A note on strongly regular near-rings

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Introduction

Throughout this note by a near-ring we mean a zerosymmetric right near-ring. A near-ring N is said to be *strongly regular* if $a \in Na^2$ for each $a \in N$. In this note we prove that if N is a strongly regular near-ring, then N has maximum condition on ideals if and only if N has minimum condition on ideals. We also present an affirmative answer to an open problem raised by MASON (See remark on page 33 of [5]).

If S is a nonempty subset of a near-ring N, we write $l(S)=\{x \in N/xS=\{0\}\}$ and $r(S)=\{x \in N/Sx=\{0\}\}$. It is easy to verify that if N is a near-ring without nonzero nilpotent elements then $l(S)=r(S)$ and this is an ideal of N. For terminology and notation, we refer to [7]. Recall that a near-ring N is called regular if for each $a \in N$ there exists $x \in N$ such that $a=axa$. A near-ring N is said to have the IFP if for any a, b in N, $ab=0$ implies $anb=0$ for each $n \in N$.

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For convenience, we first state a lemma whose proof follows from propositions 9.37 and 9.43 of [7].

Lemma 1. *Suppose N is a near-ring without nonzero nilpotent elements. Then*
(i) *N has the* IFP,
(ii) *For any $a, b \in N$, $ab=0$ implies $ba=0$,*
(iii) *$en=ene$ for any idempotent e and any $n \in N$.*

The following lemma has been proved in several papers (eg [8]). The converse is also true, indeed in a more general set-up (see theorem 12).

Lemma 2. *Suppose N is a strongly regular near-ring. Then*
(i) *N has no nonzero nilpotent elements,*
(ii) *N is regular.*

In ring theory it is well known that [3, proposition 2.28] if R is a strongly regular ring then every one sided ideal is two sided. But in case of near-rings MASON [5] showed that if N is a strongly regular near-ring with identity, then every N-subgroup of N is an ideal. Our next result is a generalization.

Lemma 3. *If N is a strongly regular near-ring, then $l(r(Na))=Na$ for each a in N. Moreover, each N-subgroup is an ideal of N.*

PROOF. Let $a\in N$. Since N is regular, there is an idempotent e in N such that $Na=Ne$. Clearly $Na\subset l(r(Na))$. If $x\in l(r(Na))$ then $xr(Na)=\{0\}$. Since e is an idempotent, by lemma 1 (i), we have $x-xe\in l(Ne)=r(Ne)=r(Na)$. Hence $x(x-xe)=0$. So $xe(x-xe)=0$, by lemma 1 (i) and hence $(x-xe)^2=0$. Therefore $x=xe\in Ne=Na$. Thus $Na=l(r(Na))$ is an ideal of N. It is now easy to show that each N-subgroup is an ideal of N.

Lemma 4. *If N is a strongly regular near-ring, then N is isomorphic to a subdirect product of near-rings N_i, where in each N_i, $N_i x=N_i$ for each nonzero x in N_i and each nonzero idempotent in N_i is a right identity.*

PROOF. From [2, theorem 3.2] and lemma 3, it follows that the intersection of all maximal ideals is $\{0\}$. Hence N is isomorphic to a subdirect product of simple near-rings N_i. But from lemma 3, for each $0\neq x\in N_i$, $N_i x$ is a nonzero ideal and hence $N_i x=N_i$. Thus each N_i has no zero divisors and hence each nonzero idempotent in N_i is a right identity.

Corollary 5. ([8, theorem 2]). *If in a strongly regular near-ring N, every nonzero homomorphic image contains a nonzero distributive element then N is isomorphic to a subdirect product of near-fields and hence $(N, +)$ is abelian.*

The proof of this corollary follows directly from lemma 4 and [7, theorem 8.3, p. 237].

Corollary 6. *Let N be a strongly regular near-ring. If N contains a left identity, then $(N, +)$ is abelian.*

Corollary 7. ([4, theorem 3]). *If in a near-ring N, for each x there exists $n(x)>1$ such that $x^{n(x)}=x$, then N is isomorphic to a subdirect product of near-rings N_i where N_i is either a near-field or $N_i x=N_i$ for each $0\neq x$ in N_i and each nonzero idempotent element of N_i is a right identity.*

In ring theory it is well known that [3, proposition 1.29] if R is a regular ring then for any a, b in R there is an element c in R such that $Ra+Rb=Rc$. We now obtain this result for strongly regular near-rings. But it is not known whether this result is true for regular near-rings.

Lemma 8. *If N is a strongly regular near-ring then for any a, b in N there is an element c such that $Na+Nb=Nc$.*

PROOF. Since N is regular, there exists idempotents e and f such that $Na=Ne$ and $Nb=Nf$. Now Ne is an ideal by lemma 3. So, $f=f^2=ff-f(f-fe)+f(f-fe)\in \in Ne+N(f-fe)$. Thus $Ne+Nf\subseteq Ne+N(f-fe)$. Clearly $f-fe\in Nf+Ne=Ne+Nf$. Hence $N(f-fe)\subseteq Ne+Nf$ and $Ne+N(f-fe)\subseteq Ne+Nf$. Therefore $Ne+Nf= =Ne+N(f-fe)$. Let g be an idempotent such that $N(f-fe)=Ng$. Clearly $ge=0$. By lemma 1 (ii), it follows that $eg=0$ and so by lemma 1 (iii) $e(e+g)=e(e+g)e=$

$=e^3=e$ and $g(e+g)=g(e+g)g=g^3=g$. Thus $e, g\in N(e+g)$ and so, $Ne+Ng\subseteq \subseteq N(e+g)$. Since $e+g\in Ne+Ng$, we have $N(e+g)\subseteq Ne+Ng$. Therefore $Ne+ +Ng=N(e+g)$ and $Na+Nb=Ne+Ng=N(e+g)$.

For a ring with an identity, it is well known that the minimum condition on right ideals implies the maximum condition on right ideals. Since in a strongly regular ring every one sided ideal is two sided ideal ([3, proposition 2.28]), it follows that in a strongly regular ring with identity the minimum condition on ideals implies the maximum condition on ideals. Theorems 3.2 and 2.7 of [6] shows that if N is a regular near-ring with *dcc* on N-subgroups, then it has maximum condition on N-subgroups. From lemma 3, it evidently follows that for a strongly regular near-ring the maximum condition on ideals implies the maximum condition on ideals. We now prove the converse of this result.

Theorem 9. *Suppose N is a strongly regular near-ring. If N has maximum condition on ideals, then it has minimum condition on ideals.*

PROOF. By lemma 3, for any a in N the principal ideal generated by a is Na. Since N has maximum condition on ideals, every ideal L of N is finitely generated. By lemma 8 L is a principal ideal and hence an annhilator ideal by lemma 3. Thus every ideal of N is an annhilator ideal.

Suppose $I_1\supseteq I_2\supseteq ...\supseteq I_n\supseteq ...$ is a chain of ideals. Then $l(I_1)\subseteq l(I_2)\subseteq ... ...\subseteq l(I_n)\subseteq ...$ is an ascending sequence of ideals and hence by the maximum condition there is a positive integer k such that $l(I_n)=l(I_k)$ for all $n\geqq k$. Hence $r(l(I_n))= =r(l(I_k))$ for all $n\geqq k$ and since every ideal is an annhilator ideal, it follows that $I_n=I_k$ for all $n\geqq k$. Thus N has minimum condition on ideals.

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In ring theory, it is well known that if every nilpotent element is in the centre then every idempotent is in the centre. Hence a regular ring without nilpotent elements is strongly regular. In [5], Mason raised the following question: If in a regular near-ring N each nilpotent element is in the centre, then is it true that N is strongly regular? In the following we prove an affirmative answer to this question. We now establish a generalized result. To this end we introduce the following

Definition 10. A near-ring N is said to be symmetric if $a^2=0$ implies $\langle a\rangle^2=\{0\}$, where $\langle a\rangle$ is the principal ideal generated by a.

Remark 11. The class of all symmetric near-rings contains the class of all near-rings without nilpotent elements, the class of all near-rings which have IFP and the class of all near-rings in which nilpotent elements are in the centre.

Theorem 12. *Suppose N is a regular near-ring which is also symmetric. Then N is a strongly regular near-ring.*

PROOF. We first show that N has no nonzero nilpotent elements. Let $s\in N$ with $s^2=0$. Then $\langle s\rangle^2=\{0\}$. If $s\neq 0$, then by regularity there exists a nonzero idempotent e in $\langle s\rangle$. Therefore $e=e^2\in\langle s\rangle^2=\{0\}$ and this is a contradiction. Thus N has no nonzero nilpotent elements. If $a\in N$ then $a=axa$ for some $x\in N$. Since

aa is an idempotent, by lemma 1 (iii) it follows that $ax=(axa)x=a(xa)x=$ $=x(xa)x(xa)\in Na$ and hence $a=axa\in Na^2$. Thus N is strongly regular.

By remark 11 and theorem 12 we have the following

Corollary 13. *Suppose N is a regular near-ring in which every nilpotent element is in the centre. Then N is strongly regular.*

We close this article with the following

Example 14. Let $N=\{0, 1, 2, 3, 4\}$. Define addition as moduolo 5 and multiplication as follows:

·	0	1	2	3	4
0	0	0	0	0	0
1	0	1	4	1	4
2	0	2	3	2	3
3	0	3	2	3	2
4	0	4	1	4	1

Then $(N, +, \cdot)$ is a near-ring (see Clay [1]). Clearly N is a regular near-ring with no nonzero nilpotent elements. But no nonzero idempotent is in the centre of N.

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