The gammaoperators in L^p spaces

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§ 1. Introduction

The gammaoperators [3]

$$G_n(f;\, x) = \frac{x^{n+1}}{n!}\int_0^\infty e^{-xu}u^n f\left(\frac{n}{u}\right) du \overset{\text{def}}{=} \int_0^\infty g_n(x,\, u) f\left(\frac{n}{u}\right) du \quad (x>0)$$

were introduced by M. MÜLLER and A. LUPAS and several papers have been devoted to the investigation of their approximation-theoretical properties. Concerning global uniform approximation we have proved in [7].

Theorem A. *Let f be a continuous and bounded function defined on $(0, \infty)$. For every $0<\alpha\leqq 1$ the statements*

$$|G_n(f)-f| \leqq K_1 n^{-\alpha}, \quad n = 1, 2, \dots$$

and

$$e^x|\Delta_h^2(g;\, x)| \leqq K_2 h^{2\alpha}, \quad h>0,\ x\in(-\infty, \infty),\ g(x) = f(e^x)$$

are equivalent.

Here

$$\Delta_h^2(g;\, x) = g(x-h)-2g(x)+g(x+h)$$

is the usual symmetric second difference of g.

The aim of the present article is to prove the analogous result in the L^p-metric. It is easy to see that for every function $f\in L^p(0, \infty)$ $G_n(f;\, x)$ is defined for every $x>0$ and $n\geqq 2$. Now the analogue of Theorem A in L^p is

Theorem 1. *Let $\varphi(x)=x$, $1\leqq p<\infty$, $f\in L^p(0, \infty)$ and $0<\alpha\leqq 1$. Then the relations*

(1.1) $$\|G_n(f)-f\|_{L^p(0,\infty)} = O(n^{-\alpha}) \quad (n\to\infty)$$

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and

$$\|\Delta^2_{h\varphi}(f)\|_{L^p(0,\infty)} = O(h^{2\alpha}) \quad (h \to 0+0) \tag{1.2}$$

are equivalent.

Corollary 1. *For any* $0<\alpha\leqq 1$ (1.1) *is equivalent to*

$$\Big\{\int_{-\infty}^{\infty} e^x|\Delta_h^2(g;x)|^p\,dx\Big\}^{1/p} \leqq Kh^{2\alpha}, \quad g(x)=f(e^x), \quad (h>0). \tag{1.3}$$

Corollary 2. *If* $\alpha=1$ *then for* $p>1$ (1.1) *is equivalent to the fact that* f *has a locally absolutely continuous derivative* f' *with* $x^2f''\in L^p(0,\infty)$, *and for* $p=1$ (1.1) *holds if and only if* f *is locally absolutely continuous and* $x^2f'(x)$ *is of bounded variation on* $(0,\infty)$.

Naturally, it is understood in Corollary 2 that f coincides a.e. with a function having the stated properties.

We made the assumption $0<\alpha\leqq 1$ because a simple modification of our proof shows, that $\{G_n\}$ is saturated of order $\{n^{-1}\}$, i.e. $f\in L^p(0,\infty)$ and

$$\liminf_{n\to\infty} n\|G_n(f)-f\|_{L^p(0,\infty)} = 0$$

implies that f is zero a.e.

Now let us introduce the following modified L^p-modulus of smoothness:

$$\omega(f;\delta)_p = \sup_{0\leqq h\leqq\delta}\|\Delta^2_{h\varphi}(f)\|_{L^p(0,\infty)}, \quad \varphi(x)=x, \ \delta>0.$$

This ω has the usual properties of the ordinary L^p-modulus of smoothness (i.e. when $\varphi\equiv 1$) e.g. there is a K independent of f, $\delta>0$ and $\lambda\geqq 1$ such that

$$\omega(f;\lambda\delta)_p \leqq K\lambda^2\omega(f;\delta)_p \tag{1.4}$$

(see [5, Theorem 1]).

For $0<\alpha<1$ Theorem 1 is contained in

Theorem 2. *Let* $1\leqq p<\infty$, $f\in L^p(0,\infty)$ *and*

$$E_n(f) = \|G_n(f)-f\|_{L^p(0,\infty)}.$$

Then there is a constant K_p *depending only on* p *such that the estimates*

$$E_n(f) \leqq K_p\omega\left(f;\frac{1}{\sqrt{n}}\right)_p \quad (n\geqq 2) \tag{1.5}$$

and

$$\omega\left(f;\frac{1}{\sqrt{n}}\right)_p \leqq K_p\frac{\|f\|+E_2(f)+\ldots+E_n(f)}{n} \quad (n\geqq 2) \tag{1.6}$$

hold.

We shall prove Theorem 2 in the next paragraph while the proof of Theorem 1 will be given in § 3.

§ 2. Proof of Theorem 2

In what follows $\varphi(x)=x$ and K is a positive constant depending only on p and not necessarily the same at each occurence.

I. Proof of (1.5). Let $L^p=L^p(0,\infty)$, $\|\cdot\|_p=\|\cdot\|_{L^p(0,\infty)}$ and

$$K(f;t)_p=\inf_{g\in D_p}(\|f-g\|_p+t^2\|\varphi^2g''\|_p), \tag{2.1}$$

$$D_p=\{g\in L^p|g' \text{ loc. abs. cont., } \varphi^2g''\in L^p\} \tag{2.2}$$

the so called (Peetre-) K-functional. In [6] we proved the estimate

$$K(f;t)_p\leqq K\omega(f;t)_p \quad (0<t<1). \tag{2.3}$$

Therefore, if we show that for $n\geqq 2$

$$\|G_n(f)\|_p\leqq K\|f\|_p \quad (f\in L^p) \tag{2.4}$$

and

$$\|G_n(f)-f\|_p\leqq\frac{K}{n}\|\varphi^2f''\|_p \quad (f\in D_p) \tag{2.5}$$

are satisfied then choosing a $g_n\in D_p$ with the property

$$\|f-g_n\|_p+\frac{1}{n}\|\varphi^2g_n''\|_p\leqq 2K\left(f;\frac{1}{\sqrt{n}}\right)_p$$

(see (2.1)) we obtain by (2.3) the required estimate:

$$\|G_n(f)-f\|_p\leqq\|G_n(f-g_n)\|_p+\|f-g_n\|_p+\|G_n(g_n)-g_n\|_p\leqq$$
$$\leqq K\left(\|f-g_n\|_p+\frac{1}{n}\|\varphi^2g_n''\|_p\right)\leqq KK\left(f;\frac{1}{\sqrt{n}}\right)_p\leqq K\omega\left(f;\frac{1}{\sqrt{n}}\right)_p.$$

For later application we shall prove somewhat more than (2.4), namely

$$\|G_n(f)\|_p\leqq\left(1+\frac{1}{n}\right)^{1/p}\|f\|_p \quad (f\in L^p,\ n\geqq 2). \tag{2.6}$$

A change of variable yields

$$G_n(f;x)=\int_0^\infty\frac{e^{-\tau}\tau^n}{n!}f\left(\frac{nx}{\tau}\right)d\tau. \tag{2.7}$$

Now let us use the fact that the norm of an integral is not greater than the integral of the corresponding norms (cf. [4, p. 271]) by which

$$\left\{\int_0^\infty\left|\int_0^\infty\frac{e^{-\tau}\tau^n}{n!}f\left(\frac{nx}{\tau}\right)d\tau\right|^p dx\right\}^{1/p}\leqq\int_0^\infty\frac{e^{-\tau}\tau^n}{n!}\left\{\int_0^\infty\left|f\left(\frac{nx}{\tau}\right)\right|^p dx\right\}^{1/p}d\tau=$$
$$=\|f\|_p\int_0^\infty\frac{e^{-\tau}\tau^{n+1/p}}{n!\,n^{1/p}}=\|f\|_p\frac{\Gamma(n+1/p+1)}{n!\,n^{1/p}}\leqq$$
$$\leqq\|f\|_p\frac{(\Gamma(n+1))^{1-1/p}(\Gamma(n+2))^{1/p}}{n!\,n^{1/p}}=\|f\|_p\frac{(n+1)^{1/p}}{n^{1/p}}$$

where, at the last but one step we also applied the convexity of $\log\Gamma(x)$ (see [1].)

In the proof of (2.5) we need the following lemma

Lemma 1. *If* $x>0$, $n\geqq 2$ *and*

$$h_x(t)=\frac{(t-x)^2}{x^2}\max\left(1;\ \frac{x}{t}\right)$$

then

$$G_n(h_x;\ x)\leqq\frac{10}{n}.$$

Indeed, a simple calculation gives (see e.g. [3]) that

$$G_n((t-x)^2;\ x)=\frac{x^2}{n-1}\quad(n\geqq 2)$$

and so

$$G_n(h_x;\ x)=\int_0^\infty g_n(x,u)\,x^{-2}\left(\frac{n}{u}-x\right)^2\max\left(1,\ \frac{ux}{n}\right)du=$$

$$=\int_0^{2n/x}+\int_{2n/x}^\infty\leqq 2x^{-2}G_n((t-x)^2;\ x)+4x^{-2}\frac{n+1}{n}G_{n+1}((t-x)^2;\ x)\leqq\frac{10}{n},$$

where we used the identity $(ux/n)g_n(x,u)=((n+1)/n)g_{n+1}(x,u)$ and the fact that for $u\geqq 2n/x$, $n\geqq 2$ the inequality

$$\left(\frac{n}{u}-x\right)^2\leqq 4\left(\frac{n+1}{u}-x\right)^2$$

holds.

Returning to the proof of (2.5), for an $f\in D_p$ we can write by Taylor's formula

$$f(t)-f(x)=f'(x)(t-x)+\int_x^t(t-u)f''(u)\,du= \tag{2.8}$$

$$=f'(x)(t-x)+\int_0^{t-x}\frac{t-x-u}{(x+u)^2}\left((x+u)^2f''(x+u)\right)du.$$

Here

$$\frac{|t-x-u|}{(x+u)^2}\leqq\frac{|t-x|}{x^2}\max\left(1;\ \frac{x}{t}\right)\quad(u\in(0,\ t-x))$$

by which

$$\left|\int_0^{t-x}\frac{t-x-u}{(x+u)^2}\left((x+u)^2f''(x+u)\right)du\right|\leqq\frac{(t-x)^2}{x^2}\max\left(1;\ \frac{x}{t}\right)M(\varphi^2f'';\ x)$$

where $M(g;x)$ denotes the Hardy—Littlewood-maximal function of g (see [4, p. 5]). Now for $p>1$ the maximal-inequality ([4, p. 5]), (2.8) and Lemma 1 together with the easily verifiable facts

$$G_n(t;x)\equiv x,\quad G_n(1;x)\equiv 1,\quad |G_n(g_1;x)|\leqq G_n(g_2;x)\quad\text{for}\quad |g_1|\leqq g_2$$

yield

$$\|G_n(f)-f\|_p \leqq \|G_n(h.;\,\cdot)M(\varphi^2 f'';\,\cdot)\|_p \leqq$$

$$\leqq \frac{10}{n}\|M(\varphi^2 f'')\|_p \leqq \frac{K}{n}\|\varphi^2 f''\|_p$$

which is (2.5).

For $p=1$ we choose a more direct argument. All the transformations below are permitted because of the nonnegativity of the occurring functions. Using (2.8) and $G_n(t-x;\,x)=0$ we obtain

$$\|G_n(f)-f\|_1 = \int_0^\infty \left|\int_0^\infty \left(\int_x^{nx/\tau}\left(\frac{nx}{\tau}-u\right)f''(u)\,du\right)\frac{e^{-\tau}\tau^n}{n!}\,d\tau\right|dx \leqq$$

$$\leqq \int_0^\infty \int_0^n \left(\int_x^{nx/\tau}\left(\frac{nx}{\tau}-u\right)|f''(u)|\,du\right)\frac{e^{-\tau}\tau^n}{n!}\,d\tau\,dx +$$

$$+\int_0^\infty \int_n^\infty \left(\int_{nx/\tau}^{x}\left(u-\frac{nx}{\tau}\right)|f''(u)|\,du\right)\frac{e^{-\tau}\tau^n}{n!}\,d\tau\,dx =$$

$$= \int_0^n \int_0^\infty \int_0^{nx/\tau} (\;)\,du\,dx\,d\tau + \int_n^\infty \int_0^\infty \int_{nx/\tau}^{x} (\;)\,du\,dx\,d\tau =$$

$$= \int_0^n \left(\int_0^\infty \left(\int_{\tau u/n}^{u}\left(\frac{nx}{\tau}-u\right)dx\right)|f''(u)|\,du\right)\frac{e^{-\tau}\tau^n}{n!}\,d\tau +$$

$$+\int_n^\infty \left(\int_0^\infty \left(\int_u^{\tau u/n}\left(u-\frac{nx}{\tau}\right)dx\right)|f''(u)|\,du\right)\frac{e^{-\tau}\tau^n}{n!}\,d\tau =$$

$$= \frac{1}{2}\left(\int_0^n + \int_n^\infty\right)\int_0^\infty \frac{\tau}{n}\left(\frac{nu}{\tau}-u\right)^2 |f''(u)|\frac{e^{-\tau}\tau^n}{n!}\,du\,d\tau =$$

$$= \frac{n+1}{2n}\int_0^\infty |f''(u)|\left(\int_0^\infty \left(\frac{nu}{\tau}-u\right)^2\frac{e^{-\tau}\tau^{n+1}}{(n+1)!}\,d\tau\right)du =$$

$$= \frac{1}{2n}\int_0^\infty u^2|f''(u)|\,du$$

since an easy computation gives

$$\int_0^\infty \left(\frac{nu}{\tau}-u\right)^2\frac{e^{-\tau}\tau^{n+1}}{(n+1)!}\,d\tau =$$

$$= u^2\left(\frac{n}{(n+1)}\frac{\Gamma(n)}{(n-1)!} - 2\frac{n}{(n+1)}\frac{\Gamma(n+1)}{n!} + \frac{\Gamma(n+2)}{(n+1)!}\right) = \frac{u^2}{n+1}.$$

The proof of (1.5) is thus complete.

II. Proof of (1.6). We shall use the following lemma which is a modification of a result of A. Grundmann [2].

Lemma 2. *Let* $\mathscr{B}$ *be a Banach space,* $D\subseteqq\mathscr{B}$ *a linear subspace,* $S: D\to\mathscr{B}$, $T_n:\mathscr{B}\to D$ $(n=2, 3, \dots)$ *linear operators with the properties*

$$\|ST_nf\|\leqq K_1 n\|f\| \quad (f\in\mathscr{B}) \tag{2.9}$$

and

$$\|ST_nf\|\leqq\left(1+\frac{K_1}{n}\right)\|Sf\| \quad (f\in D) \tag{2.10}$$

where K_1 *is a positive constant. Then*

$$\inf_{g\in D}\left(\|f-g\|+\frac{1}{n}\|Sg\|\right)\leqq\frac{K_2}{n}\left(\|f\|+\sum_{i=2}^{n}\|T_if-f\|\right) \quad (n=2, 3, \dots) \tag{2.11}$$

where the constant K_2 *depends only on* K_1.

Proof. Let $2^m\leqq n<2^{m+1}$, $E_i=\|T_i f-f\|$, and for $0\leqq\nu<m$ let k_ν be defined by

$$2^\nu<k_\nu\leqq 2^{\nu+1},\quad E_{k_\nu}=\min_{2^\nu<i\leqq 2^{\nu+1}} E_i.$$

Clearly, if $K\left(\frac{1}{n}\right)$ is the infimum on the left of (2.11) then

$$K\left(\frac{1}{n}\right)\leqq\|f-T_{k_{m-1}}f\|+\frac{1}{2^m}\|ST_{k_{m-1}}f\| \tag{2.12}$$

and we can see by (2.9) and (2.10) that here

$$\|ST_{k_{m-1}}f\|\leqq\|ST_{k_{m-1}}(f-T_{k_{m-2}}f)\|+\|ST_{k_{m-1}}T_{k_{m-2}}f\|\leqq$$
$$\leqq K_1k_{m-1}E_{k_{m-2}}+\left(1+\frac{K_1}{k_{m-1}}\right)\|ST_{k_{m-2}}f\|.$$

Iterating this it follows by $2^\nu<k_\nu\leqq 2^{\nu+1}$ that

$$\|ST_{k_{m-1}}f\|\leqq K_1\left(k_{m-1}E_{k_{m-2}}+\left(1+\frac{K_1}{k_{m-1}}\right)k_{m-2}E_{k_{m-3}}+\dots\right.$$
$$\left.\dots+\left(1+\frac{K_1}{k_{m-1}}\right)\dots\left(1+\frac{K_1}{k_2}\right)k_1E_{k_0}\right)+\left(1+\frac{K_1}{k_{m-1}}\right)\dots\left(1+\frac{K_1}{k_1}\right)\|ST_{k_0}f\|\leqq$$
$$\leqq K_1\prod_{\nu=0}^{m-1}\left(1+\frac{K_1}{2^\nu}\right)\left(2\sum_{\nu=0}^{m-1}2^\nu E_{k_\nu}+K_1\|f\|\right)\leqq K_2\left(\sum_{k=2}^{2^m}E_k+\|f\|\right)$$

which, together with (2.12), proves the lemma.

Now let us apply Lemma 2 with $\mathscr{B}=L^p(0,\infty)$, D as in (2.2), $Sf=\varphi^2f''$, $T_n=G_n$. If we show (2.9) and (2.10) then we obtain from the lemma that (see (2.1))

$$K\left(f;\frac{1}{\sqrt{n}}\right)_p\leqq\frac{K}{n}\left(\|f\|_p+\sum_{i=2}^{n}\|G_i(f)-f\|_p\right). \tag{2.13}$$

Since the inequality $\omega\left(f; \frac{1}{\sqrt{n}}\right)_p \leqq KK\left(f; \frac{1}{\sqrt{n}}\right)_p$ was proved in [6, Theorem 1] the estimate (1.6) follows from (2.13).

Thus, there remains to be shown that

$$\|\varphi^2 G_n''(f)\|_p \leqq Kn\|f\|_p \quad (f\in L^p,\ n \geqq 2) \tag{2.14}$$

and

$$\|\varphi^2 G_n''(f)\|_p \leqq \left(1+\frac{K}{n}\right)\|\varphi^2 f''\|_p \quad (f\in D_p,\ n \geqq 2) \tag{2.15}$$

are satisfied with a K independent of n and f.

Let us begin with (2.14). Since for fixed $x>0$ and $(h)<x/2$

$$\frac{1}{h}\left|e^{-u(x+h)}u^n f\left(\frac{n}{u}\right)-e^{-ux}u^n f\left(\frac{n}{u}\right)\right| \leqq \left|\frac{e^{-uh}-1}{-uh}\right| e^{-ux}u^{n+1}\left|f\left(\frac{n}{u}\right)\right| \leqq$$

$$\leqq Ke^{|h|u}e^{-ux}u^{n+1}\left|f\left(\frac{n}{u}\right)\right| \leqq Ke^{-ux/2}u^{n+1}\left|f\left(\frac{n}{u}\right)\right|$$

it follows from the dominated convergence theorem that

$$\frac{d\left(\int_0^\infty e^{-ux}u^n f\left(\frac{n}{u}\right)du\right)}{dx} = -\int_0^\infty e^{-ux}u^{n+1}f\left(\frac{n}{u}\right)du$$

and applying this once more we obtain

$$\begin{aligned} x^2G_n''(f;x) &= \int_0^\infty \frac{d^2g_n(x,u)}{dx^2}f\left(\frac{n}{u}\right)du = \int_0^\infty ((n-xu)^2+(n-2ux))\times \\ &\times g_n(x,u)f\left(\frac{n}{u}\right)du = n\int_0^\infty g_n^*(x,u)f\left(\frac{n}{u}\right)du+nG_n(f;x)- \\ &-2(n+1)G_{n+1}\left(f\left(\frac{n}{n+1}\cdot\right);x\right) \end{aligned} \tag{2.16}$$

where

$$g_n^*(x,u) = \frac{1}{n}(n-ux)^2g_n(x,u) = \frac{x^{n+1}}{n\cdot n!}(n-ux)^2u^ne^{-ux}.$$

Simple computations give

$$\int_0^\infty g_n^*(x,u)\,du = \frac{1}{n}\Big(\int_0^\infty [n^2g_n(x,u)-2n(n+1)g_{n+1}(x,u)+$$

$$+(n+1)(n+2)g_{n+2}(x,u)]\,du\Big) = 1+\frac{2}{n}$$

and (use that $\Gamma(x+1)=x\Gamma(x)$)

$$\int_0^\infty g_n^*(1, u)\frac{u^{1/p}}{n^{1/p}}\,du = \frac{1}{n^{1+1/p}n!}\left[n^2\Gamma\left(n+1+\frac{1}{p}\right)-2n\Gamma\left(n+2+\frac{1}{p}\right)+\right.$$

$$\left.+\Gamma\left(n+3+\frac{1}{p}\right)\right] = \frac{\Gamma(n+1+1/p)}{n!\,n^{1/p}}\,\frac{n^2-2n(n+1+1/p)+(n+1+1/p)(n+2+1/p)}{n} \leqq K$$

and so (2.6) and its proof give (2.14) (see (2.16)).

It is easy to give a formal proof for (2.15). We obtain from (2.7) that for $f\in D_p$

$$G_n''(f; x) = \int_0^\infty \frac{e^{-\tau}\tau^n}{n!}\,\frac{n^2}{\tau^2}f''\left(\frac{nx}{\tau}\right)d\tau \tag{2.17}$$

and so, by (2.6),

$$\|\varphi^2 G_n''(f)\|_p \leqq \left(1+\frac{1}{n}\right)^{1/p}\|\varphi^2 f\|_p \leqq \left(1+\frac{K}{n}\right)\|\varphi^2 f''\|_p .$$

To justify this formal approach all we have to prove is that the derivation after the integral sign in (2.17) is permitted. But

$$\frac{1}{2}\lim_{h\to 0} h^{-2}\Delta_h^2(G_n(f); x) = G_n''(f; x)$$

a.e.

$$\frac{1}{2}\lim_{h\to 0} h^{-2}\left[f\left(\frac{n(x-h)}{\tau}\right)-2f\left(\frac{nx}{\tau}\right)+f\left(\frac{n(x+h)}{\tau}\right)\right] = \frac{n^2}{\tau^2}f''\left(\frac{nx}{\tau}\right)$$

and

$$h^{-2}\Delta_h^2(G_n(f); x) = \int_0^\infty \frac{e^{-\tau}\tau^n}{n!}\,h^{-2}\left(f\left(\frac{n(x-h)}{\tau}\right)-2f\left(\frac{nx}{\tau}\right)+f\left(\frac{n(x+h)}{\tau}\right)\right)d\tau =$$

$$= \frac{1}{n!}\int_0^\infty h^{-2}\left(e^{-\tau(x-h)/x}\tau^n\left(\frac{x-h}{x}\right)^{n+1}-2e^{-\tau}\tau^n+e^{-\tau(x+h)/x}\tau^n\left(\frac{x+h}{x}\right)^{n+1}\right)f\left(\frac{nx}{\tau}\right)d\tau =$$

$$\stackrel{\text{def}}{=} \frac{1}{n!}\int_0^\infty e^{-\tau}\tau^n h^{-2}h^*_{x,n,h}(\tau)\,f\left(\frac{nx}{\tau}\right)d\tau.$$

An easy consideration gives for $|h|<\frac{x}{2}$ that we have

$$h^{-2}|h^*_{x,n,h}(\tau)| \leqq K_{x,n}(1+\tau^2+e^{\tau|h|/x}) \leqq K_{x,n}(1+\tau^2+e^{\tau/2})$$

and so, by the dominated convergence theorem, (2.17) follows from the relations above.

The proof of Theorem 2 is complete.

§ 3. Proof of Theorem 1

That (1.2) implies (1.1) for all p and α follows from Theorem 2, (1.5). Also, the implication (1.1)⇒(1.2) for $0<\alpha<1$ follows from Theorem 2, (1.6) (see also (1.4)).

Thus, there has remained to show that (1.1) implies (1.2) for $\alpha=1$ and all $p\geqq 1$. To this end we first verify that if $g\in C^2(0,\infty)$ has compact support in $(0,\infty)$ and if

$$A_n(f,g) = n\int_0^\infty (G_n(f)-f)g \quad (f\in L^p)$$

then

$$|A_n(f,g)| \leqq K_g\|f\|_p$$

with a constant K_g independent of $f\in L^p$ and $n\geqq 2$.

Indeed, with the notation

$$h(u) = g\left(\frac{1}{u}\right)$$

we obtain by a change of variable

$$|A_n(f,g)| = n\left|\int_0^\infty\int_0^\infty \frac{e^{-\tau}\tau^n}{n!}\left(f\left(\frac{nx}{\tau}\right)-f(x)\right)g(x)\,d\tau\,dx\right| =$$

$$= n\left|\int_0^\infty\int_0^\infty \frac{e^{-\tau}\tau^n}{n!}f(x)\left(\frac{\tau}{n}g\left(\frac{\tau}{n}x\right)-g(x)\right)d\tau\,dx\right| =$$

$$= \left|\int_0^\infty f(x)\left\{(n+1)\left[G_{n+1}\left(h;\frac{n}{(n+1)x}\right)-h\left(\frac{n}{(n+1)x}\right)\right]+\right.\right.$$

$$\left.\left.+\left[(n+1)h\left(\frac{n}{(n+1)x}\right)-nh\left(\frac{1}{x}\right)\right]\right\}dx\right|.$$

Let $g(x)=0$ outside the interval (a,b), $a>0$, $b<\infty$. By Theorem A

$$(n+1)\left|G_{n+1}\left(h;\frac{n}{(n+1)x}\right)-h\left(\frac{n}{(n+1)x}\right)\right|\leqq K_g$$

and for $x\geqq a/2$

$$\left|(n+1)h\left(\frac{n}{(n+1)x}\right)-nh\left(\frac{1}{x}\right)\right|\leqq \|h\|_{\sup}+n\|h'\|_{\sup}\frac{1}{(n+1)x}\leqq\frac{K_h}{a}\leqq K_g$$

(for $x<a/2$ the expression on the left hand side is zero), and these together with the preceding equality prove our statement for $p=1$.

If $p>1$ then we argue as follows. Let

$$p_n(x) = (n+1)\left|G_{n+1}\left(h;\frac{n}{(n+1)x}\right)-h\left(\frac{n}{(n+1)x}\right)\right|+\left|(n+1)h\left(\frac{n}{(n+1)x}\right)-nh\left(\frac{1}{x}\right)\right|.$$

We have just seen that $p_n(x)$ is bounded, $|p_n(x)| \leqq Kg$, which gives

$$\int_0^{2b} p_n(x)\,dx \leqq K_g.$$

For $x > 2b$ we have $h\left(\frac{n}{(n+1)x}\right) = h\left(\frac{1}{x}\right) = 0$ and so

$$\int_{2b}^{\infty} p_n(x)\,dx = \int_{2b}^{\infty} (n+1)\left|\int_0^{\infty} \frac{e^{-\tau}\tau^{n+1}}{(n+1)!}\, g\left(\frac{x\tau}{n}\right) d\tau\right| dx \leqq$$

$$\leqq \int_{2b}^{\infty} \int_{an/x}^{bn/x} \frac{e^{-\tau}\tau^{n+1}}{n!} \left|g\left(\frac{x\tau}{n}\right)\right| d\tau\, dx = \int_0^{n/2} \frac{e^{-\tau}\tau^{n+1}}{n!} \int_{an/\tau}^{bn/\tau} \left|g\left(\frac{x\tau}{n}\right)\right| dx\, d\tau \leqq$$

$$\leqq K_g \int_0^{n/2} \frac{e^{-\tau}\tau^{n+1}}{n!}\, d\tau \leqq K_g n \frac{e^{-n/2}(n/2)^{n+1}}{n!} = o(1)$$

where, at the last step we used the Stirling formula. Thus,

$$\int_0^{\infty} p_n(x)\,dx \leqq K_g, \quad |p_n(x)| \leqq K_g$$

which imply by the Hölder-inequality

$$|A_n(f, g)| \leqq \int_0^{\infty} |f(x)|\, p_n(x)\,dx \leqq \|f\|_p \|p_n\|_q \leqq$$

$$\leqq \|f\|_p \|p_n\|_{\sup}^{1-1/q} \|p_n\|_1^{1/q} \leqq K_g \|f\|_p \quad \left(\frac{1}{p} + \frac{1}{q} = 1\right)$$

what was to be proved.

An easy consideration gives (cf. [8, Lemma 5.5] and [3]) that if $f \in C^2(0, \infty) \cap \cap L^p(0, \infty)$ then

$$\lim_{n\to\infty} n\big(G_n(f; x) - f(x)\big) = \frac{1}{2} x^2 f''(x)$$

uniformly on compact subsets of $(0, \infty)$. Thus, if $f \in C^2(0, \infty) \cap L^p(0, \infty)$ and $g \in \in C^2(0, \infty)$ has compact support in $(0, \infty)$ then

$$A(f, g) \stackrel{\text{def}}{=} \lim_{n\to\infty} A_n(f, g) = \frac{1}{2} \int_0^{\infty} x^2 f''(x)\, g(x)\,dx$$

exists and integrating by parts twice we can see that

$$A(f, g) = \frac{1}{2} \int_0^{\infty} \big(x^2 g(x)\big)'' f(x)\,dx.$$

Since we verified that $|A_n(f, g)| \leqq K_g \|f\|_p$ and $C^2(0, \infty) \cap L^p(0, \infty)$ is dense in $L^p(0, \infty)$, it follows that $A(f, g)$ exists for every $f \in L^p(0, \infty)$ and $g \in C^2(0, \infty)$, support $g \subseteqq (0, \infty)$, furthermore,

$$A(f, g) = \frac{1}{2} \int_0^\infty \left(x^2 g(x)\right)'' f(x)\, dx.$$

Now let us suppose that $\|G_n(f)-f\|_p = O(n^{-1})$. By weak-compactness there is a subsequence $\{n_k\}$ of the natural numbers such that $\{n_k(G_{n_k}(f)-f)\}$ converges weakly to a $h \in L^p$ for $1<p<\infty$ and to a bounded signed Borel measure μ for $p=1$.

Let us consider first the case $p>1$. We have

$$(3.1) \qquad \frac{1}{2} \int_0^\infty f(x)\left(x^2 g(x)\right)'' dx = A(f, g) = \lim_{k\to\infty} A_{n_k}(f, g) = \int_0^\infty h(x)\, g(x)\, dx$$

for every $g \in C^2(0, \infty)$, support $g \subseteqq (0, \infty)$. One solution of (3.1) is given by

$$f_1(x) = \int_x^\infty \int_\tau^\infty \frac{2h(u)}{u^2}\, du\, d\tau$$

and, as a standard argument gives, the associated homogeneous equation $\frac{1}{2} \int_0^\infty f(x)\left(x^2 g(x)\right)'' dx = 0$ $\left(g \in C^2(0, \infty),\ \text{support } g \subseteqq (0, \infty)\right)$ has only solutions that are linear (a.e.) (sketch of this last assertion: putting

$$\left(x^2 g(x)\right)'' = \begin{cases} \ \ 1 & \text{if } \ y \leqq x \leqq y+a \ \text{ or } \ y+3a \leqq x \leqq y+4a \\ -1 & \text{if } \ y+a < x < y+4a \\ \ \ 0 & \text{otherwise} \end{cases}$$

into the preceding equality and differentiating with respect to a it follows that $f(y+a)-3f(x+3a)+2f(y+4a)=0$ for $a \geqq 0$ and almost every $y>0$. Then the same holds everywhere for the continuous function $F_n(x) = \int_x^{x+1/n} f(u)\, du$. If F_n is non-linear then subtracting a linear function from it we can suppose that for some $c, d>0$ $F_n(c)=F_n(d)=0$ but $F_n(x)>0$ for $x \in (c, d)$. If $y+3a \in (c, d)$ is a maximum point of F_n and a is small then $y+a$ and $y+4a$ must again be maximum points — contradiction. Thus, each F_n is linear and so $f(x) = \lim_{n\to\infty} n \int_x^{x+1/n} f(u)\, du$ is also a linear function (a.e.)). Thus,

$$f(x) = ax+b+\int_x^\infty \int_\tau^\infty \frac{2h(u)}{u^2}\, du\, d\tau \quad \text{(a.e.)}$$

by which $x^2 f''(x) = 2h(x) \in L^p(0, \infty)$. Now (1.2) follows from this by [5, Theorem 2].

Similarly, in the case $p=1$ we have

$$f(x) = ax+b+\int_x^\infty \int_\tau^\infty \frac{2\, d\mu(u)}{u^2}\, d\tau.$$

Here $a=0$ since the last term on the right tends to zero as $x\to\infty$ and $f\in L^1(0,\infty)$; and so

$$\frac{1}{2}x^2 f'(x) = x^2\int_x^\infty \frac{d\mu(u)}{u^2}.$$

We shall prove that $\frac{1}{2}x^2 f'(x)$ is of bounded variation on $(0,\infty)$ and then [5, Theorem 2] yields (1.2). If $|\mu|$ denotes the total variation of μ then for all points $0<x_1<\ldots<x_{n+1}<\infty$ we have

$$\sum_{i=1}^n\left|\frac{1}{2}x_i^2 f'(x_i)-\frac{1}{2}x_{i+1}^2 f'(x_{i+1})\right| \le \sum_{i=1}^n x_i^2\int_{x_i}^{x_{i+1}}\frac{d|\mu|(u)}{u^2}+$$

$$+\sum_{i=1}^n (x_{i+1}^2-x_i^2)\int_{x_{i+1}}^\infty \frac{d|\mu|(u)}{u^2} \le \sum_{i=1}^n\int_{x_i}^{x_{i+1}} d|\mu|(u)+$$

$$+\sum_{i=2}^n x_i^2\int_{x_i}^{x_{i+1}}\frac{d|\mu|(u)}{u^2}+x_{n+1}^2\int_{x_{n+1}}^\infty\frac{d|\mu|(u)}{u^2} \le 2|\mu|\quad(0,\infty)<\infty$$

which proves that $\frac{1}{2}x^2 f''(x)$ has finite total variation on $(0,\infty)$.

The proof of Theorem 1 is complete.

Finally, we prove the corollaries. Corollary 2 is an immediate consequence of Theorem 1 and [5, Theorem 2]. To prove Corollary 1 let $g(x)=f(e^x)$ $\big(x\in(-\infty,\infty)\big)$. Making the change of variable $\log x=u$ we obtain that for $0<h<1$

$$\Big\{\int_0^\infty |\Delta_{hx}^2(f;\,x)|^p\,dx\Big\}^{1/p} \le \Big\{\int_{-\infty}^\infty e^u|\Delta^2_{\log(1+h)}(g;\,u)|^p\Big\}^{1/p}+$$

$$+\Big\{\int_{-\infty}^\infty e^u\big|g\big(u-\log(1-h)\big)-g\big(u+\log(1+h)\big)\big|^p\,du\Big\}^{1/p}=I_1(h)+I_2(h).$$

If we assume (1.3) then $I_1(h)\le Kh^{2\alpha}$. Let $0<\tau<1$ and let m be chosen according to $\frac{1}{2}\le 2^m\tau\le 1$. Since

$$g(x+2\tau)-g(x)-2\big(g(x+\tau)-g(x)\big)=\Delta_\tau^*(g;\,x)$$

where

$$\Delta_\tau^*(g;\,x)=g(x)-2g(x+\tau)+g(x+2\tau)$$

is the second order forward difference of g, we obtain

$$\Big\{\int_{-\infty}^\infty e^u|g(u+\tau)-g(u)|^p\Big\}^{1/p} \le \frac{1}{2}\Big\{\int_{-\infty}^\infty e^u|g(u+2\tau)-g(u)|^p\,du\Big\}^{1/p}+$$

$$+\frac{1}{2}\Big\{\int_{-\infty}^\infty e^u|\Delta_\tau^*(g;\,u)|^p\,du\Big\}^{1/p}$$

the iteration of which yields

$$\left\{\int_{-\infty}^{\infty} e^u |g(u+\tau)-g(u)|^p\,du\right\}^{1/p} \leqq \sum_{i=1}^{m-1} \frac{1}{2^i}\left\{\int_{-\infty}^{\infty} e^u\,|\Delta^*_{2^{i-1}\tau}(g;\,u)|^p\,du\right\}^{1/p} +$$

$$+\frac{1}{2^m}\left\{\int_{-\infty}^{\infty} e^u\,|\Delta^*_{2^{m-1}\tau}(g;\,u)|^p\,du\right\}^{1/p} \leqq$$

$$\leqq K\sum_{i=1}^{m-1} 2^{-i} I_1(2^{i-1}\tau)+K2^{-m}\|f\|_p \leqq K\sum_{i=1}^{m} 2^{-i}(2^i\tau)^{2\alpha} \leqq$$

$$\leqq \begin{cases} \tau^{2\alpha} & \text{if } \ 0<2\alpha<1 \\ \left(\log\dfrac{1}{\tau}\right)\tau & \text{if } \ 2\alpha=1 \\ \tau & \text{if } \ 1<2\alpha\leqq 2 \end{cases} \leqq K\tau^{\alpha}.$$

Putting here $\tau=-\log(1-h)-\log(1+h)\sim h^2$ we get

$$I_2(h) \leqq K\left\{\int_{-\infty}^{\infty} e^u\,|g(u+\tau)-g(u)|^p\right\}^{1/p} \leqq K\tau^{\alpha} \leqq Kh^{2\alpha}.$$

Collecting our estimates we can see that (1.3) implies (1.2). That (1.2) also implies (1.3) can be proved by the same method (using the transformation $e^u=x$) and the proof is complete.

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