On the Lebesgue function on infinite interval, I

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1. The approximation of the Lebesgue function on a finite interval was extensively discussed by G. GRÜNWALD and P. TURÁN [1]. In this paper we extend their investigation to infinite interval.

Moreover we establish the convergence of the Lagrange interpolation polynomial on the infinite interval.

Let a weight function

$$p(x) \geqq p > 0 \tag{1.1}$$

be given, such that

$$\int_{-1}^{1} p(x)\,dx < \infty. \tag{1.2}$$

Clearly, there exists a unique orthonormal polynomial system $\{\omega_n(x)\}_{n=0}^{\infty}$ such that

$$\int_{-1}^{1} \omega_n(x)\omega_m(x)\,p(x)\,dx = \delta_{n,m} = \begin{cases} 0, & n \neq m \\ 1, & n = m \end{cases}.$$

A special case of this system is the system of the so called Jacobi polynomials, where

$$p(x) = (1-x)^{\alpha}(1+x)^{\beta}, \quad 0 > \alpha,\ \beta > -1.$$

Let $\omega_n(x_{\nu,n}) = 0$, where

$$-1 < x_{n,n} < \ldots < x_{\nu+1,n} < x_{\nu,n} < \ldots < x_{1,n} < 1. \tag{1.3}$$

The Lagrange interpolation polynomial $L_n(x)$ for prescribed values $\{y_{\nu,n}\}_{n=1}^{\infty}$ such that

$$L_n(x_{\nu,n}) = y_{\nu,n} \quad (\nu = 1, 2, \ldots, n, \quad n = 1, 2, 3, \ldots)$$

can be explicitly written as

$$L_n(x) = \sum_{\nu=1}^{n} y_{\nu,n}\, l_{\nu,n}(x) \tag{1.4}$$

where as usual

$$l_{\nu,n}(x) = \frac{\omega_n(x)}{\omega_n'(x_{\nu,n})(x - x_{\nu,n})} \quad l_{\nu,n}(x_{j,n}) = \delta_{\nu j}.$$

The Lebesgue function is given as

$$\lambda_n(x) \overset{\text{def}}{=} \sum_{\nu=1}^{n} |l_{\nu,n}(x)| \tag{1.5}$$

and the Lebesgue constant is defined as

$$\lambda_n = \max_{x\in[-1,1]} \{\lambda_n(x)\}.$$

G. GRÜNWALD and P. TURÁN [1] proved that if (1.1), is true then

$$\lambda_n(x) \leqq C_1\sqrt{n}\,, \quad x\in(-1,1)$$

and

$$\lambda_n(x) \leqq C_2 n, \quad x\in[-1,1],$$

where C_1 and C_2 are suitable constants.

2. In this paper the approximation of $\lambda_n(x)$ on the interval $[0,\infty)$ will be considered.

Let a weight function $p(x)$ such that

$$\int_0^\infty x^k p(x)\,dx < \infty \quad (k=0,1,2,\ldots) \tag{2.1}$$

be given.

Clearly, there exists a unique system of orthonormal polynomials $\{\omega_n^*(x)\}_{n=0}^\infty$ such that $\omega_n^*(x)$ has exactly n simple zeros, i.e.

$$0 < x_1 < x_2 < \ldots < x_\nu < x_{\nu+1} < \ldots < x_n < \infty \quad (n=1,2,3,\ldots). \tag{2.2}$$

It is plausible to choose our x_ν-points as the zeros of n-th Laguerre polynomial defined by

$$\omega_n^*(x) = \frac{1}{n!}\,x^{-\alpha}e^x(e^{-x}x^{n+\alpha})^{(n)} \quad (\alpha > -1), \tag{2.3}$$

then in this case

$$p(x) = x^\alpha e^{-x}.$$

We shall now assert

Theorem 1. *Let* $p(x)x^{-\alpha}e^x \geqq m > 0$, $\alpha > -1$ *and*

$$\int_0^\infty x^k p(x)\,dx < \infty \quad (k=0,1,2,\ldots).$$

Then the Lebesgue function using the nodes (2.2) *satisfies the following condition:*

$$\lambda_n(x) = \begin{cases} O(1)\,x^{-\frac{\alpha}{2}+\frac{1}{4}}\,n^{\frac{1}{4}} & \left(0 < x \leqq A,\ \alpha > -\frac{1}{2}\right), \\ O(1)\,n^{\frac{1}{4}} & \left(0 \leqq x \leqq A,\ -1 < \alpha \leqq -\frac{1}{2}\right), \end{cases}$$

where A *is an arbitrary fix real number.*

Remark. *If* $p(x)=x^\alpha e^{-x}$, *then* $m=1$.

3. A very useful role will be played by the following statement.

Lemma 1. *The Cotes numbers belonging to the system* (2.2) *satisfy the relations*

$$(3.1)\qquad \int_0^\infty l_\nu(x)\,p(x)\,dx = \int_0^\infty l_\nu(x)^2 p(x)\,dx \quad (1 \leqq \nu \leqq n,\ n = 1, 2, 3, \ldots).$$

PROOF. Let

$$I = \int_0^\infty l_\nu(x)[l_\nu(x)-1]\,p(x)\,dx, \quad l_\nu(x) = \frac{\omega_n^*(x)}{\omega_n^{*\prime}(x_\nu)(x_\nu - x)}.$$

Clearly $l_\nu(x_j)=0$, $\nu\neq j$ and $l_\nu(x_\nu)=1$, $j, \nu=1, 2, \ldots, n$, $n=1, 2, 3, \ldots$. Also

$$I = \int_0^\infty \omega_n^*(x)\,q_{n-2}(x)\,p(x)\,dx,$$

where $q_{n-2}(x)$ is a polynomial of degree $n-2$. Hence from the orthogonality of $\{\omega_n^*(x)\}_{n=0}^\infty$, I is equal to zero and the lemma is proved.

It is also easy to show that the orthogonality of $\{\omega_n^*(x)\}_{n=0}^\infty$ implies the equation

$$(3.2)\qquad \int_0^\infty l_i(x)\,l_j(x)\,p(x)\,dx = \begin{cases} 0, & i \neq j, \\ \mu_i, & i = j, \end{cases}$$

where from (3.1) μ_i $(i=1, 2, \ldots, n,\ n=1, 2, 3, \ldots)$ are the Cotes numbers.

4. We shall now present the proof of Theorem 1.

Observe that

$$(4.1)\qquad \sum_{\nu=1}^n l_\nu(x) \equiv 1,$$

then from Lemma 1:

$$(4.2)\qquad \sum_{\nu=1}^n \int_0^\infty l_\nu^2(x)\,p(x)\,dx = \sum_{\nu=1}^n \int_0^\infty l_\nu(x)\,p(x)\,dx = \int_0^\infty p(x)\,dx.$$

This integral is independent of n.

Also, from (3.2) and (4.2)

$$(4.3)\qquad \int_0^\infty \Big\{\sum_{\nu=1}^n l_\nu(x)\Big\}^2 p(x)\,dx = \int_0^\infty \Big\{\sum_{\nu=1}^n l_\nu(x)^2\Big\}\,p(x)\,dx = \int_0^\infty p(x)\,dx < \infty.$$

Define $\varepsilon_\nu = \operatorname{sign}\{l_\nu(x_0)\}$, where $x_0 \geqq 0$ is an arbitrary fix value, and let

$$(4.4)\qquad \Psi_n(x) = \sum_{\nu=1}^n \varepsilon_\nu l_\nu(x).$$

The polynomial $\psi_n(x)$ is of degree $n-1$ and can be expanded by the Fourier series of Laguerre orthonormal polynomials

$$(4.5)\qquad \Phi_k(x) = \Gamma(\alpha+1)^{-\frac{1}{2}} \binom{n+\alpha}{n}^{-\frac{1}{2}} L_k^{(\alpha)}(x) \quad (k = 0, 1, 2, \ldots).$$

Thus

$$\Psi_n(x) = \sum_{k=0}^{n-1} c_k \Phi_k(x) \tag{4.6}$$

and from Cauchy's inequality

$$[\Psi_n(x)]^2 \leqq \sum_{k=0}^{n-1} c_k^2 \sum_{k=0}^{n-1} \{\Phi_k(x)\}^2. \tag{4.7}$$

That is,

$$\int_0^\infty [\Psi_n(x)]^2 x^\alpha e^{-x}\, dx = \int_0^\infty \Big\{\sum_{k=0}^{n-1} c_k \Phi_k(x)\Big\}^2 x^\alpha e^{-x}\, dx = \sum_{k=0}^{n-1} c_k^2 \leqq \tag{4.8}$$

$$\leqq \frac{1}{m} \int_0^\infty \Big[\sum_{\nu=1}^{n} \varepsilon_\nu l_\nu(x)\Big]^2 p(x)\, dx.$$

From (3.1) and (4.3) the last term in (4.8) gives

$$\sum_{k=0}^{n-1} c_k^2 \leqq \frac{1}{m} \int_0^\infty p(x)\, dx < \infty. \tag{4.9}$$

This value, which is independent of n, clearly satisfies relation

$$[\Psi_n(x)]^2 = O(1) \sum_{k=0}^{n-1} [\Phi_k(x)]^2. \tag{4.10}$$

We have that (SZEGŐ [2])

$$L_n^{(\alpha)}(x) = O(1) x^{-\frac{\alpha}{2}+\frac{1}{4}} n^{\frac{\alpha}{2}-\frac{1}{4}} \quad \Big(0 \leqq x \leqq A,\ \alpha \geqq -\frac{1}{2}\Big) \tag{4.11}$$

and

$$L_n^{(\alpha)}(x) = O(1) n^{\frac{\alpha}{2}-\frac{1}{4}} \quad \Big(0 \leqq x \leqq A,\ -1 < \alpha \leqq -\frac{1}{2}\Big). \tag{4.12}$$

Relations (4.5), (4.11) and (4.12) give directly

$$\sum_{k=0}^{n-1} [\Phi_k(x)]^2 = \Gamma(\alpha+1)^{-1} \sum_{k=0}^{n-1} \binom{k+\alpha}{k}^{-1} [L_k^{(\alpha)}(x)]^2 = \tag{4.13}$$

$$= O(1) x^{-\alpha+\frac{1}{2}} n^{\frac{1}{2}} \quad \Big(x \in (0, A],\ \alpha \geqq -\frac{1}{2}\Big),$$

$$\sum_{k=0}^{n-1} [\Phi_k(x)]^2 = O(1) n^{\frac{1}{2}} \quad \Big(x \in [0, A],\ -1 < \alpha \leqq -\frac{1}{2}\Big). \tag{4.14}$$

Hence, from (4.4), (4.7), (4.9) and (4.10) we have

$$\Psi_n(x_0) = \lambda_n(x_0) = \sum_{\nu=1}^{n} |l_\nu(x_0)| = \tag{4.15}$$

$$= \begin{cases} O(1)\, x_0^{-\frac{\alpha}{2}+\frac{1}{4}} n^{\frac{1}{4}} & \left(x_0 \in (0, A),\ \alpha \geqq -\dfrac{1}{2}\right), \\ O(1)\, n^{\frac{4}{1}} & \left(x_0 \in [0, A],\ -1 < \alpha \leqq \dfrac{1}{2}\right). \end{cases}$$

Thus, the theorem is proven, because x_0 is an arbitrary value.

5. It is interesting at this point to consider a function

$$f(x) = e^{\alpha x}\varphi(x) \quad (a > 0,\ x \geqq 0), \tag{5.1}$$

where $\varphi(x) \in \mathrm{Lip}_M \gamma \left(\dfrac{1}{2} < \gamma \leqq 1\right)$.

Let us define polynomials $L_n(x; f)$ of degree $n-1$, satisfying the following equalities, on the roots of Laguerre polynomials

$$\begin{gathered} L_n(x_\nu;\ f) = e^{-\alpha x_\nu} f(x_\nu) = \varphi(x_\nu) = y_\nu \\ (\nu = 1, 2, 3, \ldots, n,\ n = 1, 2, 3, \ldots). \end{gathered} \tag{5.2}$$

The polynomials $L_n(x; f)$ have the explicit forms

$$L_n(x;\ f) = \sum_{\nu=1}^{n} y_\nu l_\nu(x), \tag{5.3}$$

where

$$l_\nu(x) = \frac{L_n^{(\alpha)}(x)}{L_n^{(\alpha)\prime}(x_\nu)(x - x_\nu)}. \tag{5.4}$$

We shall now prove the following statement.

Theorem 2. *If $f(x)$ satisfies the condition* (5.1). *Then*

$$|f(x) - e^{\alpha x} L_n(x;\ f)| = \begin{cases} O(1)\, x^{\frac{\alpha}{2}-\frac{1}{4}} n^{\frac{1}{4}-\frac{\gamma}{2}} & \left(x \in (0, A],\ \alpha \geqq -\dfrac{1}{2}\right), \\ O(1)\, n^{\frac{1}{4}-\frac{\gamma}{2}} & \left(x \in [0, A],\ -1 < \alpha \leqq -\dfrac{1}{2}\right), \end{cases}$$

where A is an arbitrary fix real number.

Proof. Let $x_\nu = x_\nu(\alpha)$ denote the roots of polynomials $L_n^{(\alpha)}(x)$, then the following inequalities are true (Szegő [2])

$$\begin{gathered} c_1 \frac{\nu^2}{2n+\alpha+1} < x_\nu < c_2 \frac{\nu^\nu}{2n+\alpha+1} \\ (\nu = 1, 2, \ldots, n,\ n = 1, 2, 3, \ldots), \end{gathered} \tag{5.5}$$

where c_1 and c_2 constants are independent of ν and n.

Let $\varphi(x)\in \mathrm{Lip}_M\,\gamma$, $\frac{1}{2}<\gamma\leqq 1$, $x\geqq 0$ and $x=\frac{x_n}{2}(u+1)$, $x_n=x_n(\alpha)$ then

$$g(u)\stackrel{\text{def}}{=}\varphi\left(\frac{x_n}{2}(u+1)\right)=\varphi(x). \tag{5.6}$$

Obviously, if $x\in[0,x_n]$ then $u\in[-1,1]$ and since $|u'-u''|\leqq\delta, u', u''\in[-1,1]$ we have

$$\omega(\delta;\, g)=\omega\left(\frac{x_n}{2}\delta;\ \varphi\right)=O(1)\left(\frac{x_n}{2}\delta\right)^{\gamma}, \tag{5.7}$$

where $\omega(\cdot\,;g)$ and $\omega(\cdot\,;\varphi)$ denote the moduli of continuity of g and φ, respectively.

If $u\in[-1,1]$, $u=\cos\vartheta$, then $\Psi(\vartheta)\stackrel{\text{def}}{=}g(\cos\vartheta)$ is an even function, so its Jackson-mean [3] $J_m(\vartheta,\Psi)$ is a pure cosine polynomial of order $2m-2$, therefore

$$J_m(\arccos u,\, \Psi)=Q_{2m-2}(u;\ g) \tag{5.8}$$

is an algebraic polynomial of degree $2m-2$ and

$$\begin{aligned} &Q_{2m-2}(u,g)=\\ &=\frac{3}{\pi m(2m^2+1)}\int_0^{\pi/2}\{g[\cos(\arccos u+2t)]+g[\cos(\arccos u-2t)]\}\left(\frac{\sin mt}{\sin t}\right)^4 dt. \end{aligned} \tag{5.9}$$

It is well known that

$$\frac{6}{\pi m(2m^2+1)}\int_0^{\pi/2}\left(\frac{\sin mt}{\sin t}\right)^4 dt=1. \tag{5.10}$$

By using relations (5.9), (5.10) and (5.7) it is easy to verify that

$$|Q_{2m-2}(u,g)-g(u)|=O(1)\left(\frac{x_n|u|}{m^2}\right)^{\gamma}+\left(\frac{x_n\sqrt{1-u^2}}{m}\right)^{\gamma}. \tag{5.11}$$

From (5.11) and (5.6) if $u=\dfrac{2x-x_n}{x_n}$ and $x\in[0,x_n]$ we have

$$\begin{aligned} \left|Q_{2m-2}\left(\frac{2x-x_n}{x_n};\ \varphi\right)-\varphi(x)\right| &= O(1)\left\{\left(\frac{|2x-x_n|}{m^2}\right)^{\gamma}+\left(\frac{\sqrt{x_nx-x^2}}{m}\right)^{\gamma}\right\}=\\ &=O(1)\left\{\left(\frac{x_n}{m^2}\right)^{\gamma}+\left(\frac{\sqrt{x_nx}}{m}\right)^{\gamma}\right\}, \end{aligned} \tag{5.12}$$

where equality

$$Q^*_{2m-2}(x;\ \varphi)\equiv Q_{2m-2}\left(\frac{2x-x_n}{x_n};\ \varphi\right)$$

implies that it is a polynomial of degree $2m-2$.

If $m=\left[\frac{n}{2}\right]$, then $2m-2\leqq n-1$ and in this case we have

$$Q^*_{2m-2}(x;\ \varphi)\equiv \sum_{\nu=1}^{n} Q_{2m-2}(x_\nu;\ \varphi)\, l_\nu(x). \tag{5.13}$$

From (5.12), (5.13) and (4.12) we have, that for $x\in(0, A]$, $\alpha\geqq -\frac{1}{2}$, $m=\left[\frac{n}{2}\right]$, $x_n = x_n(\alpha) = O(1)n$

$$|\varphi(x)-L_n(x;\ \varphi)| \leqq |\varphi(x)-Q^*_{2m}(x;\ \varphi)| + \sum_{\nu=1}^{n} |Q^*_{2m-2}(x_\nu)-\varphi(x_\nu)|\,|l_\nu(x)| \leqq$$

$$\leqq O(1)n^{-\frac{\gamma}{2}} + O(1)n^{-\frac{\gamma}{2}} \sum_{0<x_\nu<2A} |l_\nu(x)| + \tag{5.14}$$

$$+O(1)n^{-\frac{\gamma}{2}}\Big\{\sum_{x_\nu>2A} |L_n^{(\alpha)}(x)|\,|L_n^{(\alpha)\prime}(x_\nu)|^{-1}\Big\}.$$

The following inequality is true, if $1\leqq m\leqq 2n-1$ (SZEGŐ [2])

$$\sum_{\nu=1}^{n} x_\nu^{\frac{m}{2}-\frac{1}{2}} |L_n^{(\alpha)\prime}(x_\nu)|^{-1} \leqq \sqrt{n}\,\Big\{\sum_{\nu=1}^{n} x_\nu^{m-1}[L_n^{(\alpha)\prime}(x_\nu)]^{-2}\Big\}^{\frac{1}{2}} = O(1)n^{\frac{1}{2}-\frac{\alpha}{2}}. \tag{5.15}$$

Using (5.1), Theorem 1, (5.14), (4.12) and (5.15) we get

$$|f(x)-e^{\alpha x}L_n(x;\ f)| = e^{\alpha x}|\varphi(x)-L_n(x;\ \varphi)| =$$

$$=\begin{cases} O(1)x^{\frac{\alpha}{2}-\frac{1}{4}}n^{\frac{1}{4}-\frac{\gamma}{2}}, & \text{for } \alpha\geqq -\frac{1}{2},\ x\in(0, A],\\ O(1)n^{\frac{1}{4}-\frac{r}{2}}, & \text{for } -1<\alpha\leqq -\frac{1}{2},\ x\in[0, A]. \end{cases}$$

This completes the proof of Theorem 2.

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