

On the cotangent bundle of a differentiable manifold

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Abstract. We study some geometric properties of the cotangent bundle T^*M of a differentiable manifold M , endowed with an arbitrary asymmetric nonlinear connection. It is obtained a pseudo-Riemannian metric G on T^*M , of Riemann extension type, for which we study the conditions under which it is either flat, or projectively flat, or conformally flat or locally symmetric. Further, by using an almost complex structure J or an almost product structure P on T^*M , defined by the same nonlinear connection and an M -tensor field on T^*M , we obtain some results concerning the property of (T^*M, J, G) (of (T^*M, P, G)) to be a Kaehler manifold with Norden metric (to be a para-Kaehler manifold).

Introduction

In [5] (see also [6], [7], [8]) the present authors have studied the properties of a *pseudo-Riemannian* metric G on the cotangent bundle T^*M of a manifold M by using a *symmetric nonlinear connection* on this bundle. The pseudo-Riemannian metric G on T^*M is very much similar to Riemann extension considered in [10], [11].

The purpose of the present paper is to study some properties of a similar pseudo-Riemannian metric G on the *cotangent bundle* T^*M by using an arbitrary nonlinear connection on this bundle. We get that the considered pseudo-Riemannian metric G on T^*M is determined only by the symmetric part of the considered nonlinear connection on T^*M and we show that the geometric properties of pseudo-Riemannian manifold

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(T^*M, G) which depend only on the pseudo-Riemannian metric G and on the Levi Civita connection $\tilde{\nabla}$ of G depend also only on the symmetric part of the considered nonlinear connection on T^*M . Hence, for the study of such geometric properties of (T^*M, G) we can assume from the beginning that the considered nonlinear connection on T^*M is symmetric. Then, we consider some additional structures on T^*M and we shall see that their properties do not depend only on metric G and its Levi Civita connection $\tilde{\nabla}$ but also on the skew-symmetric part of the considered nonlinear connection. So, we can define an almost complex structure J and an almost product structure P on T^*M and we then get the conditions under which (T^*M, J, G) is a Kaehlerian manifold with Norden metric (see [2], [7]) and the conditions under which (T^*M, P, G) is a parakaehlerian manifold (see [1], [8]). Some classes of manifolds whose cotangent bundles carry parakaehlerian structures are also presented (Theorems 9, 12, 14, 16, 18). Remark that the assumption that the considered nonlinear connection on T^*M is not necessarily symmetric it is essential in order to obtain these classes of manifolds.

The manifolds, tensor fields and geometric objects we consider in this paper, are assumed to be differentiable of class C^∞ . We use the well known summation convention, the range for the indices i, j, k, h, l, s, t being always $\{1, 2, \dots, n\}$. We shall denote by $\Gamma(T^*M)$ the module of smooth vector fields on T^*M .

1. The pseudo-Riemannian manifold (T^*M, G)

Let M be an n -dimensional smooth manifold and denote by $\pi : T^*M \rightarrow M$ its cotangent bundle with fibres the cotangent spaces to M . Then T^*M is a $2n$ -dimensional smooth manifold and some local charts induced naturally from local charts on M , may be used. Namely, if $(U, x^i); i = 1, \dots, n$ is a local chart on M , then the local chart $(\pi^{-1}(U), q^i, p_i); i = 1, \dots, n$ is defined on T^*M , where $q^i = x^i \circ \pi; i = 1, \dots, n$, i.e. the first n local coordinates of a cotangent vector from $\pi^{-1}(U)$ are the local coordinates of its base point, thought of as functions on $\pi^{-1}(U) \subset T^*(M)$ and $p_i; i = 1, \dots, n$ are the vector space coordinates with respect to the natural frame (dx^1, \dots, dx^n) in T^*M defined by the local chart $(U, x^i); i = 1, \dots, n$. The M -tensor fields and the linear M -connections may be considered on T^*M and the usual tensor fields and linear connections on the base manifold M may be thought of naturally as M -tensor fields and linear M -connections on T^*M (see [10], [5]). Let $VT^*M = \text{Ker } \pi_* \subset TT^*M$ be the vertical distribution over T^*M . Then VT^*M is involutive with fibre dimension n and the local vector fields

$(\partial^i = \frac{\partial}{\partial p_i})$; $i = 1, \dots, n$ define a local frame in VT^*M . A nonlinear connection on T^*M is defined by a complementary distribution HT^*M (called the horizontal distribution) to VT^*M in TT^*M . A local frame in HT^*M , related to the induced local chart $(\pi^{-1}(U), q^i, p_i)$ is defined by the local vector fields $(\delta_i = \frac{\delta}{\delta q^i})$; $i = 1, \dots, n$, where:

$$\delta_i = \frac{\delta}{\delta q^i} = \frac{\partial}{\partial q^i} - N_{ij} \frac{\partial}{\partial p_j}.$$

The functions $N_{ij} = N_{ij}(q, p)$; $i, j = 1, \dots, n$, are the connection coefficients of the considered nonlinear connection in the induced local chart $(\pi^{-1}(U), q^i, p_i)$; $i = 1, \dots, n$, and we assume that this nonlinear connection is not necessarily symmetric. Then we have

$$(1) \quad TT^*M = VT^*M \oplus HT^*M$$

and (∂^i, δ_i) ; $i = 1, \dots, n$, is a local frame in T^*M adapted to the direct sum decomposition (1). The system of local 1-forms $(\delta p_i, dq^i)$; $i = 1, \dots, n$, where

$$\delta p_i = dp_i + N_{ji} dq^j$$

is the dual local frame of the local frame (∂^i, δ_i) ; $i = 1, \dots, n$. Then:

$$(2) \quad \left[\frac{\partial}{\partial p_i}, \frac{\delta}{\delta q^j} \right] = \Phi_{jk}^i \frac{\partial}{\partial p_k}; \quad \left[\frac{\delta}{\delta q^i}, \frac{\delta}{\delta q^j} \right] = -R_{kij} \frac{\partial}{\partial p_k}$$

where

$$(3) \quad \Phi_{jk}^i = -\frac{\partial N_{jk}}{\partial p_i}; \quad R_{kij} = \frac{\delta N_{jk}}{\delta q^i} - \frac{\delta N_{ik}}{\delta q^j}$$

and the integrability of the differential system on T^*M defined by HT^*M is equivalent to the vanishing of the components R_{kij} on $\pi^{-1}(U)$.

Remark that the components Φ_{ij}^k define a linear M -connection while the components R_{kij} define an M -tensor field of type (0,3) on T^*M .

Consider the following pseudo-Riemannian metric G on T^*M of Riemann extension type

$$(4) \quad G = 2\delta p_i dq^i = 2dp_i dq^i + (N_{ji} + N_{ij}) dq^j dq^i$$

where δp_i is defined by an arbitrary nonlinear connection N_{ij} on T^*M . Remark that the pseudo-Riemannian metric G depends only on the symmetric part of the considered nonlinear connection on T^*M , the distributions VT^*M and HT^*M on T^*M are both isotropic with respect to G and the signature of G is (n, n) .

Denote by $\tilde{\nabla}$ the Levi Civita connection of the considered pseudo-Riemannian metric G on T^*M .

Then the following result is proved by a straightforward computation.

Proposition 1. *The local coordinate expression of $\tilde{\nabla}$ in the local frame (∂^i, δ_i) adapted to the direct sum decomposition (1) is:*

$$\begin{aligned}\tilde{\nabla}_{\partial^i} \partial^j &= 0; & \tilde{\nabla}_{\delta_i} \partial^j &= -\frac{1}{2}(\Phi_{ik}^j + \Phi_{ki}^j) \partial^k; & \tilde{\nabla}_{\partial^i} \delta_j &= \frac{1}{2}(\Phi_{jk}^i - \Phi_{kj}^i) \partial^k; \\ \tilde{\nabla}_{\delta_i} \delta_j &= \frac{1}{2}(\Phi_{ij}^k + \Phi_{ji}^k) \delta_k + \frac{1}{2}(R_{ijk} - R_{jki} - R_{kij}) \partial^k\end{aligned}$$

where Φ_{jk}^i and R_{kij} are given by (3).

Remark. From Proposition 1 it follows that the essential coefficients of the local coordinate expression of $\tilde{\nabla}$ in the local adapted frame (∂^i, δ_i) ; $i = 1, \dots, n$ are only the symmetric part and the skew-symmetric part of the linear M -connection defined by Φ_{jk}^i on T^*M and the M -tensor field on T^*M defined by $\frac{1}{2}(R_{ijk} - R_{jki} - R_{kij})$. On the other hand, taking into account that the pseudo-Riemannian metric G depends only on the symmetric part of the considered nonlinear connection defined by the coefficients N_{ij} on T^*M we can consider the symmetric nonlinear connection defined by the connection coefficients \bar{N}_{ij} on T^*M determined by the symmetric part of the coefficients N_{ij} , i.e. $\bar{N}_{ij} = \frac{1}{2}(N_{ij} + N_{ji})$. We obtain another horizontal distribution $\overline{HT^*M}$ defined by the coefficients \bar{N}_{ij} and the local vector fields $(\bar{\delta}_i = \frac{\bar{\delta}}{\delta q^i})$; $i = 1, \dots, n$, where

$$\bar{\delta}_i = \frac{\bar{\delta}}{\delta q^i} = \frac{\partial}{\partial q^i} - \bar{N}_{ij} \frac{\partial}{\partial p_j}$$

define a local frame in $\overline{HT^*M}$. Then $(\partial^i, \bar{\delta}_i)$; $i = 1, \dots, n$ is a local frame in T^*M adapted to the direct sum decomposition:

$$(5) \quad TT^*M = VT^*M \oplus \overline{HT^*M}.$$

Denote by $(\bar{\delta}p_i, dq^i)$; $i = 1, \dots, n$ the local dual frame of the local frame $(\partial^i, \bar{\delta}_i)$; $i = 1, \dots, n$. Then we have

$$\bar{\delta}p_i = dp_i + \bar{N}_{ij} dq^j$$

and the pseudo-Riemannian metric G defined by (4) on T^*M becomes

$$G = 2\bar{\delta}p_i dq^i = 2(dp_i + \bar{N}_{ij} dq^j) dq^i.$$

If we denote by T_{ij} the components defined by the skew-symmetric part of the componets N_{ij} , i.e. $T_{ij} = \frac{1}{2}(N_{ij} - N_{ji})$, then the components T_{ij} define an M -tensor field on T^*M and we have

$$\delta_i = \bar{\delta}_i - T_{ij}\partial^j$$

and from Proposition 1 we get by a straightforward computation

Proposition 2. *The local coordinate expression of the Levi Civita connection $\tilde{\nabla}$ of G in the local frame $(\partial^i, \bar{\delta}_i)$; $i = 1, \dots, n$ adapted to the direct sum decomposition (5) is:*

$$\begin{aligned} \tilde{\nabla}_{\partial^i}\partial^j &= 0; & \tilde{\nabla}_{\bar{\delta}_i}\partial^j &= -\bar{\Phi}_{ik}^j\partial^k; & \tilde{\nabla}_{\partial^i}\bar{\delta}_j &= 0; \\ \tilde{\nabla}_{\bar{\delta}_i}\bar{\delta}_j &= \bar{\Phi}_{ij}^k\bar{\delta}_k + \bar{R}_{ijk}\partial^k \end{aligned}$$

where the coefficients $\bar{\Phi}_{ij}^k$ and \bar{R}_{ijk} are given by

$$(6) \quad \bar{\Phi}_{ij}^k = -\partial^k\bar{N}_{ij}; \quad \bar{R}_{kij} = \bar{\delta}_i\bar{N}_{jk} - \bar{\delta}_j\bar{N}_{ik}.$$

Remark. The result from Proposition 2 is well known from [5], [6] for the case where the considered nonlinear connection on T^*M is symmetric. Also, from Proposition 2 it follows that the essential coefficients of the local coordinate expression of $\tilde{\nabla}$ in the local adapted frame $(\partial^i, \bar{\delta}_i)$; $i = 1, \dots, n$ are only $\bar{\Phi}_{ij}^k$ and \bar{R}_{kij} given by (6) and they are expressed by using only the symmetric part of the arbitrary nonlinear connection defined by the coefficients N_{ij} .

Since the geometric properties of (T^*M, G) which depend only on the metric G and its Levi Civita connection $\tilde{\nabla}$ are independent of the choice of either the horizontal distribution HT^*M or $\overline{HT^*M}$ on T^*M , we have, from Proposition 2:

Theorem 3. *The geometric properties of the pseudo-Riemannian manifold (T^*M, G) to be either flat, or projectively flat, or conformally flat or locally symmetric which depend only on the metric G and its Levi Civita connection $\tilde{\nabla}$ are expressed only in the terms of the symmetric part of the considered nonlinear connection on T^*M . Hence, in order to study such geometric properties of (T^*M, G) we can assume from the begining that the considered nonlinear connection on T^*M is symmetric.*

The above properties of (T^*M, G) have been studied by the present authors in [5], [6], assuming that the considered nonlinear connexion on T^*M is symmetric.

2. An almost complex structure on (T^*M, G)

In this section we consider an arbitrary nonlinear connection on T^*M and study an additional structure naturally defined on T^*M .

We may define an almost complex structure J on T^*M and study the conditions under which the pseudo-Riemannian metric G given by (4) is a Norden metric for the defined almost complex structure J . Next we get necessary and sufficient conditions for (T^*M, J, G) to be a Kaehlerian manifold with Norden metric (see [2], [7]).

Assume that the components $g_{jk}; j, k = 1, \dots, n$, define a nondegenerate M -tensor field of type $(0, 2)$ on T^*M . Denote by $g^{jk}; j, k = 1, \dots, n$, the components of its inverse matrix, i.e.

$$(7) \quad g_{ih}g^{kh} = g_{hi}g^{hk} = \delta_i^k.$$

Then the components $g^{jk}; j, k = 1, \dots, n$, define an M -tensor field of type $(2, 0)$ on T^*M . Define the automorphism J of TT^*M expressed in local frame adapted to the direct sum decomposition (1) by

$$(8) \quad J\left(\frac{\delta}{\delta q^i}\right) = g_{ji} \frac{\partial}{\partial p_j}; \quad J\left(\frac{\partial}{\partial p_i}\right) = -g^{ij} \frac{\delta}{\delta q^j}.$$

It follows by a straightforward computation that J defines an almost complex structure on T^*M (see also [7]). We have

Proposition 4. *The pseudo-Riemannian metric G defined by (4) is a Norden metric for the almost complex structure J defined by (8) if and only if the M -tensor field g_{jk} is symmetric, i.e. $g_{jk} = g_{kj}$.*

PROOF. It follows easily by direct verification for the local vector fields δ_i, ∂^i that $G(JX, JY) = -G(X, Y); \forall X, Y \in \Gamma(T^*M)$, if and only if $g_{jk} = g_{kj}$.

In the following assume that g_{ij} is a nondegenerate symmetric M -tensor field on T^*M and denote by (T^*M, J, G) the almost complex manifold T^*M with the almost complex structure J defined by (8) and with Norden metric G given by (4). According to the terminology from [2], we have that (T^*M, J, G) is a Kaehlerian manifold with Norden metric if $\tilde{\nabla}J = 0$, or equivalently, the tensor field F of type $(0, 3)$ defined by

$$(9) \quad F(X, Y, Z) = G((\tilde{\nabla}_X J)Y, Z); \quad X, Y, Z \in \Gamma(T^*M)$$

vanishes identically on T^*M .

By using (8), (9) and Proposition 1 we get by a straightforward computation

Proposition 5. *The local coordinate expression of F in the local frame adapted to the direct sum decomposition (1) is:*

$$\begin{aligned}
 F\left(\frac{\partial}{\partial p_i}, \frac{\partial}{\partial p_j}, \frac{\partial}{\partial p_k}\right) &= -\frac{\partial g^{jk}}{\partial p_i}; \\
 F\left(\frac{\partial}{\partial p_i}, \frac{\partial}{\partial p_j}, \frac{\delta}{\delta q^k}\right) &= F\left(\frac{\partial}{\partial p_i}, \frac{\delta}{\delta q^k}, \frac{\partial}{\partial p_j}\right) = \frac{1}{2}g^{jh}(\Phi_{kh}^i - \Phi_{hk}^i); \\
 F\left(\frac{\partial}{\partial p_i}, \frac{\delta}{\delta q^j}, \frac{\delta}{\delta q^k}\right) &= \frac{\partial g_{jk}}{\partial p_i}; \\
 F\left(\frac{\delta}{\delta q^i}, \frac{\partial}{\partial p_j}, \frac{\partial}{\partial p_k}\right) &= -\frac{\delta g^{jk}}{\delta q^i} - \frac{1}{2}g^{jh}(\Phi_{ih}^k + \Phi_{hi}^k) - \frac{1}{2}g^{kh}(\Phi_{ih}^j + \Phi_{hi}^j); \\
 F\left(\frac{\delta}{\delta q^i}, \frac{\partial}{\partial p_j}, \frac{\delta}{\delta q^k}\right) &= F\left(\frac{\delta}{\delta q^i}, \frac{\delta}{\delta q^k}, \frac{\partial}{\partial p_j}\right) = -\frac{1}{2}g^{jh}(R_{ihk} - R_{hki} - R_{kjh}); \\
 F\left(\frac{\delta}{\delta q^i}, \frac{\delta}{\delta q^j}, \frac{\delta}{\delta q^k}\right) &= \frac{\delta g_{jk}}{\delta q^i} - \frac{1}{2}g_{jh}(\Phi_{ik}^h + \Phi_{ki}^h) - \frac{1}{2}g_{kh}(\Phi_{ij}^h + \Phi_{ji}^h).
 \end{aligned}$$

Examining these relations it follows that the condition $F = 0$ which must be fulfilled for (T^*M, J, G) to be a Kaehlerian manifold with Norden metric is reduced to:

$$\begin{aligned}
 (i) \quad \frac{\partial g_{jk}}{\partial p_i} &= 0; \quad (ii) \quad \Phi_{ij}^k = \Phi_{ji}^k; \quad (iii) \quad R_{kij} - R_{ijk} - R_{jki} = 0; \\
 (iv) \quad \frac{\delta g_{jk}}{\delta q^i} - \frac{1}{2}g_{jh}(\Phi_{ik}^h + \Phi_{ki}^h) - \frac{1}{2}g_{kh}(\Phi_{ij}^h + \Phi_{ji}^h) &= 0.
 \end{aligned}
 \tag{10}$$

From (10) (i) it follows that the componets g_{ij} are independent of p_k . Thus the M -tensor field defined by g_{ij} is obtained from a tensor field on the base manifold, defining a (pseudo-) Riemannian metric on M . From (10) (ii) it follows that the linear M -connection defined by Φ_{ij}^k on T^*M is symmetric, such that the condition (10) (iv) becomes:

$$\frac{\partial g_{jk}}{\partial q^i} - \Phi_{ij}^h g_{hk} - \Phi_{ik}^h g_{jh} = 0.
 \tag{11}$$

From (11) it follows that the symmetric linear M -connection defined by Φ_{ij}^k on T^*M is in fact the Levi Civita connection ∇ of g_{ij} , thought of as a linear M -connection on T^*M . Thus, from (11) and the first relation (3) we have

$$(i) \quad \Phi_{ij}^k = \{^k_{ij}\}; \quad (ii) \quad N_{ij} = -p_k \{^k_{ij}\} + a_{ij}
 \tag{12}$$

where $\{^k_{ij}\}$ are the Christoffel symbols defined by g_{ij} and the components a_{ij} ; $i, j = 1, \dots, n$, define an arbitrary tensor field of type $(0, 2)$ on the base manifold M thought of as an M -tensor field on T^*M . Using (12) (ii) in the second relation (3), we get by a straightforward computation

$$(13) \quad R_{kij} = -p_h R_{kij}^h + \nabla_i a_{jk} - \nabla_j a_{ik}$$

where R_{kij}^h denotes the local coordinate components of the curvature tensor R of the Levi Civita connection ∇ of g_{ij} on M and $\nabla_i a_{jk}$ are the local components of the covariant derivative of the tensor field on M defined by a_{jk} with respect to ∇ . Then, using (13) and the first Bianchi identity for the curvature tensor R_{kij}^h on M , the condition (10) (iii) is reduced to:

$$(14) \quad -p_h R_{kij}^h + \nabla_i c_{jk} - \nabla_j c_{ik} - \nabla_k b_{ij} = 0$$

where c_{ij} and b_{ij} denote respectively the symmetric part and the skew-symmetric part of the arbitrary tensor field a_{ij} , i.e.

$$c_{ij} = \frac{1}{2}(a_{ij} + a_{ji}); \quad b_{ij} = \frac{1}{2}(a_{ij} - a_{ji}).$$

From (14) it follows that $R_{kij}^h = 0$, i.e. the Levi Civita connection ∇ of g_{ij} is flat and the components c_{jk} and b_{jk} defined respectively as the symmetric part and the skew-symmetric part of a_{jk} must satisfy the condition

$$(15) \quad \nabla_i c_{jk} - \nabla_j c_{ik} = \nabla_k b_{ij}.$$

Hence we may state

Theorem 6. *The almost complex manifold with Norden metric (T^*M, J, G) is Kaehlerian with Norden metric if and only if J and G are defined by a flat (pseudo-) Riemannian structure on the base manifold M and a nonlinear connection given by (12) (ii) where a_{ij} is a tensor field on M which satisfies condition (15), c_{ij} and b_{ij} denoting respectively the symmetric and the skew-symmetric part of a_{ij} .*

Remarks. (i) The condition (15) implies that the 2-form with the components b_{ij} defined as the skew-symmetric part of a_{ij} is closed. Then if the 2-form with the components b_{ij} is closed the condition (15) is equivalent to $\nabla_i a_{jk} - \nabla_j a_{ik} = 0$. Thus, Theorem 6 becomes: The almost complex manifold with Norden metric (T^*M, J, G) is Kaehlerian with Norden metric if and only if J and G are defined by a (pseudo-) Riemannian structure on the base manifold M and a nonlinear connection given by (12) (ii), such that the horizontal distribution HT^*M is involutive and the 2-form defined by b_{ij} is closed.

(ii) In the case $b_{ij} = 0$ (i.e. the considered nonlinear connection defined by N_{ij} on T^*M is symmetric) we get the results obtained in [7].

(iii) If we assume that c_{ij} define a Codazzi tensor field with respect to the connection ∇ on M , i.e. $\nabla_i c_{jk} = \nabla_j c_{ik}$ and the 2-form defined by b_{ij} is parallel with respect to ∇ , then the condition (15) is identically verified. It is known that the n -dimensional torus T^n has a flat Riemannian metric, hence every Codazzi tensor field on T^n and every parallel 2-form on T^n define a Kaehlerian structure with Norden metric on T^*T^n .

3. An almost product structure on (T^*M, G)

Consider the following almost product structure P on T^*M naturally defined by the direct sum decomposition (1), i.e.

$$(16) \quad P \left(\frac{\partial}{\partial p_i} \right) = \frac{\partial}{\partial p_i}; \quad P \left(\frac{\delta}{\delta q^i} \right) = -\frac{\delta}{\delta q^i}.$$

It follows easily that $G(PX, PY) = -G(X, Y); \forall X, Y \in \Gamma(T^*M)$, therefore (T^*M, G, P) is an almost parahermitian manifold (see [1], [8]). Define the 2-form Ω associated with the almost parahermitian structure (G, P) on T^*M by

$$(17) \quad \Omega(X, Y) = G(PX, Y); \quad X, Y \in \Gamma(T^*M).$$

According to the terminology from [1] (see also [8]) we have that (T^*M, G, P) is a parakaehlerian manifold if $\tilde{\nabla}\Omega$ vanishes identically on T^*M . Using Proposition 1 we obtain by a straightforward computation

$$(18) \quad \begin{aligned} (\tilde{\nabla}_{\partial^i}\Omega)(\partial^j, \partial^k) &= (\tilde{\nabla}_{\partial^i}\Omega)(\partial^j, \delta_k) = (\tilde{\nabla}_{\delta_i}\Omega)(\partial^j, \partial^k) = (\tilde{\nabla}_{\delta_i}\Omega)(\partial^j, \delta_k) = 0; \\ (\tilde{\nabla}_{\partial^i}\Omega)(\delta_j, \delta_k) &= \Phi_{kj}^i - \Phi_{jk}^i; \\ (\tilde{\nabla}_{\delta_i}\Omega)(\delta_j, \delta_k) &= R_{kij} + R_{jki} - R_{ijk}. \end{aligned}$$

Then the condition $\tilde{\nabla}\Omega = 0$ which must be fulfilled for (T^*M, G, P) to be a parakaehlerian manifold is reduced to:

$$(19) \quad (i) \Phi_{jk}^i = \Phi_{kj}^i; \quad (ii) R_{ijk} = R_{jki} + R_{kij}.$$

From (19) (i) it follows that the linear M -connection defined by Φ_{ij}^k on T^*M is symmetric. Hence we have

Proposition 7. *The almost parahermitian manifold (T^*M, G, P) is a parakaehlerian manifold if and only if the linear M -connection Φ_{ij}^k and the M -tensor field R_{kij} defined by the considered nonlinear connection N_{ij} satisfy the conditions (19).*

In order to obtain some classes of manifolds whose cotangent bundles carry parakaehlerian structures, we write the components N_{ij} of the considered nonlinear connection on T^*M in the form:

$$(20) \quad N_{ij} = \overset{0}{N}_{ij} + b_{ij}$$

where $\overset{0}{N}_{ij}$ are the components of a symmetric nonlinear connection on T^*M , i.e. $\overset{0}{N}_{ij} = \overset{0}{N}_{ji}$ and b_{ij} are the components of a skew-symmetric M -tensor field of type (0,2) on T^*M , i.e. $b_{ij} = -b_{ji}$. The decomposition (20) is always possible. The condition (19) (i) shows that

$$\frac{\partial b_{ij}}{\partial p_k} = 0$$

i.e. b_{ij} are the components of a tensor field on the base manifold M , thought of as an M -tensor field on T^*M . Considering N_{ij} of the form (20) where b_{ij} are independent of the cotangential coordinates p_k we have

$$\Phi_{ij}^k = -\partial^k N_{ij} = -\partial^k \overset{0}{N}_{ij}$$

and we get by a straightforward computation

$$(21) \quad R_{kij} = \overset{0}{R}_{kij} + \nabla_i b_{jk} - \nabla_j b_{ik}$$

where $\overset{0}{R}_{kij}$ and $\nabla_i b_{jk}$ are defined by:

$$(22) \quad \overset{0}{R}_{kij} = \overset{0}{\delta}_i \overset{0}{N}_{jk} - \overset{0}{\delta}_j \overset{0}{N}_{ik}$$

$$(23) \quad \nabla_i b_{jk} = \frac{\partial b_{jk}}{\partial q^i} - \Phi_{ij}^h b_{hk} - \Phi_{ik}^h b_{jh}$$

and where $\overset{0}{\delta}_i$ is given by

$$\overset{0}{\delta}_i = \frac{\partial}{\partial q^i} - \overset{0}{N}_{ih} \frac{\partial}{\partial p_h}.$$

Since we have $\sum_{(ijk)}^0 R_{kij} = 0$ it follows that the condition (19) (ii) is expressed simply by:

$$(24) \quad \sum_{(ijk)}^0 R_{ijk} = \nabla_i b_{jk}$$

where $\sum_{(ijk)}$ denotes the sum consisting of three terms obtained by cyclic permutations of i, j, k . Then we have

$$\sum_{(ijk)} \nabla_i b_{jk} = \sum_{(ijk)} \partial_i b_{jk} = 0$$

which shows that the 2-form defined by the coefficients b_{ij} is closed. Then, by using (21) it follows that (24) is equivalent to

$$(25) \quad R_{kij} = \nabla_k b_{ij} + \nabla_i b_{jk} - \nabla_j b_{ik} = 0.$$

Hence we may state

Theorem 8. *The almost parahermitian manifold (T^*M, G, P) is parakaehlerian if and only if the nonlinear connection defined by N_{ij} is flat and the linear M -connection defined by $\Phi_{jk}^i = -\partial^i N_{jk}$ is symmetric.*

In the following we shall consider some particular nonlinear connections of the form (20) where the components b_{ij} are independent of p_k and the linear M -connection defined by $\Phi_{jk}^i = -\partial^i N_{jk}$ is symmetric and we shall study the conditions under which (T^*M, G, P) is a parakaehlerian manifold.

First, let M be a manifold with a torsion-free linear connection D and denote by Γ_{ij}^k the connection coefficients of D . Consider the nonlinear connection defined by the components N_{ij} on T^*M given by

$$(26) \quad N_{ij} = -\Gamma_{ij}^k p_k + c_{ij} + b_{ij}$$

where the components c_{ij} (respectively b_{ij}) define a symmetric (respectively a skew-symmetric) tensor field of type (0,2) on the base manifold M thought of as a symmetric (respectively skew-symmetric) M -tensor field on T^*M . The considered components N_{ij} are of the form (20), where b_{ij}

are independent of p_k and $\overset{0}{N}_{ij}$ are given by

$$\overset{0}{N}_{ij} = -p_k \Gamma_{ij}^k + c_{ij}.$$

Then, by using (22) the condition (24) (or equivalently (25)) becomes

$$(27) \quad -p_h R_{kij}^h + D_i c_{jk} - D_j c_{ik} - D_k b_{ij} = 0$$

where R_{kij}^h denotes the local coordinate components of the curvature tensor of D on M . Hence we may state

Theorem 9. *Let D be a torsion-free linear connection on M and consider on T^*M the nonlinear connection defined by (26), where Γ_{ij}^k are the connection coefficients of D and c_{ij} (respectively b_{ij}) are the components of a symmetric (respectively skew-symmetric) tensor field on M . Then (T^*M, G, P) is a parakaehlerian manifold if and only if D is flat and the components c_{ij} and b_{ij} satisfy the condition*

$$(28) \quad D_i c_{jk} - D_j c_{ik} - D_k b_{ij} = 0.$$

Remark. If c_{ij} define a Codazzi tensor field with respect to D ; i.e. $D_i c_{jk} = D_j c_{ik}$ and the 2-form b_{ij} is parallel with respect to D , i.e. $D_k b_{ij} = 0$, then the condition (28) is identically verified.

By taking into account of Theorems 3 and 9 from this paper and of Theorem 9 from [5], by using the Ricci identity we obtain

Corollary 10. *Let D be a torsion-free linear connection on M and consider on T^*M the nonlinear connection defined by (26). Let (T^*M, G, P) be the almost parahermitian manifold with G defined by (4) and P defined by (16). Then if (T^*M, G, P) is a parakaehlerian manifold we have that the pseudo-Riemannian manifold (T^*M, G) is locally symmetric.*

Remark that, in general, the converse of the above assertion is not true. More precisely, by using Theorems 3 and 9 from this paper and Theorem 3 in [5] we have

Corollary 11. *In the same hypothesis as in Corollary 10, we have that (T^*M, G, P) is a parakaehlerian manifold if and only if the pseudo-Riemannian manifold (T^*M, G) is flat and the components c_{ij} and b_{ij} satisfy the condition (28).*

Now, let M be a manifold with a torsion-free linear connection D having the connection coefficients Γ_{ij}^k and consider the nonlinear connection defined by the components N_{ij} on T^*M given by

$$(29) \quad N_{ij} = -\Gamma_{ij}^k p_k + k p_i p_j + c_{ij} + b_{ij}$$

where the components c_{ij} (respectively b_{ij}) define a symmetric (respectively skew-symmetric) tensor field on M thought of as a symmetric (respectively skew-symmetric) M -tensor field on T^*M and k is a nonzero constant. The components N_{ij} given by (29) are of the form (20) where

$\overset{0}{N}_{ij}$ are given by

$$\overset{0}{N}_{ij} = -p_k \Gamma_{ij}^k + k p_i p_j + c_{ij}$$

and the components b_{ij} are independent of p_k . Using (22), (23) we get by a straightforward computation that the condition (24) (or equivalently (25)) becomes

$$(30) \quad \begin{aligned} & -(R_{kij}^h + k(a_{ik}\delta_j^h - a_{jk}\delta_i^h + a_{ij}\delta_k^h - a_{ji}\delta_k^h))p_h \\ & + D_i c_{jk} - D_j c_{ik} - D_k b_{ij} = 0 \end{aligned}$$

where R_{kij}^h are the local coordinate components of the curvature tensor of D on M and the components a_{ij} are defined by

$$a_{ij} = c_{ij} + b_{ij}.$$

Then the condition (30) is equivalent to the following two relations:

$$(31) \quad \begin{aligned} (i) \quad & R_{kij}^h = k(a_{jk}\delta_i^h - a_{ik}\delta_j^h - a_{ij}\delta_k^h + a_{ji}\delta_k^h); \\ (ii) \quad & D_i c_{jk} - D_j c_{ik} - D_k b_{ij} = 0. \end{aligned}$$

From (31) (i) it follows

$$(32) \quad a_{jk} = \frac{1}{k(n^2 - 1)}(nR_{jk} + R_{kj})$$

where the components $R_{jk} = R_{kij}^i$ define the Ricci tensor field obtained from the curvature tensor field of D . Remark that R_{jk} is not necessarily symmetric. Then the condition (31) (i) becomes

$$(33) \quad \begin{aligned} R_{kij}^h &= \frac{1}{n^2 - 1} \{ \delta_i^h (nR_{jk} + R_{kj}) - \delta_j^h (nR_{ik} + R_{ki}) \} \\ & - \frac{1}{n - 1} (R_{ij} - R_{ji}) \delta_k^h \end{aligned}$$

therefore the connection D must be projectively flat. From (33) and the second Bianchi identity we also get

$$D_i a_{jk} = D_j a_{ik}$$

and since $b_{ij} = \frac{1}{2k(n+1)}(R_{ij} - R_{ji})$ it follows $\sum_{(i,j,k)} D_i b_{jk} = 0$, i.e. the 2-form

defined as the skew-symmetric part of a_{ij} is closed. Then the condition (31) (ii) is identically verified.

Hence we may state

Theorem 12. *Let M be a smooth manifold with $\dim M > 2$ and let D be a torsion-free linear connection on M . Consider the nonlinear connection on T^*M defined by (29), where Γ_{ij}^k are the connection coefficients*

of D and the components c_{ij} (resp. b_{ij}) are defined as the symmetric part (resp. the skew-symmetric part) of a_{ij} given by (32). Then the almost parahermitian manifold (T^*M, G, P) with G defined by (4) and P defined by (16) is a parakaehlerian manifold if and only if the connection D is projectively flat.

Remark. If c_{ij} is a nondegenerate symmetric tensor field of type $(0,2)$ on M then it defines a (pseudo-) Riemannian metric g on M and assuming that D is the Levi Civita connection of g and $b_{ij} = 0$, then the condition (31) (ii) is identically verified and from (33) we get: Let (M, g) be a (pseudo-) Riemannian manifold with D the Levi Civita connection of g . Consider the nonlinear connection on T^*M defined by (29) where Γ_{ij}^k are the connection coefficients of D , c_{ij} are the local coordinate components of g and $b_{ij} = 0$. Then (T^*M, G, P) is a parakaehlerian manifold if and only if (M, g) has constant sectional curvature k (see Theorem 2 in [8]).

By using Theorems 3 and 12 from this paper and Theorem 6 from [6] we obtain

Corollary 13. *Let M be a smooth manifold with $\dim M > 2$ and let D be a torsion-free linear connection on M . Consider on T^*M the nonlinear connection defined by (29), where Γ_{ij}^k are the connection coefficients of D and c_{ij} (resp. b_{ij}) are the symmetric part (resp. the skew-symmetric part) of a_{ij} given by (32). Then the almost parahermitian manifold (T^*M, G, P) with G defined by (4) and P defined by (16) is a parakaehlerian manifold if and only if the pseudo-Riemannian manifold (T^*M, G) is locally symmetric.*

Another parakaehlerian structures on cotangent bundles can be obtained in the cases of complex and quaternion manifolds.

Let (M, F) be a *complex manifold* with the almost complex structure defined by the tensor field F of type $(1, 1)$ such that $F^2 = -I$ and denote by D a torsion-free almost complex connection on M , i.e. we have $DF = 0$. Consider on T^*M the nonlinear connection defined by the components

$$(34) \quad N_{ij} = -\Gamma_{ij}^k p_k + k(p_i p_j - F_i^k F_j^h p_k p_h) + c_{ij} + b_{ij}$$

where Γ_{ij}^k are the connection coefficients of D on M , F_i^h are the components of F , k is a nonzero constant and the components c_{ij} (resp. b_{ij}) define a symmetric (resp. skew-symmetric) tensor field on M thought of as a symmetric (resp. skew-symmetric) M -tensor field on T^*M . In this case we have that N_{ij} is of the form (20) where b_{ij} are independent of p_k and the components N_{ij}^0 are given by

$$(35) \quad N_{ij}^0 = -p_k \Gamma_{ij}^k + k(p_i p_j - F_i^k F_j^h p_k p_h) + c_{ij}.$$

We obtain by a straightforward computation that the condition (24) which must be fulfilled for (T^*M, G, P) to be a parakaehlerian manifold becomes:

$$(36) \quad -\{R_{kij}^h + k(a_{ik}\delta_j^h - a_{jk}\delta_i^h + a_{ij}\delta_k^h - a_{ji}\delta_k^h + a_{jl}F_i^l F_k^h - a_{il}F_j^l F_k^h - a_{il}F_k^l F_j^h + a_{jl}F_k^l F_i^h)\}p_h + D_i c_{jk} - D_j c_{ik} - D_k b_{ij} = 0$$

where R_{kij}^h are the local coordinate components of the curvature tensor of D on M and a_{ij} are defined by

$$a_{ij} = c_{ij} + b_{ij}.$$

The condition (36) is equivalent to

$$(37) \quad \begin{aligned} (i) \quad & R_{kij}^h = k(a_{jk}\delta_i^h - a_{ik}\delta_j^h - a_{ij}\delta_k^h + a_{ji}\delta_k^h + a_{il}F_j^l F_k^h \\ & - a_{jl}F_i^l F_k^h + a_{il}F_k^l F_j^h - a_{jl}F_k^l F_i^h); \\ (ii) \quad & D_i c_{jk} - D_j c_{ik} - D_k b_{ij} = 0 \end{aligned}$$

From (37) (i), we obtain by a straightforward computation

$$(38) \quad a_{jk} = \frac{1}{k(n+2)}R_{jk} + \frac{1}{k(n^2-4)}\{R_{jk} + R_{kj} - F_j^i F_k^h (R_{ih} + R_{hi})\}$$

Thus, in order for the condition (37) (i) be fulfilled it is necessary that the connection D on M be H -projectively flat (see [13]). By taking into account (38), from the expression (37) (i) of the curvature tensor field of D and using the second Bianchi identity we obtain

$$D_i a_{jk} = D_j a_{ik}.$$

On the other hand, from (38) we have $b_{ij} = \frac{1}{2}(a_{ij} - a_{ji}) = \frac{1}{2k(n+2)}(R_{ij} - R_{ji})$, hence $\sum_{(i,j,k)} D_i b_{jk} = 0$. It follows then that c_{ij} and b_{ij} defined as the symmetric part respectively the skew-symmetric part of a_{ij} expressed by (38) satisfy identically the condition (37) (ii).

Hence we may state

Theorem 14. *Let M be a complex manifold with real dimension $n > 2$, the tensor field F defining the complex structure on M and the torsion-free almost complex connection D . Consider the nonlinear connection on T^*M defined by (34) where Γ_{ij}^k are the connection coefficients of D and c_{ij} (resp. b_{ij}) are the symmetric part (resp. the skew-symmetric part) of a_{ij} given by (38), R_{ij} denoting the Ricci tensor obtained from the curvature tensor of D . Then the almost parahermitian manifold (T^*M, G, P) with G defined by (4) and P defined by (16) is a parakaehlerian manifold if*

and only if the connection D is H -projectively flat with respect to the considered complex structure of M .

Remark. If c_{ij} is a positive definite symmetric tensor field on M , then it defines a Riemannian metric g on M . We suppose that (M, g, F) is a Kaehler manifold and that D is the Levi Civita connection of g such that $DF = 0$. If we assume that $b_{ij} = 0$, then the condition (37) (ii) is identically verified and from (37) (i) and (38) we get: Consider on T^*M the nonlinear connection defined by (34) where Γ_{ij}^k are the connection coefficients of D , c_{ij} are the local coordinate components of g and $b_{ij} = 0$. Then (T^*M, G, P) is a parakaehlerian manifold if and only if the Kaehler manifold (M, g, F) has constant holomorphic sectional curvature $4k$ (see Theorem 3 in [8]).

By using Theorems 3 and 14 from this paper and Theorem 7 from [6] we obtain

Corollary 15. *Let M be a complex manifold with real dimension $n > 2$, the tensor field F defining the complex structure on M and the torsion-free almost complex connection D . Consider on T^*M the nonlinear connection defined by (34), where Γ_{ij}^k are the connection coefficients of D and c_{ij} (resp. b_{ij}) are the symmetric (resp. the skew-symmetric part) of a_{ij} given by (38), R_{ij} denoting the Ricci tensor obtained from the curvature tensor of D . Then the almost parahermitian manifold (T^*M, G, P) with G defined by (4) and P defined by (16) is a parakaehlerian manifold if and only if the pseudo-Riemannian manifold (T^*M, G) is locally symmetric.*

Consider now (M, S) a quaternion manifold. Then M is a $4m$ -dimensional manifold, S is a subbundle with fibre dimension 3 in the bundle of the tensors of type $(1, 1)$ on M and, locally, S has a canonical base (F_1, F_2, F_3) such that:

$$F_\alpha^2 = -I; \quad F_\alpha \circ F_\beta = -F_\beta \circ F_\alpha = F_\gamma$$

where $\alpha = 1, 2, 3$ and (α, β, γ) is any cyclic permutation of $(1, 2, 3)$. Suppose that D is a torsion-free linear connection on M adapted to the considered quaternary structure i.e. locally we have

$$DF_\alpha = -\eta_\beta \otimes F_\gamma + \eta_\gamma \otimes F_\beta$$

where η_1, η_2, η_3 are locally defined 1-forms associated with the adapted linear connection D .

Consider on T^*M the nonlinear connection defined by the components

$$(39) \quad N_{ij} = -\Gamma_{ij}^k p_k + k \left(p_i p_j - \sum_{\alpha=1,2,3} (F_\alpha)_i^k (F_\alpha)_j^h p_k p_h \right) + c_{ij} + b_{ij}$$

where Γ_{ij}^k are the connection coefficients of D , $(F_\alpha)_i^k$ are the local coordinate components of F_α , k is a nonzero constant and c_{ij} (resp. b_{ij}) define a symmetric (resp. skew-symmetric) tensor field of type $(0,2)$ on M thought of as a symmetric (resp. skew-symmetric) M -tensor field on T^*M . In this case the considered nonlinear connection is of the form (20) where b_{ij} are independent of p_k and the components $\overset{0}{N}_{ij}$ are given by

$$(40) \quad \overset{0}{N}_{ij} = -\Gamma_{ij}^k p_k + k \left(p_i p_j - \sum_{\alpha=1,2,3} (F_\alpha)_i^k (F_\alpha)_j^h p_k p_h \right) + c_{ij}.$$

We get by a straightforward computation that the condition (24) for (T^*M, G, P) to be a parakaehlerian manifold becomes

$$(41) \quad - \left\{ R_{kij}^h + k \left[a_{ik} \delta_j^h - a_{jk} \delta_i^h + a_{ij} \delta_k^h - a_{ji} \delta_k^h + \sum_{\alpha=1,2,3} ((F_\alpha)_i^l (F_\alpha)_k^h a_{jl} - (F_\alpha)_j^l (F_\alpha)_k^h a_{il} - (F_\alpha)_k^l (F_\alpha)_i^h a_{jl}) \right] \right\} p_h + D_i c_{jk} - D_j c_{ik} - D_k b_{ij} = 0$$

where R_{kij}^h are the local coordinate components of the curvature tensor of D on M and a_{ij} are given by

$$a_{ij} = b_{ij} + c_{ij}.$$

The condition (41) is equivalent to

$$(42) \quad (i) \quad R_{kij}^h = k \left\{ a_{jk} \delta_i^h - a_{ik} \delta_j^h - a_{ij} \delta_k^h + a_{ji} \delta_k^h + \sum_{\alpha=1,2,3} \left[- (F_\alpha)_i^l (F_\alpha)_k^h a_{jl} + (F_\alpha)_j^l (F_\alpha)_k^h a_{il} + (F_\alpha)_k^l (F_\alpha)_j^h a_{il} - (F_\alpha)_k^l (F_\alpha)_i^h a_{jl} \right] \right\};$$

$$(ii) \quad D_i c_{jk} - D_j c_{ik} - D_k b_{ij} = 0.$$

From (42) (i), we obtain after a straightforward computation

$$(43) \quad a_{jk} = \frac{1}{4k(m+1)} R_{jk} + \frac{1}{8km(m+1)(m+2)} (R_{jk} + R_{kj}) + \frac{1}{16km(m+2)} \left[R_{jk} + R_{kj} - \sum_{\alpha=1,2,3} (F_\alpha)_j^h (F_\alpha)_k^i (R_{hi} + R_{ih}) \right].$$

Thus, in the case of almost quaternion structure, the condition (42) (i) is fulfilled if and only if the curvature invariant is trivial (see [9]). But this

condition, together with the existence of the torsion-free linear connection D adapted to the considered almost quaternion structure are the conditions under which the almost quaternion structure is integrable, i.e. M is a quaternion manifold. From (42) (i), (43) and the second Bianchi identity we also obtain

$$(i) D_i a_{jk} = D_j a_{ik}; \quad (ii) \sum_{(i,j,k)} D_i b_{jk} = 0$$

which imply that the condition (42) (ii) is identically satisfied.

Hence we may state

Theorem 16. *Assume that the smooth manifold M with $\dim M > 4$ carries an almost quaternary structure with a torsion-free linear connection D , adapted to this structure. Consider on T^*M the nonlinear connection defined by (39) where Γ_{ij}^k are the connection coefficients of D and c_{ij} (resp. b_{ij}) are the symmetric part (resp. the skew-symmetric part) of a_{ij} given by (43), R_{ij} denoting the Ricci tensor obtained from the curvature tensor of D . Then the almost parahermitian manifold (T^*M, G, P) with G defined by (4) and P defined by (16) is a parakaehlerian manifold if and only if the curvature invariant of the considered almost quaternary structure is trivial (equivalently, the almost quaternary structure is integrable).*

Remark. If c_{ij} is a positive definite symmetric tensor field on M then it defines a Riemannian metric g on the quaternion manifold (M, S) . Suppose that the linear connection D is the Levi Civita connection of g . If we assume that $b_{ij} = 0$ then the condition (42) (ii) is identically verified and we get: If we consider on T^*M the nonlinear connection defined by (39) where Γ_{ij}^k are the connection coefficients of D , c_{ij} are the local coordinate components of g and $b_{ij} = 0$, then (T^*M, G, P) is a parakaehlerian manifold if and only if the quaternion Kaehler manifold (M, g, S) has constant Q -sectional curvature $4k$ (see Theorem 4 in [8]). We also remark that the quaternion projective spaces are manifolds with the properties from Theorem 16.

By using Theorems 3 and 16 from this paper and Theorem 8 from [6] we obtain

Corollary 17. *Assume that the smooth manifold M with $\dim M > 2$ carries an almost quaternary structure with a torsion-free linear connection D , adapted to this structure. Consider on T^*M the nonlinear connection defined by (39), where Γ_{ij}^k are the connection coefficients of D and c_{ij} (resp. b_{ij}) are the symmetric part (resp. the skew-symmetric part) of a_{ij} given by (43), R_{ij} denoting the Ricci tensor obtained from the curvature tensor of D . Then the almost parahermitian manifold (T^*M, G, P) with*

G defined by (4) and P defined by (16) is a parakaehlerian manifold if and only if the pseudo-Riemannian manifold (T^*M, G) is locally symmetric.

Consider now a real manifold M carrying a tensor-product structure (see [3], [4]). Such a structure is defined on a manifold M with $\dim M = rs$ and is obtained as follows. Let S be a tensor field of type $(2, 2)$ on M satisfying the conditions:

$$(i) S_{ij}^{ik} = r\delta_j^k; \quad (ii) S_{ij}^{hi} = s\delta_j^h;$$

$$(iii) S_{kh}^{ij} = S_{hk}^{ji}; \quad (iv) S_{ih}^{st}S_{jk}^{hl} = S_{jh}^{tl}S_{ki}^{hs} = S_{kh}^{tl}S_{ij}^{ht};$$

(*) for every point $x \in M$ there exists a neighborhood U of x and a vector field $X \neq 0$ defined on U such that $S(X \otimes X) = X \otimes X$.

The integrability conditions for this structure is fulfilled in the case $r > 2, s > 2$ iff there exists a torsion-free linear connection D on M adapted to the structure, i.e. $DS = 0$. Moreover, the curvature tensor field of such a connection has a special expression.

Tensor-product structures are defined on the real Grassmann manifolds $G_r(R^{r+s})$.

From the above conditions for the tensor field S we get easily the following properties of this tensor field:

$$S_{st}^{ij}S_{kh}^{st} = \delta_k^i\delta_h^j, \quad S_{jt}^{is}S_{sh}^{tk} = rS_{jh}^{ik}, \quad S_{th}^{is}S_{ks}^{hj} = sS_{kh}^{ij}, \quad S_{kt}^{is}S_{hs}^{tj} = \delta_k^i\delta_h^j.$$

We shall show that in the case of a manifold M carrying an integrable tensor product structure there is a nonlinear connection on T^*M such that (T^*M, G, P) is a parakaehlerian manifold.

Let D be a torsion-free linear connection adapted to the considered tensor-product structure, i.e. $DS = 0$. Consider on T^*M the nonlinear connection defined by

$$(44) \quad N_{ij} = -\Gamma_{ij}^k p_k + \frac{1}{2}(S_{ij}^{hk} + S_{ji}^{hk})p_h p_k + c_{ij} + b_{ij}$$

where Γ_{ij}^k are the connection coefficients of D and c_{ij} (resp. b_{ij}) define a symmetric (resp. skew-symmetric) tensor field of type $(0, 2)$ on M thought of as a symmetric (resp. skew-symmetric) M -tensor field on T^*M . In this case the considered nonlinear connection is of the same form (20) where b_{ij} are independent of p_k and the components $\overset{0}{N}_{ij}$ are given by

$$\overset{0}{N}_{ij} = -\Gamma_{ij}^k p_k + \frac{1}{2}(S_{ij}^{hk} + S_{ji}^{hk})p_h p_k + c_{ij}.$$

We get by a straightforward computation that the condition (24) which must be fulfilled for (T^*M, G, P) to be a parakaehlerian manifold becomes

$$(45) \quad \begin{aligned} & -p_h \{ R_{kij}^h - a_{il}S_{jk}^{lh} - a_{il}S_{jk}^{hl} + a_{jl}S_{ik}^{lh} + a_{jl}S_{ik}^{hl} \} \\ & + D_i c_{jk} - D_j c_{ik} - D_k b_{ij} = 0 \end{aligned}$$

where R_{kij}^h denotes the curvature tensor of D and a_{ij} are defined by

$$a_{ij} = c_{ij} + b_{ij}.$$

The condition (45) is equivalent to

$$(46) \quad (i) R_{kij}^h = a_{il}(S_{jk}^{lh} + S_{jk}^{hl}) - a_{jl}(S_{ik}^{lh} + S_{ik}^{hl}); \quad (ii) D_i c_{jk} - D_j c_{ik} = D_k b_{ij}$$

From (46) (i) we get

$$(47) \quad \begin{aligned} a_{ij} = & \frac{1}{2(r+s)}(R_{ij} - R_{ji}) \\ & + \frac{1}{(r+s)^2 - 4} \left\{ \frac{r+s}{2}(R_{ij} + R_{ji}) + R_{hk}(S_{ij}^{hk} + S_{ij}^{kh}) \right\}. \end{aligned}$$

Replacing the expression (47) of a_{ij} in (46) (i), next by using the second Bianchi identity we obtain

$$D_i a_{jk} - D_j a_{ik} = 0; \quad \sum_{(i,j,k)} D_i b_{jk} = 0$$

which imply that the condition (46) (ii) is identically verified.

Hence we may state

Theorem 18. *Let M be a smooth manifold with $\dim M = rs$; $r > 2$, $s > 2$ carrying a tensor-product structure defined by the tensor field S and let D be a torsion-free linear connection on M , adapted to this structure. Consider on T^*M the nonlinear connection defined by (44), where Γ_{ij}^k are the connection coefficients of D and c_{ij} (resp. b_{ij}) are the symmetric part (resp. the skew-symmetric part) of a_{ij} given by (47), R_{ij} denoting the Ricci tensor obtained from the curvature tensor of D . Then the almost parahermitian manifold (T^*M, G, P) with G defined by (4) and P defined by (16) is a parakaehlerian manifold.*

Finally, we define other almost parahermitian structures on T^*M as follows. Consider h_{ij} ; $i, j = 1, \dots, n$, the components of a nondegenerate M -tensor field of type $(0, 2)$ on T^*M and denote by h^{ij} the components of the inverse of the matrix (h_{ij}) ; $i, j = 1, \dots, n$. On TT^*M define the

automorphism Q expressed in the local frame adapted to the direct sum decomposition (1) by

$$(48) \quad Q \left(\frac{\delta}{\delta q^i} \right) = h_{ik} \frac{\partial}{\partial p_k}; \quad Q \left(\frac{\partial}{\partial p_i} \right) = -h^{ik} \frac{\delta}{\delta q^k}.$$

The following result is obtained by a straightforward computation

Proposition 19. *The automorphism Q given by (48) defines an almost product structure on T^*M if and only if the M -tensor field h_{ij} is skew-symmetric, i.e. $h_{ij} = -h_{ji}$.*

Assume that M is an even-dimensional manifold whose cotangent bundle T^*M carries a nondegenerate skew-symmetric M -tensor field h_{ij} of type $(0, 2)$. We may check easily that the pseudo-Riemannian metric G defined by (4) on T^*M and the almost product structure Q are related by

$$G(QX, QY) = -G(X, Y); \quad \forall X, Y \in \Gamma(T^*M)$$

thus:

Proposition 20. *(T^*M, G, Q) is an almost parahermitian manifold.*

Now it is possible to obtain some classes of almost parahermitian manifolds on (T^*M, G, Q) according to the classification in [1] (see [8] for the case when the considered nonlinear connection on T^*M is symmetric).

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