Problems connected with the constant regression of quadratic statistics on linear statistics

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1. Introduction and the results

Denote by $R(n, m)$ the set of n by m matrices with real entries. $B^* \in R(m, n)$ denotes the transpose of $B \in R(n, m) \cdot e \in R(n, 1)$ is the vector with all components 1.

Let the trace of $A=(a_{jk}) \in R(n, n)$ be different from zero, and let

$$Q = x^*Ax, \quad x = (x_j) \in R(n, 1).$$

The quadratic form

$$P(Q) = \frac{1}{n!} \sum \sum_{j,k=1}^{n} a_{jk} x_{i_j} x_{i_k}$$

is said to be the adjoint quadratic form of Q ([1], p. 799), where the first summation runs over all permutations $i_1, \ldots, i_n$ without repetition of the first n integers.

After a short computation we get that ([2])

$$P(Q) = x^*Cx, \quad x \in R(n, 1),$$

where $C \in R(n, n)$ is the matrix with diagonal elements

$$a = \frac{1}{n} \operatorname{tr} A \neq 0, \tag{1.1}$$

and the remaining entries are equal to

$$b = \frac{e^*Ae - \operatorname{tr} A}{n(n-1)}. \tag{1.2}$$

C is said to be the associated matrix of A. Since the eigenvalues of C are ([2], Theorem 4.3.)

$$a+(n-1)b, \quad a-b \tag{1.3}$$

C is positive semidefinite if and only if

$$n \operatorname{tr} A = e^*Ae, \quad \text{or} \quad e^*Ae = 0, \tag{1.4}$$

and positive definite if and only if

$$n \operatorname{tr} A > e^* Ae > 0.$$

Conditions (1.1) and (1.4) involve that we suppose that $a>0$ in our treatment. We shall see, we use several time the fact in the following that C is a positive definite or semidefinite matrix. For that we did not deal with the case of $a=0$, since C is positive semidefinite by virtue of (1.4) if $b=0$, i.e. all entries of C are equal to zero in this case.

Denote by $V_n \subset R(n, 1)$ the set of vectors with components different from zero satisfying the following condition: Using notation $\alpha^\nu=(\alpha_j^\nu)\in V_n$ if $\alpha=(\alpha_j)\in V_n$, then let

$$S_\nu = e^* \alpha^\nu \neq 0 \quad (\nu = 0, 1, 2, \ldots). \tag{1.5}$$

The aim of this paper is to prove the following Theorems.

Theorem 1.1. *Let $A\in R(n, n)$ with positive trace, and with positive definite or semidefinite associated matrix be given. Let $\alpha\in V_n$. Let $X\in R(n, 1)$ be a random vector variable. Let the components of X be a sample from a population, which has zero expectation, and a finite variance different from zero. Suppose that $Q=X^*AX$ has constant regression on $Y=\alpha^*X$. In this case the components of X are normally distributed random variables if and only if $e^*Ae=0$, and $\alpha=\gamma e$, where $\gamma\neq 0$ is an arbitrary constant.*

Author dealt in his papers [2] and [3] with the question, what sufficient conditions must be satisfied by vectors $\alpha\in V_n$ in order that statistics Q let have constant regression on Y, if the components of X are normally disributed. Theorem 1.1. says that this is possible if and only if $\alpha=e$ except for a constant factor. This result suggests us that this is the same in generally too. I.e. Q has constant regression on Y if and only if $\alpha=e$ except for a constant factor. The following two theorems show us that this statement does not hold.

Theorem 1.2. *Let $A\in R(n, n)$ with positive trace, and with positive definite or semidefinite associated matrix be given. Moreover let $\operatorname{tr} A=e^*Ae$. Let $\alpha\in V_n$. Assume that $X\in R(n, 1)$ is a random vector variable. Let the components of X be a sample from a population, which has zero expectation, and a finite variance $\sigma^2>0$. In this case $Q=X^*AX$ has constant regression on $Y=\alpha^*X$ if and only if the components of X are uniformly distributed random variables with two discontinuity points σ and $-\sigma$.*

Theorem 1.3. *Let $A\in R(n, n)$ with positive trace be given. Let $\alpha\in V_n$ be a vector with components $+1$, and -1. Let $X\in R(n, 1)$ be a random vector variable. Let the components of X be a sample from a population with symmetric distribution function on the origin, and with a finite variance $\sigma^2>0$. Let*

$$p = 1+\frac{b}{a}\left(\frac{S_1^2}{n}-1\right). \tag{1.6}$$

Then the statistic $Q=X^*AX$ has constant regression on $Y=\alpha^*X$

a) if and only if $1/p$ is a positive integer. In this case the characteristic function $f(t)$ of the components of X is the following.

$$f(t) = \left[\cos\left(t\sigma\sqrt{p}\right)\right]^{1/p}, \quad t \in R(1, 1).$$

b) If and only if p is an arbitrary negative number. In this case

$$f(t) = \left[\operatorname{ch}\left(t\sigma\sqrt{-p}\right)\right]^{1/p}, \quad t \in R(1, 1).$$

c) In the case of $p=0$ if and only if

$$f(t) = \exp\left(-\frac{\sigma^2}{2} t^2\right).$$

From Theorems 1.1. and 1.3. we get the following result (cp. [9], Theorem 3).

Corollary 1.1. *Let $A = R(n, n)$ with positive trace, and with positive definite, or semidefinite associated metrix be given. Let $X \in R(n, 1)$ be a random vector variable. Let the components of X be a sample from a population, which has zero expectation, and a finite variance $\sigma^2 > 0$. Then $Q = X^* A X$ has constant regression on $Y = e^* X$ either if $e^* A e = 0$, while the components of X are normally distributed random variables, or if $e^* A e > 0$ and if*

$$\frac{1}{p} = \frac{\operatorname{tr} A}{e^* A e}$$

is a positive integer. In this case the characteristic function of the components of X is

$$\left[\cos\left(t\sigma\sqrt{p}\right)\right]^{1/p}, \quad t \in R(1, 1).$$

In his paper [4] the author proved the following (Corollary 2). If the assumption of Corollary 1.1. of this paper is satisfied, and if the components of X are infinitely divisible random variables, then Q has constant regression on $e^* X$ if and only if $e^* A e = 0$, i.e. the components of X are normally distributed random variables. Theorem 5 of paper [5] asserts the same statement without the condition of the infinitely divisibility. From Corollary 1.1. of this paper the just mentioned Theorem is incomplete. Professor D. N. SHANBHAG was kind to call my attention to this circumtance in a letter.

These investigations would be completely closed, if Theorem 1.3. would be extended in the more generally case too, if $\alpha \in V_n$ is an arbitrary vector, and if the condition that the distribution function of the components of X is a symmetric function on the origin is not satisfied. To do this it needs the solution of a more complicated differential equation, as which was solved connected with the proof of Theorem 1.3. It seems it is very difficult to solve the just mentioned more general differential equation.

In section 2 we deal with Lemmata, which are necessary to the proof of the Theorems, and of Corollary 1.1. In section 3 we give the proof of the Theorems, and of Corollary 1.1.

2. Lemmata

Lemma 2.1. *Let $\alpha=(\alpha_j)\in V_n$, and let k and l be arbitrary positive add integers Then*

$$S_k S_l \leqq n S_{k+l} \tag{2.1}$$

with equality if and only if $\alpha=\gamma e$, where $\gamma\neq 0$ is an arbitrary real constant.

PROOF. From the statement 368 of [6] it can show the following inequality. If the real numbers a_j, b_j $(j=1, \ldots, n)$ satisfy inequality

$$a_1 \geqq \ldots \geqq a_n, \quad b_1 \geqq \ldots \geqq b_n, \tag{2.2}$$

then

$$\sum_{j=1}^{n} a_j \sum_{j=1}^{n} b_j \leqq n \sum_{j=1}^{n} a_j b_j, \tag{2.3}$$

with equality if and only if either $a_1=a_n$, or $b_1=b_n$.

Without loss of the generality let $\alpha_1\geqq \ldots \geqq \alpha_n$. Then the numbers

$$a_j = \alpha_j^k, \; b_j = \alpha_j^l \quad (j=1, \ldots, n)$$

satisfy inequalities (2.2). Thus inequality (2.3), i.e. inequality (2.1.) holds with equality if and only if $\alpha_1=\alpha_n$.

Lemma 2.2. *Let λ and ϱ be real numbers, and suppose that $\varrho>0$. Then*

$$f(t) = [\text{ch}\,(\lambda t)]^{-\varrho}, \quad t\in R(1, 1)$$

s an infinitely divisible characteristic function.

PROOF. Lemma 2 of the paper [9] says that if a, ϱ and λ are three real numbers, and suppose that $\varrho>0$, then

$$f_a(\lambda t) = [\text{ch}\,(\lambda t)+ia\,\text{sh}\,(\lambda t)]^{-\varrho}, \quad t\in R(1, 1)$$

is an infinitely divisible characteristic function. Since

$$\text{ch}\,(2\lambda t) = \text{ch}^2\,(\lambda t)+\text{sh}^2\,(\lambda t),$$

identity

$$f(\lambda t) = f_1\left(\frac{\lambda}{2}\,t\right) f_{-1}\left(\frac{\lambda}{2}\,t\right)$$

gives us the proof of the statement.

Lemma 2.3. *Let λ, and ϱ be real numbers, and suppose that $\varrho>0$. Then*

$$[\cos\,(\lambda t)]^{\varrho}, \quad t\in R(1, 1) \tag{2.4}$$

is a characteristic function if and only if ϱ is a positive integer.

PROOF. It is obviously that $\cos(\lambda t)$ is the characteristic function of the purly discrete uniform distribution with two discontinuity points λ, and $-\lambda$, respectively.

Lemma 2.3.1. *Let m be a real number, and assume that $m>0$. In this case* $(\cos t)^{m+1}$ *is a characteristic function if and only if* $(\cos t)^m$ *is a characteristic function.*

PROOF. Since $\cos t$ is a characteristic function, it is obvious if $(\cos t)^m$ is a characteristic function, then $(\cos t)^{m+1}$ is a characteristic function too.

Let us suppose that $(\cos t)^{m+1}$ is a characteristic function. Let M be an arbitrary positive integer, moreover let t_j $(j=1, \ldots, M)$ be real numbers different from one another. Then we get from the Bochner's Theorem that the symmetric matrix

$$A_{m+1} = \left([\cos (t_k - t_l)]^{m+1}\right)_{k,l=1}^{M}$$

is positive definite or semidefinite. Using identity

$$\cos (t_k - t_l) = \frac{1}{2}[e^{i(t_k - t_l)} + e^{i(t_l - t_k)}],$$

we get that

$$A_{m+1} = \frac{1}{2}[DA_m D^* + D^* A_m D]$$

where D is the diagonal matrix with diagonal elements e^{it_k} $(k=1, \ldots, M)$. and D^* denotes now the conjugate transpose of D. Thus

$$x^* A_{m+1} x = x^* (D^* A_m D) x \geqq 0, \quad x \in R(M, 1).$$

Since $D^* A_m D$ is a Hermitian matrix, we conclude from the last identity that $D^* A_m D$, thus A_m too is a positive definite, or semidefinite matrix. Since M, and t_j $(j=1, \ldots, M)$ are arbitrary, using Bochner's Theorem again, we get that $(\cos t)^m$ is a characteristic function in conformity with our statement.

It is obvious the following Lemma.

Lemma 2.3.2. *If ϱ is a positive integer, then* (2.4) *is a characteristic function.*

Lemma 2.3.3. *If $\varrho = N$ is a positive integer, then characteristic function* (2.4.) *has exactly N indecomposable factors.* ([7], 5.1.)

PROOF. Under the assumption of the Lemma, (2.4) is the characteristic function of the purly discrete uniform distribution with discontinuity points $(N-2k)\lambda$ $(k=0, 1, \ldots, N)$. Since these discontinuity points are the consecutive terms of a finite arithmetic sequence, the statement of the Lemma follows directly ([7], p. 124).

Lemma 2.3.4. *If $\varrho = r/s$, where $r<s$ are relative prime integers, then* (2.4) *is not a characteristic function.*

PROOF. If (2.4) would be a characteristic function, then we conclude from identity

$$[f(t)]^s = [\cos (\lambda t)]^r$$

that characteristic function $[\cos (\lambda t)]^r$ has at least $s>r$ indecomposable factors contradicting to the statement of Lemma 2.3.3.

Lemma 2.3.5. *If the rational number ϱ is not an integer, then* (2.4) *is not a characteristic function.*

PROOF. If $0<\varrho<1$, the statement is contained in Lemma 2.3.4. Let $\varrho=N+(r/s)$, where $N\geqq 1$ is an integer, moreover $r<s$ are relative prime integers. Let us suppose that (2.4) is a characteristic function. Using Lemma 2.3.1. we get that $[\cos(\lambda t)]^{r/s}$ is a characteristic function too, contradicting to Lemma 2.3.4.

Lemma 2.3.6. *If* $\varrho=\alpha$ *is an irrational number, then* (2.4) *is not a characteristic function.*

PROOF. Let $[\alpha]$ be the largest integer not greater then α, and let $\{\alpha\}=\alpha-[\alpha]$. It is obvious that $0<\{\alpha\}<1$. We can suppose that $0<\alpha<1$. Namely $(\cos t)^{\alpha}$, and $(\cos t)^{\{\alpha\}}$ are characteristic functions, or not in the same time by the Lemma 2.3.1.

Let now $\varrho=\alpha_1$, where $0<\alpha_1<1$ is an irrational number. If (2.4) is a characteristic function, then $[\cos(\lambda t)]^{M\alpha_1}$ is a characteristic function too for arbitrary positive integer M. Using Lemma 2.3.1., we get that $[\cos(\lambda t)]^{\alpha_M}$ is a characteristic function too, where $\alpha_M=\{M\alpha_1\}$. If we apply the well-known Theorem of *H.* WEIL [10], we get that the elements of the sequence $(\alpha_M)_{M=1}^{\infty}$ are everywhere dense on the interval $[0, 1]$. Therefore if $0<r<1$ is a rational number, there is a subsequence $(\alpha_M^{(r)})_{M=1}^{\infty}$ of $(\alpha_M)_{M=1}^{\infty}$ such that $\alpha_M^{(r)}\to r$, if $M\to\infty$. Thus we obtain from the Continuity Theorem that

$$\lim_{M\to\infty}[\cos(\lambda t)]^{\alpha_M^{(r)}}=[\cos(\lambda t)]^r$$

is a characteristic function. But this result contradicts to Lemma 2.3.4.

Lemmata 2.3.2., 2.3.4., 2.3.5. and 2.3.6., respectively, give us the proof of Lemma 2.3.

3. The proofs

3.1. We say that $Q=X^*AX$ has constant regression on $Y=\alpha^*X$ if the identity

$$E(Qe^{itY})=E(Q)E(e^{itY}),\quad t\in R(1, 1)$$

holds ([8], 6.2.). It can show ([1], Lemma 4.1.) if Q has constant regression on Y, then $P(Q)$ has constant regression on Y too. Thus the second characteristic ([7], 2.4.) $\varphi(t)=\lg f(t)$ satisfies the differential equation

$$a\sum_{j=1}^{n}\varphi''(\alpha_j t)+a\sum_{j=1}^{n}[\varphi'(\alpha_j t)]^2+b\sum_{\substack{j,k=1\\ j\neq k}}^{n}\varphi'(\alpha_j t)\,\varphi'(\alpha_k t)=-a\sigma^2 n \tag{3.1}$$

in the neighbourhood of the origin ([2]), where $\varphi''_{(0)}=-\sigma^2$. It can show by the help of (3.1) that all derivatives of $\varphi(t)$ exist. Namely

$$\varphi_{(0)}^{(2k+1)}=0\quad(k=0, 1, \ldots),$$

moreover the recursion-formula

$$\varphi_{(0)}^{(2v+2)}=\begin{cases}-\dfrac{1}{aS_{2v}}\displaystyle\sum_{k=1}^{v}\binom{2v}{2k-1}\varphi_{(0)}^{(2k)}\,\varphi_{(0)}^{(2(v-k+1))}\,(\alpha^{2k-1})^*\,C\alpha^{2(v-k)+1},\\ \qquad (v=1, 2, \ldots)\\ \varphi''_{(0)}=-\sigma^2\end{cases} \tag{3.2}$$

holds.

It can prove ([1], Theorem 4.3.), if C is a positive definite, or semidefinite matrix, then the characteristic function of the components of X is an antire function. In our case the representation

$$\varphi(t) = -\sum_{\nu=1}^{\infty} \frac{\varphi_{(0)}^{(2\nu)}}{(2\nu)!} t^{2\nu}, \quad t \in R(1,1) \tag{3.3}$$

holds in a neighbourhood of the origin.

Since the diagonal elements of matrix C are equal to a, and the remaining entries are equal to b, we get that

$$(\alpha^{2k-1})^* C \alpha^{2(\nu-k)+1} = (a-b) S_{2\nu} + b S_{2k-1} S_{2(\nu-k)+1} =$$
$$= [a+(n-1)b] S_{2\nu} - b[n S_{2\nu} - S_{2k-1} S_{2(\nu-k)+1}].$$

Thus formula (3.2) can be expressed in the following way.

$$\varphi_{(0)}^{(2\nu+2)} = -\left[1+(n-1)\frac{b}{a}\right] \sum_{k=1}^{\nu} \binom{2\nu}{2k-1} \varphi_{(0)}^{(2k)} \varphi_{(0)}^{(2(\nu-k+1))} + \tag{3.4}$$
$$+\frac{b}{a} \sum_{k=1}^{\nu} \binom{2\nu}{2k-1} \varphi_{(0)}^{(2k)} \varphi_{(0)}^{(2(\nu-k+1))} \left[n - \frac{S_{2k-1} S_{2(\nu-k)+1}}{S_{2\nu}}\right] \quad (\nu = 1, 2, \ldots),$$

where

$$n - \frac{S_{2k-1} S_{2(\nu-k)+1}}{S_{2\nu}} \geqq 0 \quad (k = 1, \ldots, \nu)$$

by Lemma 2.1., with equality if and only if $\alpha = \gamma c$ with parameter $\gamma \neq 0$.

3.2. *The proof of Theorem* 1.1.

In order that (3.3) let be the second characteristic of a normally distributed random variable, it is necessary the satisfaction of the condition $\varphi_{(0)}^{(4)} = 0$, i.e. the satisfaction of the equality

$$\alpha^* C \alpha = (a-b) \sum_{j=1}^{n} \alpha_j^2 + b\left(\sum_{j=1}^{n} \alpha_j\right)^2 = 0 \tag{3.5}$$

by $\alpha = (\alpha_j) \in V_n$ from (3.2). In this case C is a singular matrix, i.e. either $a = b$, or $a + (n-1)b = 0$ using (1.3). In consequence of (1.3) we have $b \neq 0$ in the both cases.

If $a = b$, then $\sum_{j=1}^{n} \alpha_j = 0$ by (3.5) contradicting to the condition (1.5).

Let $a + (n-1)b = 0$, i.e. $a - b = -nb$. Thus we get using (3.5) that $\varphi_{(0)}^{(4)} = 0$ if and only if

$$\left(\sum_{j=1}^{n} \alpha_j\right)^2 = n \sum_{j=1}^{n} \alpha_j^2,$$

i.e. $\alpha = \gamma e$ from the Schwarz inequality, where $\gamma \neq 0$ is a constant. In this case recursionformula (3.4) is reducible to

$$(3.6)\qquad \begin{cases} \varphi''_{(0)} = -\sigma^2, \\ \varphi^{(2\nu+2)}_{(0)} = -\dfrac{e^*Ae}{\operatorname{tr} A} \displaystyle\sum_{k=1}^{n} \binom{2\nu}{2k-1} \varphi^{(2k)}_{(0)} \varphi^{(2(\nu-k+1))}_{(0)} \quad (\nu = 1, 2, \dots). \end{cases}$$

In consequence of (1.1) and (1.2) we obtain

$$a+(n-1)b = \frac{e^*Ae}{n}.$$

Thus we have

$$\varphi^{(2\nu+2)}_{(0)} = 0 \quad (\nu = 1, 2, \dots)$$

by (3.6) if condition $\varphi^{(4)}_{(0)}=0$ is satisfied, i.e. in a neighbourhood of the origin

$$\varphi(t) = -\frac{\sigma^2}{2} t^2.$$

Since $f(t)$ is an entire function, we get that

$$(3.7)\qquad f(t) = \exp\left\{-\frac{\sigma^2}{2} t^2\right\}, \quad t \in R(1, 1),$$

which was to be proved.

3.3. *The proof of Theorem* 1.2.

If $b=0$, then differential equation (3.1) reduces to

$$(3.8)\qquad \sum_{j=1}^{n} \varphi''(\alpha_j t) + \sum_{j=1}^{n} [\varphi'(\alpha_j t)]^2 + n\sigma^2 = 0.$$

Moreover we get from (3.4) that the solution of (3.8) is independent of $\alpha \in V_n$. I.e. differential equation (3.8) reduce to

$$\varphi''(t) + [\varphi'(t)]^2 + \sigma^2 = 0,$$

which has the only solution $f(t)=\cos(\sigma t)$, since $\varphi(t)$ is the second characteristic of an uniformly distributed discret random variable with discontinuity points σ and $-\sigma$, respectively.

3.4. *The proof of Theorem* 1.3.

Since $\varphi(t)$ is an even real function, we get that $\varphi'(t)$ is an add, and $\varphi''(t)$ is an even function. Using this remark in (3.1) we obtain the differential equation

$$(3.9)\qquad y' + py^2 + \sigma^2 = 0$$

with p defined by (1.6) by the notation $y=y(t)=\varphi'(t)$. In following we give the solution of (3.9) under the initial condition

$$(3.10)\qquad y(0) = 0, \quad y'(0) = -\sigma^2 < 0$$

in a neighbourhood of the origin in the cases if $p>0$, and if $p<0$, respectively.

3.4.1. Let $p>0$. Separating the variables, (3.9) can be written in the form

$$-\frac{1}{\sigma^2}\frac{dy}{1+\left(y\frac{1}{\sigma}\sqrt{b}\right)^2}=dt. \tag{3.11}$$

Since $y(0)=0$, we receive that $\varphi(t)$ is an even function, consequently one is a real function ([4], Theorem), thus $y(t)$ is real function too.

Let us integrate both sides of (3.11) we have

$$y(t)=-\frac{\sigma}{\sqrt{p}}\operatorname{tg}\left[\sigma\sqrt{p}(t+\gamma)\right],$$

where γ is an arbitrary real constant. Let us differentiate the last function, we get according to the initial condition (3.10) that $y'(0)=-\sigma^2$ if and only if $\gamma=0$. Taking this into consideration, we obtain that

$$\varphi(t)=\lg f(t)=\lg\left\{C\left[\cos\left(\sigma\sqrt{p}\,t\right)\right]^{1/p}\right\}.$$

Using the initial condition $f(0)=1$ ((3.10)), we have $C=1$. Thus

$$f(t)=\left[\cos\left(\sigma\sqrt{p}\,t\right)\right]^{1/p} \tag{3.12}$$

in a neighbourhood of the origin. Since (3.12) is an entire function the representation (3.12) holds if t runs over the whole real line.

Using Lemma 2.3. we get the proof of the statement a).

3.4.2. Let $p=-q<0$. Separating the variables, (3.9) can be written in the form

$$\frac{1}{\sigma^2}\frac{dy}{\left(\frac{\sqrt{q}}{\sigma}y\right)^2-1}=dt, \tag{3.13}$$

where $y(t)$ is a real function. After the integration of the both sides of (3.13) we obtain that

$$y(t)=-\frac{\sigma}{\sqrt{q}}\operatorname{th}\left(t\sigma\sqrt{q}+\gamma\right),$$

where γ is an arbitrary real constant. Let us differentiate the last function, we get accordin to the inicial condition (3.10) that $y(0)=-\sigma^2$ if and only if $\gamma=0$. Thus

$$\varphi(t)=\lg f(t)=\lg\left\{c\left[\operatorname{ch}\left(\sigma\sqrt{q}\,t\right)\right]^{-1/q}\right\}.$$

Using the initial condition $f(0)=1$ ((3.10)), we have that $c=1$. Thus

$$f(t)=\left[\operatorname{ch}\left(\sigma\sqrt{q}\,t\right)\right]^{-1/q} \tag{3.14}$$

in a neighbourhood of the origin. Since (3.14) is an entire function, representation (3.14) holds if $t\in R(1, 1)$.

Taking Lemma 2.2. ibto consideration, we obtain of the proof of statement b).

3.4.3. The statement c) is trivial.

3.5. *The proof of the Corollary* 1.1.

The first statement follows immediatelly from Theorem 1.1. The conditions of Theorem 1.3. are satisfied too. Namely the characteristic function of the components of X is a real even function, and $p>0$.

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