lambda) = 9/2 - S(P, \lambda)$. In other words, $S(P', \lambda)$ is the reflection of $S(P, \lambda)$ about 9/4. Clearly $S(P, \lambda)$ and $S(P', \lambda)$ are homeomorphic.

We next consider $\bar{P} = P \cup P' = \{0, 1, 2, 7, 8, 9\}$. First note that $\bar{P}$ is symmetric (about 9/4). Also note that $S(P, \lambda) \cup S(P', \lambda) \subset S(\bar{P}, \lambda)$. We will show, in the next section, that $S(\bar{P}, \lambda) = [0, 13/6] \cup [7/3, 9/2]$.

## 4. Conditions for which the set of $P$-sums is a set of subsums

Let $G = \langle g_1, g_2, \ldots, g_M \rangle$ be an ordered $M$-tuple of real numbers with $|g_1| \le |g_2| \le \ldots \le |g_M|$, and let $\lambda = \langle \lambda_n \rangle$ be a sequence of real numbers. $[G]\lambda$ will denote the sequence $\alpha = \langle \alpha_n \rangle$ where

$$\alpha_n = g_{M-r+1}\lambda_{q+1} \text{if } n = qM + r \text{ and } 1 \le r \le M.$$

(i.e., $\alpha = \langle g_M\lambda_1, g_{M-1}\lambda_1, \ldots, g_1\lambda_1, g_M\lambda_2, g_{M-1}\lambda_2, \ldots, g_1\lambda_2, \ldots \rangle$). Also set

$$\sigma G = \left\{ \sum_{k=1}^{M} \varepsilon_k g_k : \varepsilon_k = 0 \text{ or } 1 \right\}$$

and $|G| = \{|g_1|, |g_2|, \ldots, |g_M|\}$. Also $\sum G$ and $\sum |G|$ will denote the sum of the elements in $G$ and $|G|$ respectively.

If the set $P = \sigma G$ for some $M$-tuple $G$, we say $P$ is generated by $G$. The next theorem tells us that if $P$ is generated by an $M$-tuple $G$, then $S(P, \lambda)$ is the set of subsums of $[G]\lambda$.

**Proposition 4.1.** $S(\sigma G, \lambda) = S([G]\lambda) = S([|G|]|\lambda|) + \sum |G|(s - \bar{s})/2.$

Proof. $S(\sigma G, \lambda) = \left\{ \sum \varepsilon_n \lambda_n : \varepsilon_n \in \sigma G \right\} = \left\{ \sum (\sum_{k=1}^{M} \varepsilon'_{n_k} g_k)\lambda_n : \varepsilon'_{n_k} = 0 \text{ or } 1 \right\} = \left\{ \sum \varepsilon''_n \alpha_n : \varepsilon''_n = 0 \text{ or } 1 \right\} = S([G]\lambda)$ where $\langle \alpha_n \rangle = [G]\lambda$ as defined above.

The second equality follows easily from Proposition 2.2 after noting that the sequence $|[G]\lambda|$ is the same as $[|G|]|\lambda|$ and that the sum of the sequences $[G]\lambda$ and $[|G|]|\lambda|$ are $\sum Gs$ and $\sum |G|\bar{s}$ respectively.

**Corollary 4.2.** *If $P_N = \{0, 1, 2, \dots, N\}$ and $G_1$ is the $N$-tuple $\langle 1, 1, \dots, 1\rangle$, then $P_N$ is generated by $G_1$ and*

$$S(P_N, \lambda) = S([G_1]\lambda) = S([G_1]|\lambda|) + N(s - \bar{s})/2.$$

PROOF. The corollary follows from Proposition 4.1 by noting that $P_N = \sigma(G_1)$ and $\sum G_1 = N$.

*Remark 4.1.* If the elements of $P$ form an arithmetic progresion, say $P = \{a, a+m, a+2m, \dots, a+Nm\} = mP_N + a$, then by Proposition 2.1 and Corollary 4.2,

$$S(P, \lambda) = mS([G_1]\lambda) + mN(s - \bar{s})/2 + as.$$

Of course, in the cases where the set of $P$-sums is a set of subsums, property B in Section 1 can be used to give necessary and sufficient conditions for the set to be a finite union of intervals (or an interval) and property C can be used to give sufficient conditions for the set to be homeomorphic to the Cantor set. The next theorem does the former for $S(P_N, \lambda)$ with $P_N = \{0, 1, 2, \dots, N\}$ as above. The interval case of the theorem can be found as Theorem 1 in [7]. However the proof found there is much longer than, and different from, the proof given here.

**Theorem 4.3.** *$S(P_N, \lambda)$ is a finite union of intervals if and only if $|\lambda_n| \le N\bar{r}_n$ for $n$ sufficiently large. (Also $S(P_N, \lambda)$ is an interval if and only if $|\lambda_n| \le N\bar{r}_n$ for all $n$.) Furthermore $S(P_N, \lambda)$ is homeomorphic to the Cantor set if $|\lambda_n| > N\bar{r}_n$ for $n$ sufficiently large.*

PROOF. By Corollary 4.2, $S(P_N, \lambda)$ is a horizontal shift of $S([G_1]|\lambda|)$. Now let the $n$th term of $[G_1]|\lambda|$ be $\alpha_n$. Then by property B in Section 1, $S([G_1]|\lambda|)$ is an interval if and only if $\alpha_n \le \sum_{k=n+1}^{\infty} \alpha_k$ for all $n$. Now if $N \nmid n$, $\alpha_n = \alpha_{n+1}$ so the inequality is clearly satisfied. On the other hand if $N|n$ and $\alpha_n = |\lambda_j|$, then $\sum_{k=n+1}^{\infty} \alpha_k = N(|\lambda_{j+1}| + |\lambda_{j+2}| + \dots) = N\bar{r}_j$. The results about intervals now follow easily. The result about the Cantor set follows directly from the last part of Theorem 2.4.

We now continue with the example in the last section with $\lambda_n = 1/3^n$ and $\bar{P} = \{0, 1, 2, 7, 8, 9\}$. If we take $G = \langle 1, 1, 7\rangle$, then it is easy to see that $\bar{P}$ is generated by $G$, i.e. $\bar{P} = \sigma G$. Hence by Proposition 4.1, $S(\bar{P}, \lambda) = S([G]\lambda)$ where $[G]\lambda$ is the sequence

$$\frac{7}{3}, \frac{1}{3}, \frac{1}{3}, \frac{7}{3^2}, \frac{1}{3^2}, \frac{1}{3^2}, \dots$$

or, placing the terms of the sequence in decreasing order,

$$\frac{7}{3}, \frac{7}{3^2}, \frac{1}{3}, \frac{1}{3}, \frac{7}{3^3}, \frac{1}{3^2}, \frac{1}{3^2}, \dots .$$

It is easy to check that each term, except the first, is less than the sum of its tail. Hence by property B and Remark 1.2 in Section 1, $S(\bar{P}, \lambda) = S([G]\lambda) = [0, 13/6] \cup [7/3, 9/2]$.

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J. E. NYMANN
DEPARTMENT OF MATHEMATICAL SCIENCES
THE UNIVERSITY OF TEXAS AT EL PASO
EL PASO, TEXAS 79968–0514
USA

RICARDO A. SÁENZ
DEPARTMENT OF MATHEMATICAL SCIENCES
THE UNIVERSITY OF TEXAS AT EL PASO
EL PASO, TEXAS 79968–0514
USA

*(Received June 6, 1996, revised November 16, 1996)*