Polynomial values in linear recurrences

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1. Introduction

Let $A_1, \ldots, A_k$ and $G_0, G_1, \ldots, G_{k-1}$ be integers. We have for the n-th term of a k-order linear recurrence

$$G_n = A_1 G_{n-1} + \ldots + A_k G_{n-k} \quad \text{for} \quad n = k, k+1, \ldots. \tag{1}$$

Let $\alpha_1, \ldots, \alpha_t$ be the distinct roots of the characteristic polynomial of the recurrence

$$X^k - A_1 X^{k-1} - \ldots - A_k. \tag{2}$$

Throughout this paper we assume that α_1 *has multiplicity one*. Then for $n \geqq 0$ we have

$$G_n = E_1 \alpha_1^n + P_2(n)\alpha_2^n + \ldots + P_t(n)\alpha_t^n, \tag{3}$$

where $P_i(n)$ is a polynomial with degree less than the multiplicity of α_i in the characteristic polynomial of G_n, and where E_1 and the coefficients of $P_i(n)$ are elements of the field $Q(\alpha_1, \ldots, \alpha_t)$.

Finally let $T(x) = B_m x^m + \ldots + B_0$ be a polynomial with integer coefficients. Its degree will be denoted by deg T, while its height, max $\{|B_i|;\ i = 0, \ldots, m\}$ by $\mathscr{H}(T)$.

The Diophantine equation

$$G_n = Ex^q + T(x)E \neq 0, \quad \text{integer} \tag{4}$$

was investigated by several authors. Naturally, most of the results are known for $T(x) = 0$.

Shorey and Stewart [4] proved for general linear recurrences that (4) has finitely many solutions in q, assuming $|\alpha_1| > |\alpha_j|$, $j = 2, \ldots, t$. Under some other restriction on G_n, recently P. Kiss [2] was able to generalize their result when deg $T < c_1 q$.

For nondegenerate second order linear recurrences Shorey and Stewart [4] derived much more, namely (4) has finitely many solutions in integers $|x| > 1$, $q \geqq 2, n$. The second author investigated in [3] for nondegenerate second order linear recurrences the slightly more general equation

$$G_n = wx^q \tag{5}$$

with $w \in S$, where S is the set of nonzero integers composed solely of primes from some fixed finite set. He showed assuming yet $(A_1, A_2)=1$ that (5) has finitely many solutions in integers $|x|>1$, $q \geqq 2$, n and $w \in S$.

STEWART [7] was dealing with the case $T(x)=c$, where c is a fixed integer. For nondegenerate second order recurrences with $|A_2|=1$ he proved the finiteness of the solution in integers $|x|>1$, $q>2$, c, n of (4).

All the above mentioned results are effective.

In the present paper we shall derive some results for (4), when $T(x)$ is a polynomial with some restriction.

2. Main results

Theorem 1. *Let* G_n *satisfy* (3), $\alpha_1, \alpha_2 \neq 1$, $|\alpha_1|>|\alpha_2|>|\alpha_j|$, $j=3, \ldots, t$ *and* $G_n - E_1\alpha_1^n \neq 0$ *for* $n>c_2$. *Further let* $\mathscr{H}(T)<H_1$ *and* $\deg T \leqq qc_3$, where $H_1>0$ *real number. Then all integer solutions* $n, |x|>1$, $q \geqq 2$ *of the equation* (4) *satisfy* $q<c_4$, *where* c_2, c_3 *and* c_4 *are effectively computable constants depending on* E, G_n *and* H_1.

For second order linear recurrences we prove a more precise result.

Theorem 2. *Let* G_n *be a nondegenerate second order linear recurrence with* $|A_2|=1$. *Further let* $\mathscr{H}(T)<H_2$ *and* $\deg T \leqq \min\{q(1-\gamma), q-3\}$ *where* H_2 *and* $\gamma<1$ *are positive real numbers. Then all integer solutions* $n, |x|>1$, $q \geqq 2$ *of* (4) *satisfy*

$$\max\{n, |x|, q\} < c_5,$$

where c_5 *is an effectively computable constant depending on* E, G_n, γ *and* H_2.

Remark. Theorem 2 is in the restriction of $\deg T$ best possible. Let L_n denote a Lucas sequence, i.e. $L_0=2$, $L_1=\alpha+\beta$ and $L_n=(\alpha+\beta)L_{n-1}+L_{n-2}$, where $\alpha\beta=-1$ and $\alpha+\beta$ integer. Then, as is well known, $L_n=\alpha^n+\beta^n$. Further it is easy to see that $L_{2n}=L_n^2+(-1)^n 2$. This means that both equations $L_n=x^2+2$ and $L_n=x^2-2$ have infinitely many integer solutions n, x. Therefore in Theorem 2 the assumption $\deg T \leqq q-3$ is necessary.

3. Auxiliary results

The most important result we use is Lemma 6 of [4].

Lemma. *Let* α *be a real algebraic number larger than one from the field* k. *Let* $[K:Q]=D$, E, A *and* B *be elements of* K, $EAB \neq 0$, *finally* δ *a positive real number. If* $Ex^q=A\alpha^n+B$ *with* $|B|<\alpha^{n(1-\delta)}$ *and* n, x, q *integers larger than one then* $q<c_6$ *is a constant, effectively computable in terms of* D, E, A, α *and* δ.

The following theorem was proved by C. L. SIEGEL [5] for the first time but in noneffective form. Using the upper estimate for linear forms of logarithms of algebraic numbers A. BAKER [1] proved it in effective form.

Theorem A. *Let $F(x)$ be a monic polynomial of degree n and with integer coefficients. Let it have at least three simple zeros. Then the integer solutions x, y of the equation $Ay^2=F(x)$ where A is an integer, satisfy* $\max\{|x|, |y|\}<c_7$ *a constant effectively computable in terms of A and the coefficients of $F(x)$.*

4. Proofs

In the sequel $c_8, c_9, \ldots$ will denote positive numbers effectively computable in terms of E, G_n, H_i and γ.

PROOF OF THEOREM 1. We may assume α_1 to be positive by changing if necessary the sign of E_1. Further since α_1 is an algebraic integer with absolute value strictly larger then all its conjugates, on taking norm we see that either $\alpha_1>1$ or α_1 is one of 0 or 1. But these two cases were excluded so we may assume $\alpha_1>1$. Put

$$B_1(n) = P_2(n)\alpha_2^n+\ldots+P_t(n)\alpha_t^n$$

$$D_1 = \max\{\deg P_i(n);\ i = 2, \ldots, t\}.$$

It is easy to show that

(6) $$2|B_1| < c_8 n^{D_1}|\alpha_2|^n.$$

Assume that for a polynomial $T(x)$ with $\mathscr{H}(T)\leqq H_1$ and $\deg T<qc_3$, $n, |x|>1$, q is a solution of (4). We shall give an estimate for c_3 in the proof.

Write (4) in the form

$$Ex^q = E_1\alpha_1^n+B_1(n)-T(x).$$

Assume first that

(7) $$B_1(n)-T(x) = 0$$

in which case

(8) $$E_1\alpha_1^n = Ex^q$$

also holds. We distinguish two cases.

If $1>|\alpha_2|\geqq|\alpha_j|$, $j=3, \ldots, t$, then $\lim_{n\to\infty} B_1(n)=0$ which means $|B_1(n)|<1$ for $n>c_8$. Further $T(x)$ is a polynomial with integer coefficients, therefore (7) has for $n>c_8$ no solutions. By (8) $q=c_9\dfrac{n}{\log|x|}<c_{10}$ because of $|x|\geqq 2$.

If $|\alpha_2|>1$, then write

$$B_1(n) = P_2(n)\alpha_2^n\left(1+\sum_{i=3}^{t}\frac{P_i(n)}{P_2(n)}\left(\frac{\alpha_i}{\alpha_2}\right)^n\right).$$

The quantity in the brackets tends to 1 if n tends to infinity, so for $n>c_{11}$ $|B_1(n)|>|P_2(n)||\alpha_2|^n(1-\varepsilon')>|\alpha_2|^{n(1-\varepsilon)}$. On the other hand $T(x)\leqq m H_1|x|^m$. From (8) we have $|x|=(|E_1/E||\alpha_1|^n)^{1/q}$, hence $|T(x)|\leqq c_{12}\, m\, |\alpha_1|^{nm/q}<|\alpha_1|^{c_{13}nm/q}$. By (7) $|\alpha_2|^{n(1-\varepsilon)}<|B_1(n)|=|T(x)|<|\alpha_1|^{c_{13}nm/q}$. This implies $m>c_{14}\dfrac{\log|\alpha_2|}{\log|\alpha_1|}q$. Therefore if $c_3\leqq c_{13}\dfrac{\log|\alpha_2|}{\log|\alpha_1|}$ then (7) and (8) have only finitely many solutions in $n, q, |x|>1$ which are effectively computable.

In the sequel we assume that $B_1(n)-T(x)\neq 0$. Put $\delta=(1-\vartheta)/2$, where $\vartheta=0$ if $\alpha_2\leqq 1$, and $\vartheta=\frac{\log|\alpha_2|}{\log|\alpha_1|}$ otherwise. Since $\alpha_1>1$ we have by (6) $|B_1|<\frac{1}{2}\alpha_1^{n(1-\delta)}$ for $n>c_{14}$. We choose c_{15} such that $|E||x|^{c_{15}}>2E_1+1$. If q is large enough then $\frac{|T(x)|}{|x|^{q-c_{15}}}<1$. Write (4) in the form

$$Ex^{c_{15}}=\frac{E_1\alpha_1^n+B_1(n)}{x^{q-c_{15}}}-\frac{T(x)}{x^{q-c_{15}}}.$$

Taking absolute values, and applying the above estimates we have

$$2|E_1|+1<|Ex^{c_{15}}|\leqq\left|\frac{E_1\alpha_1^n+B_1(n)}{x^{q-c_{15}}}\right|+1.$$

Hence $2|E_1||x|^{q-c_{15}}\leqq|E_1\alpha_1^n+B_1(n)|\leqq 2|E_1|\alpha_1^n$, so $|x|^{q-c_{15}}<\alpha_1^n$. Further $|T(x)|\leqq\leqq m\,H_1|x|^m<|x|^{c_{16}q}$, with $c_{16}<1$. Hence $|T(x)|<\alpha_1^{c_{16}qn/(q-c_{15}}<\frac{1}{2}\alpha_1^{n(1-\delta)}$ when δ is small enough. So we have $|B_1(n)+T(x)|\leqq\alpha_1^{n(1-\delta)}$.

Note that if $n<c_{17}$ and (4) holds then $q<c_{18}$ as required. Of course since

$$|x|^m(Ex^{q-m}-mH_1)\leqq|E|\,|x|^q-|T(x)|\leqq|Ex^q+T(x)|=|G_n|\leqq c_{19}\alpha_1^n$$

the required inequality for q holds.

Finally by the Lemma if $n>c_{20}$ and (4) holds then $q<c_1$ as was stated.

PROOF OF THEOREM 2. The assumption $|A_2|=1$ means $|\alpha_1\alpha_2|=1$. We show that α_1 and α_2 are real numbers. Of course if one of them had a nonzero immaginary part then $|\alpha_1|=|\alpha_2|$ would hold since they are roots of a polynomial with integer coefficients. This imply $|\alpha_1|=|\bar{\alpha}_2|=1$.

But α_1/α_2 cannot be a root of unity by nondegeneracy. Thus α_1, α_2 are real numbers and $|\alpha_1|>1>|\alpha_2|$ holds since $|\alpha_1\alpha_2|=1$. Further the equation $A_2\alpha_2^n=T(x)$ has only finitely many solutions, since with n large enough $0<|T(x)|=|A_2\alpha_2^n|<1$. Therefore $n<c_{21}$, which implies q, $|x|<c_{22}$.

In the sequel we assume $A_2\alpha_2^n\neq T(x)$. Now we shall prove $q<c_{23}$. Assume that (4) has a solution $n, q, |x|>1$ such that $q\geqq c_{23}$, with $\frac{\log qH_2}{q\log 2}<\frac{\gamma}{2}$, and $q\frac{\gamma}{4}>1+\frac{\gamma}{4}$. Then

$$|T(x)|<mH_2|x|^m<|x|^{m+\frac{\log H_2 m}{\log 2}}<|x|^{q(1-\gamma)+\frac{\log qH_2}{\log 2}}.$$

Applying the assumption we have

$$q(1-\gamma)+\frac{\log qH_2}{\log 2}<q(1-\gamma)+q\frac{\gamma}{2}=q\left(1-\frac{\gamma}{2}\right)<q-1.$$

Hence $\frac{|T(x)|}{|x|^{p-1}}<1$. From this follows as in the proof of Theorem 1 $|x|^{q-1}<\alpha_1^n$, and $|T(x)|<\alpha_1^{(1-\delta)}$ with $\delta=\frac{\gamma}{4}$. Finally it is abvious that $P_2(n)|\alpha_2|^n<\alpha_1^{n(1-\delta)}$. Now applying the Lemma we conclude that q is bounded by an effectively computable constant.

Now let q and $T(x)$ be fixed with deg $T<q(1-\gamma)$ and consider the equation

$$G_n = Ex^q+T(x) = T_1(x).$$

It is well known that $G_{n+1}^2-A_1G_{n+1}G_n+A_2G_n^2=CA_2^n$, with $C=G_1^2-A_1G_1G_0+ +A_2G_0^2$ (see for example [3] Lemma 1). From this follows $DG_n^2+4CA_2^n=z^2$, with $D=A_1^2-4A_2\neq 0$ and z an integer. Replacing G_n^2 by $Ex^q+T(x)$, and taking $|A_2|=1$ into account we have

$$R(x) = D\big(Ex^q+T(x)\big)^2\pm 4C = DT_1^2(x)\pm 4C = z^2.$$

This is an elliptic equation, and by means of Theorem A it has finitely many solutions in x, z when $R(x)$ has at least three simple zeros. Let $\big(R(x), R'(x)\big)=Q(x)$. It is well known that a root ω of $R(x)$ has multiplicity at least two if and only if ω is a root of $Q(x)$. Further $\big(R(x), T_1(x)\big)=1$ because of $c\neq 0$, and $R'(x)= =2D\,T_1(x)\,T_1'(x)$ so $\deg Q(x)\leqq q-1$. This means that if either $\deg Q(x)<q-1$ or it has at least one multiple root then $R(x)$ has at least three simple zeros and we are ready.

Hence the only wrong case is when $Q(x)=T_1'(x)$, and $R(x)=T_1'(x)^2S(x)$ with a polynomial $S(x)$ with rational coefficients of degree two with not any multiple roots. Let $S(x)=s_2x^2+s_1x+s_0$ and consider the equation

$$D\big(Ex^q+T(x)\big)^2\pm 4C = \big(2qEx^{q-1}+T'(x)\big)^2(s_2x^2+s_1x+s_0).$$

The coefficient of x^{2q-1} and that of x^{2q-2} on the left hand side is 0, because of $\deg T(x)<q-3$, while the coefficient of x^{2q-1} on the right hand side is $4q^2E^2s_1$, and that of x^{2q-2} is $4q^2E^2s_0$. This means $s_1=s_0=0$, and $S(x)=s_2x^2$ wich is a contradiction.

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