$*$-structures and orthomodular lattices

BY I. G. KALMÁR (Debrecen)

The aim of this paper is to discuss the relation between orthomodular lattices and a special algebraic structure called $*$-structure. This problem originates from quantum logic, where it is supposed that the propositions (events) of a physical system form an orthomodular lattice.

Similar ideas were proposed by FOULIS [1] and later by POOL [5], [6]. Foulis discovered the lattice-semigroup connection, and he has shown that every orthomodular lattice generates, in a given way, a Baer*-semigroup, and every Baer* semigroup contains a subset that is an orthomodular lattice. However, as proved by FOULIS [1], this connection is not "reflexive" because starting from a Baer*-semigroup, constructing the orthomodular lattice of its closed projections, and constructing the associated Baer*-semigroup, the two Baer*-semigroups are not isomorphic.

Pool's works [5], [6] continued this line and studied the connection between the quantum logic structure and the existence of states that one can interpret in terms of idealized measurements. More exactly, it was proved that his axiomatic system for the state-transformation functions implies a $*$-semigroup structure on theese functions.

Our representation of orthomodular lattices by $*$-structures is based on the $*$ product:

Let $\mathscr{L}(\vee, \wedge, \perp, 0, 1)$ be an orthomodular lattice with first and last elements 0 and 1, respectively. Then for all $a, b \in \mathscr{L}$ one can define

$$a * b = (a \vee b^{\perp}) \wedge b.$$

For a detailed examination of orthomodular lattices and of the $*$ product we refer to MAEDA [4] and KALMBACH [3]. Here we recall only some basic notions and results.

Two elements a and b of $\mathscr{L}$ are *orthogonal* $(a \perp b)$ if $a \leqq b^{\perp}$. We say that $a, b \in \mathscr{L}$ are *compatible* and write $a \leftrightarrow b$ if there exists a Boolean sublattice **B** in $\mathscr{L}$ containing a and b.

Proposition 1. *If one of the three elements a, b, c of an orthomodular lattice $\mathscr{L}$ is compatible with each of the two others, then*

$$(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$$

and

$$(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c).$$

Proposition 2. *The following statements are true in an orthomodular lattice* $\mathcal{L}$:

(a) $a \leqq b \Rightarrow a \leftrightarrow b$
(b) $a \perp b \Rightarrow a \leftrightarrow b$
(c) $a \leftrightarrow b \Rightarrow a \leftrightarrow b^{\perp}$
(d) $a \leftrightarrow b \Rightarrow a * b = a \wedge b$
(e) $\mathcal{L}$ *is distributive if and only if* $a \leftrightarrow b$ *for all* $a, b \in \mathcal{L}$
(f) $a \leftrightarrow b_i \Rightarrow a \leftrightarrow \bigvee_i b_i, \ a \leftrightarrow \bigwedge_i b_i$
(g) $(a \vee b)^{\perp} = a^{\perp} \wedge b^{\perp}, \ (a \wedge b)^{\perp} = a^{\perp} \vee b^{\perp}$ *for all* $a, b \in \mathcal{L}$.

Proposition 3. (See also [2, Proposition 6.2]) *In an arbitrary orthomodular lattice* $\mathcal{L}$ *the* $*$ *product prossesses the following properties: if* $x, y, z \in \mathcal{L}$, *then*

(a) $z * z = z$
(b) $z * x = 0$ *if* $z \perp x$
(c) $(z * x) * y = 0$ *if* $x \perp y$
(d) $(z * x) * x = z * x$
(e) $z * (z * x) = z * x$
(f) $z * x = z \wedge x \Leftrightarrow x \leftrightarrow z$
(g) $(z * x) * y = (z * y) * x = z * x$ *if* $x \leqq y$
(h) $(\bigvee_i z_i) * x = \bigvee_i (z * x)$ *if* $\bigvee_i z_i$ *exists*
(i) $(x * y) * z = x * (y * z)$ *if* $z \leftrightarrow y$

Theese properties of the $*$ product serve as a basis for our definition of a $*$-structure:
Let $\mathcal{S}$ be a nonempty set with two special elements 0 and 1. For each $x \in \mathcal{S}$ let an element $x' \in \mathcal{S}$ be given and for each $y, z \in \mathcal{S}$ an element $y \cdot z = yz \in \mathcal{S}$. Let $x \leqq y$ iff $xy = x$ and assume that the following properties hold:

(i) $x'' = x, \ x \leqq y \Rightarrow y' \leqq x' \ \forall x, y \in \mathcal{S}$
(ii) $0 \leqq x \leqq 1 \ \forall x \in \mathcal{S}$
(iii) $xy = 0 \Leftrightarrow x \leqq y'$
(iv) $xx = x \ \forall x \in \mathcal{S}$
(v) $zy = yz \Rightarrow (xy)z = x(yz)$
(vi) $xy = yx \Rightarrow xy^{\perp} = y^{\perp}x$

Then $\mathcal{S}$ (with the two operations $'$ and $\cdot$) will be called a *$*$-structure*.

We say that $x \in \mathcal{S}$ and $y \in \mathcal{S}$ commute $((x, y)C)$ if $xy = yx$. Propositions 1—2—3. imply the following

Theorem 1. *Every orthomodular lattice forms a $*$-structure, where $'$ and $\cdot$ are defined as the orthocomplementation of the lattice and the $*$ product, respectively.*

Let us remark that the operation $x, y \to x \cdot y, \ x, y \in \mathcal{S}$ is not commutative in general.

We need some technical computations:

Proposition 4. *Let* $S(', \cdot, 0, 1)$ *be a $*$-structure. Then*

(a) *if* $x, y \in \mathcal{S}$, $x \leqq y$, *then* x *and* y *commute*
(b) $x \leqq y$ *iff* $xy = x$ *is a partial ordering on* $\mathcal{S}$

(c) *Let* $x+y=(x'\cdot y')'$, $x, y\in\mathcal{S}$, *then*
 (c_1) x *and* y *commute iff* $x+y=y+x$
 (c_2) $x\leqq y$ *iff* $y+x=y$
 (c_3) *if* y *and* z *commute, then* $(x+y)+z=x+(y+z)\ \forall x\in\mathcal{S}$.
(d) $xy\leqq y\leqq x+y\ \ \forall x, y\in\mathcal{S}$
(e) $x'x=xx'=0;\ x+x'=x'+x=1\ \ \forall x\in\mathcal{S}$
(f) *if* x *and* y, x *and* z *commute, and* $y\leqq z$, *then* $xy\leqq xz$, $yz\leqq zx$, *and* $x+y\leqq$ $\leqq x+z$.

PROOF. (a) $x\leqq y\Rightarrow x\cdot y'=0$, $y'\leqq x'\Rightarrow y'\cdot x''=y'\cdot x=0$, so $xy'=y'x$ which implies $xy=yx$.
(b) $x\leqq x$ is trivial. If $x\leqq y$ and $y\leqq x$, then $x=xy=yx=y$. If $x\leqq y$, $y\leqq z$, then by (a) of Proposition 4. y and z commute, and $xz=(xy)z=x(yz)=xy=x$, so $x\leqq z$.
(c_1) x and y commute $\Leftrightarrow$ x' and y' commute $\Leftrightarrow x'y'=y'x'\Leftrightarrow(x'y')'=(y'x')'\Leftrightarrow$ $\Leftrightarrow x+y=y+x$
(c_2) $x\leqq y\Leftrightarrow y'\leqq x'\Leftrightarrow y'x'=y'\Leftrightarrow(y'x')'=y''\Leftrightarrow y+x=x$
(c_3) is trivial
(d) $(xy)y=x(yy)=xy$, and $(x+y)+y=x+(y+y)=x+y$, so by (c) $y\leqq x+y$.
(e) $x'x=xx'=0$ is trivial. $0'\geqq x\ \ \forall x\in\mathcal{S}$, so $0'\geqq 1\Rightarrow 0'=1$, $1'=0$. Hence $x+x'=$ $=x'+x=1$.
(f) Let $xy=yx$, $xz=zx$, $y\leqq z$, then $(xy)(xz)=(xy)\cdot(zx)=\big((xy)\cdot z\big)x=\big(x(yz)\big)x=$ $=(xy)x=x(yx)=x(xy)=(xx)y=xy$, so $xy\leqq xz$. With the help of the preceding inequality $y\leqq z\Rightarrow z'\leqq y'\Rightarrow x'z'\leqq x'y'\Rightarrow(x'z')'\geqq(x'y')'$ that is $x+y\leqq x+z$.

Now we can state our main theorem:

Theorem 2. *If* $\mathcal{S}(0, 1, \cdot, ')$ *is a* $*$*-structure, then* $\mathcal{S}(0, 1, \leqq, ')$ *is an orthomodular lattice with* $'$ *as an orthocomplementation,*

$$\inf(x, y) = x\wedge y = (x+y')y,$$

and

$$\sup(x, y) = x\vee y = (xy')+y.$$

PROOF. Let $x, y\in\mathcal{S}$ and $z=(x+y')$. First we prove that $z\leqq x$. Indeed, $z\leqq x\Leftrightarrow x'\leqq z'\Leftrightarrow x'z=0$. However, $x'z=x'\big((x+y')y\big)=x'\big(y(x+y')\big)=(x'y)(x+y')=0$, because $(x'y)'=x+y'$, so $(x^{\perp}y)(x+y')=0$.

By Proposition 4. (d) we have $z\leqq y$. To see $z=\inf(x, y)$, let us assume $\omega\leqq x$, $\omega\leqq y$ for an $\omega\in\mathcal{S}$. Then

$$(x+y')+(\omega+y') = x+\big(y'+(\omega+y')\big) = x+(\omega+y') = (x+\omega)+y' =$$
$$= (\omega+x)+y' = x+y',$$

so by Proposition 4. (c) $\omega+y'\leqq x+y'$. Since $(y, x+y')C$, $(y, \omega+y')C$, $(\omega, \omega+y')C$, $\omega\leqq y$, by Proposition 4. (f) we have

$$\omega = (\omega+y')\omega \leqq (\omega+y')y \leqq (x+y')y' = z.$$

Hence $\omega\leqq z$ which implies $x\wedge y=(x+y')y$.

On the other hand let $u=(xy')+y$. Then $u'=(x'+y)\cdot y'=x'\wedge y'$, so $u'\leqq x'$, $u'\leqq y'$, $u\geqq x$, $u\geqq y$. If $v\in\mathcal{S}$ and $x\leqq v$, $y\leqq v$, then $x'\geqq v'$, $y'\geqq v'$, so $u'\geqq v'$

or equivalently $u \leqq v$. This means that

$$x \vee y = (xy') + y.$$

Of course $x \wedge y = (x+y')y$, $x \vee y = (xy') + y$ imply

$$(x+y')y = (y+x')x \tag{2.1}$$

$$(xy')+y = (yx')+x. \tag{2.2}$$

Since

$$x \wedge x' = (x+x'')x' = (x+x)x' = xx' = 0$$

and

$$x \vee x' = (xx'')+x' = (xx)+x' = x+x = 1,$$

the operation $'$ is an ortocomplementation on $\mathcal{S}$. The proof will be complete if we show the weak modularity in $\mathcal{S}$. Let us assume that $x, y \in \mathcal{S}$, $(x, y)C$. Then $z = xy = yx$, and $z \leqq x$, $z \leqq y$.

If $\omega \leqq x$, $\omega \leqq y$ for some $\omega \in \mathcal{S}$, then by Proposition 4. (f) we have $xy \geqq \omega y = \omega$, so

$$x \wedge y = xy \quad \text{if} \quad (x, y)C. \tag{2.3}$$

By a similar method we get

$$x \vee y = x+y \quad \text{if} \quad (x, y)C. \tag{2.4}$$

Let now $x \leqq y$, then $(x, y)C$, $(y, x')C$, $(y \wedge x', x)C$ and consequently

$$y = x+y = (yx')+x = (y \wedge x') \vee x.$$

This means that the weak modularity holds in $\mathcal{S}$.

Proposition 5. *Let $\mathcal{S}$ be a $*$-structure and $a, b \in \mathcal{S}$. Then*

(i) $ab = (a \vee b') \wedge b$, $a+b = (a \wedge b') \vee b$, *and*
(ii) $(a, b)C \Leftrightarrow a \leftrightarrow b \Leftrightarrow a \wedge b = ab \Leftrightarrow a \vee b = a+b$, *where* $\vee$, $\wedge$ *have the same meaning as in the preceding theorem, and* $\leftrightarrow$ *is the sign of compatibility.*

PROOF. (i) If $a, b \in \mathcal{S}$, then

$$(a \vee b') \wedge b = [((ab)+b')+b']\, b = ((ab)+b')b = (ab) \wedge b = ab,$$

and

$$(a \wedge b') \vee b = ((a' \vee b) \wedge b')' = (a'b')' = a+b.$$

(ii) If $(a, b)C$, then by (2.3) $a \wedge b = ab$, and from (i) $a \wedge b = ab = (a \vee b') \wedge b$. Hence $a \leftrightarrow b$. On the other hand, if $a \leftrightarrow b$ then $a \wedge b = (a \vee b') \wedge b = ab$, and $a \wedge b = (b \vee a') \wedge a = ba$, which implies $(a, b)C$.

The implications $a \leftrightarrow b \Leftrightarrow a \wedge b = ab$ follow from Proposition 2.(d) and from (2.3). Moreover, with the help of De Morgan's laws we can get $a \wedge b = ab \Leftrightarrow a \vee b = a+b$.

If $\mathcal{S}(0, 1, ', \cdot)$ is a $*$-structure, then let us denote by $l(\mathcal{S})$ the orthomodular lattice with partial ordering defined by $x \leqq y$ iff $xy = x$, and with the orthocomplementation $'$. Conversely, if $\mathcal{L}(0, 1, \vee, \wedge, \perp)$ is an orthomodular lattice, then denote by $s(\mathcal{L})$ the $*$-structure defined by $ab = a * b$, and $a' = a^{\perp}$.

Theorem 3. *Let* $\mathscr{L}$ *and* $\mathscr{S}$ *be an orthomodular lattice and a* $*$*-structure, respectively. Then*

(i) $l(s(\mathscr{L}))=\mathscr{L}$

and

(ii) $s(l(\mathscr{S}))=\mathscr{S}$.

PROOF. (i) is trivial because the partial orderings in $\mathscr{L}$ and $l(s(\mathscr{L}))$ coincide. (ii) is a simple consequence of Proposition 5.

References

[1] D. I. FOULIS, Baer*-semigroup, *Proc. Amer. Math. Soc.* **11** (1960), 648—654.
[2] I. G. KALMÁR, Conditional probability measures on proposition systems, *Publ. Math. (Debrecen)* **30** (1983), 101—115.
[3] G. KALMBACH, Orthomodular lattices, *Academic Press*, 1983.
[4] F. MAEDA, S. MAEDA, Theory of symmetric lattices, *Springer, Berlin*, 1970.
[5] J. C. T. POOL, Baer*-semigroup and the logic of quantum mechanics, *Common. Math. Phys.* **9** (1968), 118—141.
[6] J. C. T. POOL, Semi modularity and the logic of quantum mechanics, *Commun. Math. Phys.* **9** (1968), 212—228.

I. G. KALMÁR
DEPT. OF MATH.
UNIVERSITY OF DEBRECEN
H-4010 DEBRECEN, PF. 12

(Received March 16, 1983)