Publ. Math. Debrecen
41 / 1–2 (1992), 23–33

Simplicial faces in pure and factorial state spaces of operator algebras

By ADEL M. ZAKI (Abha)

1. Introduction

It is known by [10] and [1] that for a unital C^*-algebra A, the pure and factorial state spaces of A (see §2) can be written as a union of w^*-closed faces of the state space. In this work, we investigate the question when the pure and factorial state spaces of a C^*-algebra A can be written as unions of w^*-closed simplicial faces of the quasi-state space $Q(A)$.

The answer in the case of the factorial state space $\overline{F(A)}$ is easy. In fact, A is an abelian C^*-algebra if, and only if, $\overline{F(A)}$ is a union of w^*-closed simplicial faces of $Q(A)$. By a closed face we shall always mean a w^*-closed face.

For the pure state space $\overline{P(A)}$, we prove that the following conditions are equivalent.

1. $\overline{P(A)}$ is a union of simplicial faces of $Q(A)$.
2. If ψ_1, ψ_2 are two distinct equivalent pure states of A, then $(1/2)(\psi_1 + \psi_2) \notin \overline{P(A)}$
3. $F(A) \cap \overline{P(A)} = P(A)$.

Moreover, we show that if $\overline{P(A)}$ is a union of simplicial faces of $Q(A)$, A is postliminal and for all irreducible representations π of A,

$$\pi(A)/LC(H_\pi),$$

where $LC(H_\pi)$ denotes the compact operators on a Hilbert space H_π, is abelian.

2. Preliminaries

Let A be a C^*-algebra. If S is a subset of the dual space A^*, we denote by $\overline{S}$ the closure of S in the w^*-topology. We denote the state space of A by $S(A)$. $S(A)$ is convex and w^*-compact, if A is unital. Let $P(A)$ be the set of extreme points of $S(A)$, which we call the pure states of A. The pure state space of A is $\overline{P(A)}$. A state of A is said to be factorial if the von Neumann algebra generated by $\pi_\phi(A)$ is a factor, where π_ϕ is the GNS representation associated with ϕ. The factor state space of A is $\overline{F(A)}$, where $F(A)$ is the set of factorial states. The type I factorial states will be denoted by $F_I(A)$. The quasi-state space of A is the set of all positive linear functionals on A with norm less than or equal to 1.

Recall that, two pure states ϕ_1, ϕ_2 are said to be equivalent if π_{ϕ_1}, π_{ϕ_2} are unitarily equivalent. Let F be a w^*-closed face of $S(A)$, where A is a unital C^*-algebra. Then, F is a Choquet simplex if, and only if, F does not contain two distinct equivalent pure states of A ([3; Th 2.5] and [2; cor 3]).

Let A be an arbitrary C^*-algebra. Consider the following condition which will have a special significance throughout this work: "$\overline{P(A)}$ is a union of closed simplicial faces of $S(A)$".

Suppose that A is non-unital and let $\tilde{A}$ be the C^*-algebra obtained from A by adjoining an identity. The restriction map $r : \overline{S(\tilde{A})} \to Q(A)$ is an affine homeomorphism of $S(\tilde{A})$ onto $Q(A)$ which maps $\overline{P(\tilde{A})}$ onto $\overline{P(A)}$ and $\overline{F(\tilde{A})}$ onto $\overline{F(A)}$ (see, for example [11]). Since $\overline{P(\tilde{A})}$ and $\overline{F(\tilde{A})}$ are unions of closed faces of $S(\tilde{A})$ [1;10], it follows that $\overline{P(A)}$ and $\overline{F(A)}$ are unions of closed faces of $Q(A)$. Furthermore, $\overline{P(A)}$ (respectively $\overline{F(A)}$) is a union of closed simplicial faces of $Q(A)$ if, and only if, $\overline{P(\tilde{A})}$ (respectively $\overline{F(\tilde{A})}$) is a union of closed simplicial faces of $S(\tilde{A})$.

3. Main results

We start this section by the following definition:

Definition 3.1. A C^*-algebra A is a said to satisfy the condition $(*)$ if, and only if, whenever ψ_1, ψ_2 are two distinct equivalent pure states of A then

$$(1/2)(\psi_1 + \psi_2) \notin \overline{P(A)}.$$

The following result shows the connection between the above condition $(*)$ and simplicial faces of the state space of A.

Proposition 3.2. *Let A a unital C^*-algebra. Then A satisfies $(*)$ if, and only if, $\overline{P(A)}$ is a union of closed simplicial faces of $S(A)$.*

Proof. ($\longrightarrow$) Let A satisfy $(*)$. Let $\phi \in \overline{P(A)}$ and let F_ϕ be the smallest closed face of $S(A)$ which contains ϕ. Using [10], we have $\overline{P(A)}$ is a union of closed faces of $S(A)$ and hence $F_\phi \subseteq \overline{P(A)}$.

Suppose that F_ϕ is not a Choquet simplex. Then using [3; Th 2.5] and [2; cor 3] F_ϕ contains two distinct equivalent pure states of A, ψ_1, ψ_2 say. Since F_ϕ is convex, then

$$\psi = (1/2)(\psi_1 + \psi_2) \in F_\phi$$

and hence $\psi \in \overline{P(A)}$, which contradicts $(*)$. Thus F_ϕ is Choquet simplex. Finally, $\overline{P(A)}$ is the union of the simplicial faces F_ϕ ($\phi \in \overline{P(A)}$).

($\longleftarrow$) Suppose $(*)$ does not hold. Then there exist equivalent pure states ψ_1, ψ_2 such that $\psi_1 \neq \psi_2$ and

$$(1/2)(\psi_1 + \psi_2) \in \overline{P(A)}$$

Let $\phi = (1/2)(\psi_1 + \psi_2)$. Then $\psi_1, \psi_2 \in F_\phi$. If F is a closed face of $S(A)$ such that $\phi \in F \subseteq \overline{P(A)}$ then $F_\phi \subseteq F$, so $\psi_1, \psi_2 \in F$ and F is not a Choquet simplex (see [3; Th 2.5] and [2; cor 3]). Thus $\overline{P(A)}$ is not a union of closed simplicial faces of $S(A)$.

Remark. Note that, when A is unital, $\overline{P(A)}$ is a union of simplicial faces of $S(A)$ if, and only if, it is a union of simplicial faces of $Q(A)$.

Proposition 3.3. *Let A be a non-unital C^*-algebra. Then the following are equivalent.*

(i) *$\overline{P(A)}$ is a union of closed simplicial faces of $Q(A)$*
(ii) *$\tilde{A}$ satisfies $(*)$.*
(iii) *A satisfies $(*)$.*

Proof. (i) $\longleftrightarrow$ (ii)

As observed in section 2, $\overline{P(A)}$ is a union of closed simplicial faces of $Q(A)$ if, and only if, $\overline{P(\tilde{A})}$ is a union of closed simplicial faces of $S(\tilde{A})$, and the latter condition is equivalent, to $\tilde{A}$ satisfying $(*)$ (see proposition 3.2).

(iii) $\longrightarrow$ (ii)

Let ϕ_1 and ϕ_2 be distinct equivalent pure states of $\tilde{A}$. Since the restriction map $r : S(\tilde{A}) \longrightarrow Q(A)$ is (1$-$1), then $r(\phi_1) \neq r(\phi_2)$. Moreover,

$r(\phi_1)$ and $r(\phi_2)$ are both in $P(A)$, since ϕ_1 and ϕ_2 are distinct and equivalent. It is routine to check that $r(\phi_1)$ and $r(\phi_2)$ are equivalent (see, for example [11]). Hence by assumption,

$$(1/2)\big(r(\phi_1)+r(\phi_2)\big) \notin \overline{P(A)}$$
$$(1/2)(\phi_1+\phi_2) \notin r^{-1}(\overline{P(A)}) = \overline{P(\tilde{A})}$$

(ii) $\longrightarrow$ (iii)

Suppose $\tilde{A}$ satisfies $(*)$ and let ψ_1 and ψ_2 be distinct equivalent pure states of A. We show that

$$(1/2)(\psi_1+\psi_2) \notin \overline{P(A)}$$

Let $\widetilde{\psi}_i$ be the unique pure state extension of ψ_i to $\tilde{A}$ $(i=1,2)$. Then $\widetilde{\psi}_1$ and $\widetilde{\psi}_2$ are distinct and equivalent. By assumption, we have

$$(1/2)(\widetilde{\psi}_1+\widetilde{\psi}_2) \notin \overline{P(\tilde{A})}.$$

Since r is (1–1), then

$$r\big((1/2)(\widetilde{\psi}_1+\widetilde{\psi}_2)\big) \notin r(\overline{P(\tilde{A})}) = \overline{P(A)}$$

and thus we get (iii).

Remark. Combining proposition 3.3 with the remark after proposition 3.2, we see that, for any C^*-algebra A, A satisfies $(*)$ if, and only if, $\overline{P(A)}$ is a union of simplicial faces of $Q(A)$.

The next results illustrates the relation between commutativity and closed simplicial faces.

Proposition 3.4. *Let A be an arbitrary C^*-algebra. Suppose that $\overline{P(A)}$ can be written as a union of closed simplicial faces of $Q(A)$. Then*

(i) *A is of type I.*
(ii) *For all irreducible representations π of A on a Hilbert space H_π, $\pi(A) \supseteq LC(H_\pi)$ and $\pi(A)/LC(H_\pi)$ is abelian.*

Proof. Let π be an irreducible representation of A on a Hilbert space H_π. For (i), it is enough to prove that

$$\pi(A) \supseteq LC(H_\pi) \qquad \text{(see [9])}.$$

It is known by [5; 4.1.10] that either

$$\pi(A) \supseteq LC(H_\pi) \quad \text{or} \quad \pi(A) \cap LC(H_\pi) = (0)$$

Suppose that $\pi(A) \cap LC(H_\pi) = (0)$, then π is not one dimensional and so there exist distinct equivalent pure states ψ_1 and ψ_2 of $\pi(A)$. Therefore, $\psi_1 \circ \pi$ and $\psi_2 \circ \pi$ are distinct equivalent pure states of A. Then

$$S(\pi(A)) = S(\pi(A)/\pi(A) \cap LC(H_\pi)) \subseteq \overline{VS(\pi(A))}$$

(See GLIMM's results in [7]), where $VS(\pi(A))$ denotes the set of vector states of $\pi(A)$. So

$$(1/2)(\psi_1 + \psi_2) \in \overline{VS(\pi(A))} = \overline{P(\pi(A))} \quad ([7,\ \text{Th } 2])$$

and

$$(1/2)(\psi_1 + \psi_2) \circ \pi = (1/2)(\psi_1 \circ \pi + \psi_2 \circ \pi) \in \overline{P(A)},$$

which contradicts the fact that A satisfies $(*)$. Thus

$$\pi(A) \supseteq LC(H_\pi).$$

Now, we prove that $\pi(A)/LC(H_\pi)$ is abelian, for all irreducible representations π of A. Assume the contrary, then there exists some π with $\pi(A)/LC(H_\pi)$ not abelian. Hence, there exist distinct equivalent pure states ψ_1 and ψ_2 of $\pi(A)/LC(H_\pi)$. Then $\psi_1 \circ \pi$ and $\psi_2 \circ \pi$ are distinct equivalent pure states of A. Now by [8, lemma 9].

$$(1/2)(\psi_1 + \psi_2) \in S(\pi(A)/LC(H_\pi) \subseteq \overline{P(\pi(A))}$$

Hence

$$(1/2)(\psi_1 + \psi_2) \circ \pi \in \overline{P(A)}.$$

This contadicts the fact that A satisfies $(*)$.

Next, we are going to find another condition equivalent to $\overline{P(A)}$ being a union of closed simplicial faces.

Proposition 3.5. *Let A be a C^*-algebra. The following conditions are equivalent:*

(i) $F(A) \cap \overline{P(A)} = P(A)$ *that is,* $P(A)$ *is relatively closed in* $F(A)$
(ii) A *satisfies* $(*)$.

PROOF. (i) $\longrightarrow$ (ii) Suppose that condition $(*)$ does not hold for A. Therefore, there exist two distinct equivalent pure states of A, ψ_1, ψ_2 say, such that $(1/2)(\psi_1 + \psi_2) \in \overline{P(A)}$. Furthermore, using [4; 2.1 (ii)], we get

$$(1/2)(\psi_1 + \psi_2) \in F(A)$$

Finally, since $(1/2)(\psi_1 + \psi_2)$ is not pure, we get a contradiction.

(ii) $\longrightarrow$ (i) Let ϕ be in $F(A) \cap \overline{P(A)}$. By proposition 3.4, A is necessarily of type I. Then $\phi \in F_I(A) \cap \overline{P(A)}$, (where $F_I(A)$ denotes the set of factorial states of type I). Note that, by [4, §2], we have

$$\phi = \sum_{i=1}^{\infty} \lambda_i \phi_i \quad \text{where} \quad \lambda_i > 0, \quad \sum_{i=1}^{\infty} \lambda_i = 1$$

and $\{\phi_i\}$ are equivalent pure states of A. On the other hand, since $\phi \in \overline{P(A)}$, these exists a closed simplicial face F of $Q(A)$ which lies in $\overline{P(A)}$ and contains ϕ.

We prove that ϕ is pure. Suppose not, then without loss of generality, we can assume that $\phi_1 \neq \phi_2$. To reach a contradiction, it is sufficient to show that $\phi_1,\ \phi_2 \in F$ (see [3, Th. 2.5] and [2., cor 3]). Note that $\phi = \lambda_1 \phi_1 + \psi$ where ψ is the rest of the infinite sum. By considering an (approximate) identity for A we obtain that

$$\|\psi\| = 1 - \lambda_1$$

Consider

$$\psi_\circ = \frac{1}{1-\lambda_1} \psi \in S(A).$$

So $\phi = \lambda_1 \phi_1 + (1-\lambda_1)\psi_\circ$, which implies that $\phi_1 \in F$. Similarly, we can prove that ϕ_2 is in F. This contradicts the fact that F is simplicial and so ϕ must be pure.

We end this section by summarizing the above results in the following theorem.

Theorem 3.6. *Let A be an arbitrary C^*-algebra. Then the following are equivalent:*

(i) *$\overline{P(A)}$ a union of closed simplicial faces of $Q(A)$.*
(ii) *whenever ψ_1 and ψ_2 are distinct equivalent pure states of A, then*

$$(1/2)(\psi_1 + \psi_2) \notin \overline{P(A)}$$

(iii) *$\overline{P(A)} \cap F(A) = P(A)$.*

4. Examples and related results

In this section, we consider some examples of type I C^*-algebras with and without property $(*)$.

Let A be a C^*-algebra. A point $\pi_0 \in \hat{A}$ is said to be singular [8, p160], if there is an $E \in A$ with $\pi(E)$ a projection for all π in some neighbourhood N of π_0, with $\pi_0(E)$ one dimensional, and such that for each neighbourhood M of π_0 contained in N, there exists π in M so that $\dim \pi(E) > 1$. If π_0 is not singular π_0 is called regular.

Let D be the C^*-algebra of all bounded sequences $x = (x_n)_{n\geq 1}$ of 2×2 complex matrices with coordinatewise operations and

$$\|x\| = \sup_n \|x_n\|$$

Let A be the C^*-subalgebra of D consisting of all $x = (x_n)$ such that x_n converges in norm to a matrix of the form

$$\begin{pmatrix} \lambda(x) & 0 \\ 0 & \lambda(x) \end{pmatrix}, \qquad \text{as} \quad n \to \infty.$$

By [6, Th 1.1. and following], $\hat{A}$ is homeomorphic to $N \cup \{\infty\}$, each $n \in N$ corresponding to a 2-dimensional representation π_n where $\pi_n(x) = x_n$ and ∞ to the 1-dimensional representation π_λ given by $\pi_\lambda(x) = \lambda(x)$. Define $E \in A$ such that

$$E = (E_n) \quad \text{and} \quad E_n = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{for all} \quad n.$$

That is, E is the identity of A. Notice that

$$\pi_\lambda(E) = 1 \quad \text{and} \quad \pi_n(E) = E_n$$

is a projection of dimension 2, so that π_λ is singular. On the other hand,

$$\pi_\lambda(A) = \{\lambda(a) : a \in A\}$$

is 1-dimensional. Now applying [8,Th 5], we have

$$\overline{P(A)} = P(A) = \bigcup_{\phi \in P(A)} \{\phi\}$$

and so we can see directly that A satisfies $(*)$ and that $\overline{P(A)}$ is a union of closed simplicial faces of $S(A)$ (in a trivial way).

We show next how tensoring with $M_2(C)$ can destroy property $(*)$.

Let C be the C^*-algebra of all sequences $x = (x_n)_{n\geq 1}$ of 4×4 matrices for which $\sup_n \|x_n\|$ is finite, with coordinatewise operations and

$$\|x\| = \sup_n \|x_n\|$$

Let B be the C^*-subalgebra of C consisting of all $x = (x_n)$ such that x_n converges in norm to a matrix of the form

$$\begin{pmatrix} a(x) & b(x) & 0 & 0 \\ c(x) & d(x) & 0 & 0 \\ 0 & 0 & a(x) & b(x) \\ 0 & 0 & c(x) & d(x) \end{pmatrix} \quad \text{as,} \quad n \to \infty$$

for some complex numbers $a(x)$, $b(x)$, $c(x)$ and $d(x)$. We write

$$M(x) = \begin{pmatrix} a(x) & b(x) \\ c(x) & d(x) \end{pmatrix}.$$

$\hat{B}$ is homeomorphic to $N \cup \{\infty\}$, each $n \in N$ corresponding to a 4-dimensional representation π_n given by $\pi_n(x) = x_n$ and ∞ to the 2-dimensional representation π_M given by $\pi_M(x) = M(x)$. In this example, we show that $\overline{P(B)}$ cannot be written as a union of closed simplicial faces of $S(B)$. Consider

$$e_1, e_2 \in C^2 \quad \text{and} \quad \xi_1, \xi_4 \in C^4$$

where

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \; e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \; \xi_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \xi_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Since ξ_1 and ξ_4 are orthogonal unit vectors in C^4, then they are linearly independent. Finally, using the definition of B, we can prove that.

$$(1/2)(w_{e_1} \circ \pi_M + w_{e_2} \circ \pi_M) = w^* - \lim w_{\frac{\xi_1 + \xi_4}{\sqrt{2}}} \circ \pi_n$$

where

$$w_{e_i}(A) = \langle Ae_i, e_i \rangle \quad \text{for all} \quad A \in M_2(C), \quad i = 1, 2\ .$$

Thus, there exist two distinct equivalent pure states of B, $\phi_1 = w_{e_1} \circ \pi_M$, $\phi_2 = w_{e_2} \circ \pi_M$ such that

$$(1/2)(\phi_1 + \phi_2) \in \overline{P(B)}.$$

So $(*)$ fails in this example and by proposition 3.2, $\overline{P(B)}$ is not a union of closed simplicial faces of $S(B)$.

We note that in [11. proposition 3.3.5], $\overline{P(B)}$ is explicitly determined:

$$\overline{P(B)} = \{w_\xi \circ \pi_n : n = 1, 2, \ldots \quad \text{and} \quad \xi \quad \text{is a unit vector in } C^4\}$$
$$\cup \{\psi \circ \pi_M : \psi \quad \text{is any state of } M_2(C)\}$$

It is also shown in [11, Proposition 3.3.11] that if a C^*-algebra C is defined by changing the definition of B to allow the limit matrix to be

$$\begin{pmatrix} M(x) & 0 \\ 0 & N(x) \end{pmatrix}$$

where $M(x)$, $N(x) \in M_2(C)$, then

$$\begin{aligned}\overline{P(C)} = &\{w_\xi \circ \pi_n : n = 1, 2, \dots \quad \text{and } \xi \text{ is a unit vector in } C^4\} \\ &\cup\{a(w_\xi \circ \pi_M) + (1-\alpha)(w_\eta \circ \pi_N) : 0 \le \alpha \le 1 \text{ and } \xi, \eta \text{ are} \\ &\quad \text{unit vectors in } C^2\}\end{aligned}$$

and hence $\overline{P(C)}$ is a union of simplicial faces of $S(A)$ (singletons and line segments).

Finally, we note that if A is a C^*-algebra such that $LC(H_\pi) \subseteq A \subseteq L(H_\pi)$ and $A/LC(H_\pi)$ is abelian then

$$\overline{P(A)} = \cup\{F_\xi : \xi \quad \text{is a unit vector in} \quad H_\pi\}$$

where

$$F_\xi = \{\alpha w_\xi + (1-\alpha)g : 0 \le \alpha \le 1,\ g \in S(A)/LC(H_\pi)\}$$

a simplicial closed face of $S(A)$ [11, §3]. In this connection, see proposition 3.4 (ii).

5. Simplicial faces in factorial state spaces of a C^*-algebra

In this section, we find a necessary and sufficient condition for the factorial state space of a C^*-algebra A to be a union of closed simplicial faces.

Let $F(A)$ be the set of all ϕ in $S(A)$ such that $\pi_\phi(A)'$ is a factor. We define the factorial state space of A as the w^*-closed of $F(A)$ and we denote it by $\overline{F(A)}$.

Proposition 5.1. *Let A be a unital C^*-algebra. Then A is abelian if, and only if, $\overline{F(A)}$ is a union of closed simplicial faces of $S(A)$.*

Proof. $(\longrightarrow)$ if A is abelian, then

$$\overline{F(A)} = \overline{P(A)} = P(A).$$

Hence $\overline{F(A)} = \bigcup_{\phi \in P(A)} \{\phi\}$, a union of closed simplicial faces of $S(A)$.

($\longleftarrow$) Suppose A is not abelian. Then there exists an irreducible representation π with $\dim H_\pi > 1$. Choose $\xi_1, \xi_2 \in H_\pi$ so that they are linearly independent unit vectors. Let

$$\psi_1(a) = \langle \pi(a)\xi_1, \xi_2\rangle \quad \text{and}$$
$$\psi_2(a) = \langle \pi(a)\xi_2, \xi_2\rangle \quad \text{for all} \quad a \in A.$$

It is easy to check that ψ_1, ψ_2 are distinct equivalent pure states of A. Let $\phi = (1/2)(\psi_1 + \psi_2)$. By [4, Th 2.1], we get

$$\phi \in F_I(A) \ (\subseteq \overline{F(A)})$$

Finally, ϕ does not belong to any closed simplicial faces of $S(A)$. For, suppose F is a face of $S(A)$ such that $\phi \in F$. Therefore, ψ_1, $\psi_2 \in F$ and F is not a simplex.

In the non-unital case, consider the restriction map r given by $r : S(\tilde{A}) \longrightarrow Q(A)$, where $\tilde{A}$ is the C^*-algebra obtained from A by the adjoining of an identity. Now since

$$r(F(\tilde{A})) = F(A) \cup \{0\}$$

and

$$0 \in \overline{P(A)} \subset \overline{F(A)}, \quad [5; 2.12.13], \text{ we obtain}$$
$$r(\overline{F(\tilde{A})}) = \overline{F(A)}.$$

Proposition 5.2. *Let A be a non-unital C^*-algebra. Then the following are equivalent:*

(i) *A is abelian*
(ii) *$\overline{F(A)}$ is a union of closed simplicial faces of $Q(A)$*

Proof. It is clear that A is abelian if, and only if, $\tilde{A}$ is abelian. Moreover, since r is an affine homeomorphism, $\overline{F(A)}$ is a union of closed simplicial faces of $Q(A)$ if, and only if, $\overline{F(\tilde{A})}$ is a union of closed simplicial faces of $S(\tilde{A})$. The result then follows from proposition 5.1.

Acknowledgements. This work forms part of my thesis [11] for the degree of Ph.D. at the University of Aberdeen. I would like to express my gratitude to my supervisor Dr. R.J. Archbold for his help and advice during this work. Also, many thanks to Dr. C.J.K. Batty for his useful remarks and discussions which helped me to finish this work.

References

[1] R. J. Archbold and C. J. K. Batty, On factorial states of operator algebras III, *J. Operator Theory* **15** (1986), 53–81.

[2] C. J. K. Batty, Abelian faces of state space of C^*-algebras, *Comm. Math. Phys.* **75** (1980), 43–50.

[3] C. J. K. Batty, Simplexes of States of C^*-algebras, *J. Operator Theory* **4** (1980), 3–23.

[4] C. J. K. Batty and E. J. Archbold, On factorial states of operator algebras II, *J. Operator Theory* **13** (1985), 131–142.

[5] J. Dixmier, C^*-algebras, *North-Holland*, *Amsterdam*, 1977.

[6] J. M. G. Fell, The structure of algebras of operator fields, *Acta. Math.* **106** (1961), 233–280.

[7] J. Glimm, A Stone–Weierstrass theorem for C^*-algebras, *Ann. of Math.* **72** (1960), 216–244.

[8] J. Glimm, Type I C^*-algebras, *Ann. of Math.* **73** (1961), 572–612.

[9] S. Sakai, On type I C^*-algebras, *Proc. Amer. Math. Soc.* **18** (1967), 861–863.

[10] F. W. Shultz, Pure states as a dual object for C^*-algebras, *Comm. Math. Phys.* **82** (1982), 497–509.

[11] A. M. Zaki Ph. D., Thesis, *Alerdeen University*, 1987.

A. M. ZAKI
ABHA TEACH. JUN. TRAIN. COLLEGE
ABHA, P.O. BOX 249
SAUDI ARABIA

(Received June 22, 1990)