Functional equations of sum form

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1. Introduction

Let Γ_n denote the set of all complete n-ary probability distributions, that is

$$\Gamma_n = \left\{p = (p_1, \dots, p_n) \mid p_i \geqq 0 \quad i = 1, \dots, n; \ \sum_{i=1}^{n} p_i = 1\right\}$$

and $\Gamma_n^0 = \{p = (p_1, \dots, p_n) \mid p_i > 0 \quad i = 1, \dots, n; \ \sum_{i=0}^{n} p_i = 1\}$. The characterization of entropies of degree α leads to the functional equation

$$\sum_{i=1}^{k} \sum_{j=1}^{l} f(p_i q_j) = \sum_{i=1}^{k} f(p_i) + \sum_{j=1}^{l} f(q_j) + \lambda \sum_{i=1}^{k} f(p_i) \sum_{j=1}^{l} f(q_j) \tag{1}$$

where $p \in \Gamma_k$, $q \in \Gamma_l$, $\lambda = 2^{1-\alpha} - 1$ is a constant and $f: [0, 1] \to R$ is the unknown function (see e.q. ACZÉL—DARÓCZY [3], LOSONCZI [15]). Allowing different functions in (1) we obtain

$$\sum_{i=1}^{k} \sum_{j=1}^{l} [f_{ij}(p_i q_j) - q_j g_i(p_i) - p_i h_j(q_j) - \lambda g_i(p_i) h_j(q_j)] = 0 \tag{2}$$

for $p \in \Gamma_k$, $q \in \Gamma_l$. In some cases p, q lie in Γ_k^0, Γ_l^0 respectively. Equations like (2) also arise in various characterizations of measures of information depending on two probability distributions.

The aim of this paper is to investigate a wide class of functional equations and obtain general methods for their solution.

We study the equation

$$\sum_{i=1}^{k} \sum_{j=1}^{l} F_{ij}(p_i, q) = 0 \tag{3}$$

where either $F_{ij}: [0, 1]^2 \to R$, $p \in \Gamma_k$, $q \in \Gamma_l$ or $F_{ij}: (0, 1)^2 \to R$, $p \in \Gamma_k^0$, $q \in \Gamma_l^0$ and $k, l \geqq 3$ are fixed integers. An equation of form (3) is called a *functional equation of sum form*.

In Section 2 we prove some lemmas which serve as our main tool in finding the general solution of (3) in Section 3.

In Section 4 we specialize our results for the equation

$$\sum_{i=1}^{k}\sum_{j=1}^{l}\left[f_{ij}(p_i q_j)-\sum_{t=1}^{N} g_{it}(p_i)h_{jt}(q_j)\right]=0 \quad (p\in\Gamma_k,\ q\in\Gamma_l) \tag{4}$$

(or $p\in\Gamma_k^0$, $q\in\Gamma_l^0$). This is the most important (but still quite general) special case of (3). Equation (3) has been solved if all functions F_{ij} are equal and $p\in\Gamma_k$, $q\in\Gamma_l$ (see Losonczi—Maksa [18]). *Apart from this and a result of Kannappan* [13] concerning the measurable solutions of (3) if $p\in\Gamma_k$, $q\in\Gamma_l$ *all sum form equations which have been investigated so far are of form* (4). In some cases however some of the functions g_{it}, h_{jt} are known (Kannappan [11], [12], Losonczi—Maksa [17] (while in other cases the condition $k, l\geqq 3$ is not satisfied (Daróczy [7], Daróczy—Járai [8], Losonczi [15], Maksa [19]).

We show that under measurability conditions (4) can be reduced to the equations

$$\bar{f}_{ij}(xy)-\sum_{t=1}^{N}\bar{g}_{it}(x)\bar{h}_{jt}(y)=0 \quad (x,\ y\in[0,1]) \tag{5}$$

($i=1,\ldots,k$; $j=1,\ldots,l$) where $\bar{f}_{ij}$, $\bar{g}_{it}$, $\bar{h}_{jt}$ are new unknown functions obtained from f_{ij}, g_{it}, h_{jt} by adding suitable linear functions. Concerning (5) see Aczél [1], Aczél—Chung [2], Járai [10], Losonczi [16], McKiernan [20], Székelyhidi [24], Vincze [25].

We remark that if the domain is open (i.e. $p\in\Gamma_k^0$, $q\in\Gamma_l^0$) or the conditions $k, l\geqq 3$ are not satisfied only the measurable solutions of some special cases of (4) have been found (Kannappan—Ng [14], Sahoo [22], [23], Daróczy—Járai [8]). Finding the general solution of equations of type (4) requires individual treatment for each equation (see Losonczi—Maksa [17], [18]).

In Section 5 we show how the general methods of sections 3, 4 work by solving some equations of type (4). Related equations have been investigated by several authors, among others by Aczél—Daróczy [3], Behara—Nath [5], Chaundy—McLeod [6], Kannappan [12], [13], Losonczi [15], Maksa [19], Mittal [21].

Through the paper we shall apply, if convenient, the following notations. I and Δ_n will denote either $[0, 1]$ and Γ_n or $(0, 1)$ and Γ_n^0.

If f is a real valued function defined on a set G containing I^k and $r, 2r\in I$ then the difference operator D_i^r $(i=1,\ldots,k)$ is given by

$$(D_i^r f)(x_1,\ldots,x_i,\ldots,x_k)=$$
$$=f(x_1,\ldots,x_i,\ldots,x_k)+f(x_1,\ldots,2r,\ldots,x_k)-2f(x_1,\ldots,r,\ldots,x_k) \quad (x_1,\ldots,x_k)\in G.$$

If f is a function of a single variable then we use D^r instead of D_1^r.

It is easy to check that D_i^r is a linear operator having the following properties,

$$D_i^r D_j^s = D_j^s D_i^r$$

$$(D_i^r)^2 = D_i^r$$

and

$$D_i^r f = f$$

if f is additive in the ith variable.

2. Basic lemmas

Our main tool in solving sum form equations in the next lemma which has been proved in a less general form by LOSONCZI—MAKSA [18].

Lemma 1. *Let* $k \geqq 3$ *be a fixed integer,* c *a constant. The functions* $\Phi_i\colon I \to R$ $(i=1, \dots, k,\ R$ *is the set of reals) satisfy the functional equation*

$$\sum_{i=1}^{k} \Phi_i(p_i) = c \quad (p \in \Delta_k) \tag{6}$$

if and only if there exists an additive function $a\colon R \to R$ *and constants* b_i $(i=1, \dots, k)$ *such that*

$$\Phi_i(x) = a(x) + b_i \quad (x \in I), \tag{7}$$

$$a(1) + \sum_{i=1}^{k} b_i = c \tag{8}$$

hold.

PROOF. *Case* 1: $I=(0, 1)$, $\Delta_k = \Gamma_k^0$. Let *first* $k>3$. Take an ε from $(0, 1)$ and choose two different indices n, r from the set $\{2, \dots, k\}$. Let $0<x, y, x+y<\varepsilon$. Substituting into (6)

$$p_1 = x,\ p_n = y,\ p_r = \varepsilon-(x+y),\ p_s = \frac{1-\varepsilon}{k-3} \quad (s \neq 1, n, r) \tag{9}$$

we get

$$\Phi_1(x) + \Phi_n(y) + \Phi_r\big(\varepsilon-(x+y)\big) = c_\varepsilon \tag{10}$$

where c_ε is a constant. Since the right hand side of (10) is symmetric in x, y we obtain

$$\Phi_n(x) - \Phi_1(x) = \Phi_n(y) - \Phi_1(y) \quad (0 < x, y, x+y < \varepsilon).$$

Substituting $y_1, y_2 \in (0, \varepsilon)$ here and observing that the left hand side is the same for $x \in (0, \varepsilon - \max\{y_1, y_2\})$ we conclude that

$$\Phi_n(y_1) - \Phi_1(y_1) = \Phi_n(y_2) - \Phi_1(y_2).$$

Since $\varepsilon \in (0, 1)$ is arbitrary

$$\Phi_n(y) - \Phi_1(y) = d_n \quad (y \in (0, 1);\ n = 2, \dots, k) \tag{11}$$

where d_n's are constants. Using (11) we obtain from (10)

$$\Phi_1(x) + \Phi_1(y) + \Phi_1\big(\varepsilon-(x+y)\big) = e_\varepsilon \quad (0 < x, y, x+y < \varepsilon) \tag{12}$$

where e_ε is a constant. Substituting $x = \frac{\varepsilon}{2}$ then $y = \frac{\varepsilon}{2}$ and deducting from (12) the equations so obtained we get

$$\Phi_1\big(\varepsilon-(x+y)\big) = \Phi_1\left(\frac{\varepsilon}{2}-x\right) + \Phi_1\left(\frac{\varepsilon}{2}-y\right) + e_\varepsilon \quad \left(0 < x, y < \frac{\varepsilon}{2}\right).$$

This means that the function

$$\Psi(x)=\Phi_1(x)-e_\varepsilon$$

is additive on the square $S_{\varepsilon/2}=\left\{(x, y)|0<x, y<\frac{\varepsilon}{2}\right\}$, that is

$$\Psi(u+v) = \Psi(u)+\Psi(v) \quad \text{if} \quad (u, v)\in S_{\varepsilon/2}.$$

Applying the quasi-extension theorem of Daróczy—Losonczi ([9], Theorem 4) we conclude that there exists a function $a_\varepsilon\colon R\to R$ additive on $R\times R$ such that

$$\Psi(u)-2\Psi(\alpha) = a_\varepsilon(u)-2a_\varepsilon(\alpha) \quad u\in(0, \varepsilon)$$

$$\Psi(u)-\Psi(\alpha) = a_\varepsilon(u)-a_\varepsilon(\alpha) \quad u\in\left(0, \frac{\varepsilon}{2}\right)$$

where (α, α) is a fixed point of $S_{\varepsilon/2}$. Therefore $\Psi(\alpha)=a_\varepsilon(\alpha)$ and

(13) $$\Phi_1(u) = \Psi(u)+e_\varepsilon = a_\varepsilon(u)+e_\varepsilon \quad \text{if} \quad u\in(0, \varepsilon).$$

Hence, for $0<\varepsilon_1, \varepsilon_2<1$ we have $a_{\varepsilon_1}(u)+e_{\varepsilon_1}=a_{\varepsilon_2}(u)+e_{\varepsilon_2}$ if $u\in(0, \min\{\varepsilon_1, \varepsilon_2\})$ which implies $a_{\varepsilon_1}=a_{\varepsilon_2}=a$, $e_{\varepsilon_1}=e_{\varepsilon_2}=e$ and

(14) $$\Phi_1(u) = a(u)+e$$

for $u\in(0, \varepsilon)$ thus for $u\in(0, 1)$ as well. By (11), (14)

$$\Phi_n(x) = a(x)+b_n \quad (x\in(0, 1),\ n = 1, \ldots, k)$$

which proves (7).

If $k=3$ then instead of (9) we use the substitution

$$p_1 = x, \quad p_n = y, \quad p_r = 1-(x+y)$$

where $0<x, y, x+y<1$. Repeating the calculations above with $\varepsilon=1$ (and with some omissions) we get (7).

It is easy to see that (7) is a solution of (6) if and only if (8) holds.

Case 2: $I=[0, 1]$, $\Delta_k=\Gamma_k$. By Case 1 we have (7) for $x\in(0, 1)$ and (8). Thus we only have to show that $\Phi_i(0)=b_i$, $\Phi_i(1)=a(1)+b_i$ $(i=1, \ldots, k)$ i.e. (8) holds for $x=0$ and $x=1$. Substituting e.g. $p_1=0$, $(p_2, \ldots, p_k)\in\Gamma^0_{k-1}$ into (6) and using (7) for $x\in(0, 1)$ and (8) we get

$$c = \Phi_1(0)+\sum_{i=2}^{k}[a(p_i)+b_i] = \Phi_1(0)+a(1)+\sum_{i=2}^{k} b_i$$

that is $\Phi_1(0)=b_1$. Similarly $\Phi_i(0)=b_i$ for $i=2, \ldots, k$. With $p_1=1$, $p_2=\ldots=p_k=0$ we obtain from (6)

$$\Phi_1(1)+\sum_{i=2}^{k} b_i = c$$

hence, by (8), $\Phi_1(1)=a(1)+b_1$ and in the same way $\Phi_i(1)=a(1)+b_i$ $(i=2, \ldots, k)$.

Corollary 1. *Equation* (6) *holds if and only if*

(15) $$\Phi_i(x)+\Phi_i(2r)-2\Phi_i(r) = a(x) \quad (x\in I,\ i = 1, \ldots, k)$$

or if $(D^r\Phi_i)(x)=a(x)$ *and*

$$a(1) = c + \sum_{i=1}^{k} [\Phi_i(2r) - 2\Phi_i(r)] \tag{16}$$

holds where r *is a fixed value in* I *such that* $2r\in I$.

PROOF. From (7) with $x=2r$ and $x=r$ using $a(2r)=2a(r)$ we easily obtain $b_i=2\Phi_i(r)-\Phi_i(2r)$ hence (7), (8) give (15), (16).

Corollary 2. *Suppose that one of the functions* $\Phi_1, \ldots, \Phi_k$ *is measurable. Then* (6) *holds if and only if*

$$\Phi_i(x)-\Phi_i(r)-\frac{c-\sum_{m=1}^{k}\Phi_m(r)}{1-kr}(x-r) = 0 \quad (x\in I,\ i=1, \ldots, k) \tag{17}$$

where $r\in I$ *is a fixed number with* $kr-1\neq 0$.

PROOF. If e.g. Φ_1 is measurable then by (7) so is the additive function a hence (see Aczél [1]) $a(x)=a(1)x$. With $x=r$ from (7) $\Phi_i(x)-\Phi_i(r)=a(1)(x-1)$, calculating $a(1)$ here we get exactly (17).

Lemma 2. *Let* $k, l\geqq 3$ *be fixed integers and suppose that the functions* $F_{ij}\colon I^2\to R$ $(i=1, \ldots, k;\ j=1, \ldots, l)$ *satisfy the equation*

$$\sum_{i=1}^{k}\sum_{j=1}^{l} F_{ij}(p_i, q_j) = 0 \quad (p\in \Delta_k,\ q\in \Delta_l). \tag{3}$$

Then there exist functions $a_j\colon R\times I\to R$ $(j=1, \ldots, l)$ *and* $b_i\colon I\times R\to R$ $(i=1, \ldots, k)$ *such that* $a_j(\cdot, y)$, $b_i(x, \cdot)$ *are additive for every fixed value of* $y, x\in I$ *respectively and for every fixed values* $r, s\in I$ *with* $2r, 2s\in I$ *we have*

$$(D_2^s D_1^r F_{ij})(x, y) = (D_2^s D_1^r F_{11})(x, y)+a_j(x, y)+b_i(x, y) \tag{18}$$

$$(x, y\in I,\ i = 1, \ldots, k;\ j = 1, \ldots, l)$$

moreover $a_1=b_1\equiv 0$.

PROOF. Equation (3) can be rewritten as

$$\sum_{i=1}^{k} \Phi_i(p_i, q_1, \ldots, q_l) = 0 \quad (p\in \Delta_k,\ q\in \Delta_l) \tag{19}$$

where

$$\Phi_i(p, q_1, \ldots, q_l) = \sum_{j=1}^{l} F_{ij}(p, q_j).$$

From corollary 1 formula (15) we get

$$D_1^r\Phi_i - D_1^r\Phi_1 = 0 \quad (i = 1, \ldots, k)$$

that is

$$\sum_{j=1}^{l} [(D_1^r F_{ij})(p, q_j)-(D_1^r F_{1j})(p, q_j)] = 0.$$

Applying corollary 1 again there are functions $b_2: I\times R\to R$ additive in the second variable such that

$$D_2^s(D_1^r F_{ij}-D_1^r F_{1j})(x, y) = b_i(x, y)$$

or

$$(D_2^s D_1^r F_{ij})(x, y)-(D_2^s D_1^r F_{1j})(x, y) = b_i(x, y) \tag{20}$$

with $b_1\equiv 0$, $i=1, \dots, k$.

Interchanging the roles of the indices i, j in the above reasoning we get.

$$(D_2^s D_1^r F_{ij})(x, y)-(D_2^s D_1^r F_{i1})(x, y) = a_j(x, y) \quad (j = 1, \dots, l) \tag{21}$$

where a_j's are as described in lemma 2 and $a_1\equiv 0$. From (20) and (21) our representation (18) follows immediately.

Lemma 3. *If* (3) *holds then we have*

$$(D_2^s D_1^r F_{ij})(x, y) = \bar{a}_j(x, y)+\bar{b}_i(x, y)+\bar{c}_{ij}(x, y) \tag{22}$$
$$(x, y\in I,\ i = 1, \dots, k;\ j = 1, \dots, l)$$

where $\bar{a}_j: R\times I\to R$, $\bar{b}_i: I\times R\to R$ *are functions additive in the first, second variable respectively,* $\bar{c}_{ij}: R\times R\to R$ *are biadditive functions (i.e. additive in both variables),* r, s *are fixed elements of* I *with* $2r, 2s\in I$.

Proof. Writing (3) in the form (19) and applying corollary 1 we get

$$\sum_{j=1}^{l} (D_1^r F_{ij})(x, q_j) = a(x, q_1, \dots, q_l) \tag{23}$$

where $a: R\times \Delta_l\to R$ is additive in the first variable. Let now $H=\{h_\gamma | \gamma\in S\}$ be a Hamel-basis of the real numbers over the field of rationals such that $H\subset I$. If $x=\sum_\alpha r_\alpha(x) h_\alpha(x)=\sum_\alpha r_\alpha h_\alpha$ (r_α is rational, $h_\alpha\in H$ the summation extends to a finite number of indices depending on x) we have

$$a(x, q_1, \dots, q_l) = \sum_\alpha r_\alpha a(h_\alpha, q_1, \dots, q_l)$$

thus by (23)

$$\sum_{j=1}^{l} \left[(D_1^r F_{ij})(x, q_j)-\sum_\alpha r_\alpha (D_1^r F_{ij})(h_\alpha, q_j)\right] = 0 \quad (i = 1, \dots, k).$$

We apply corollary 1 again (keeping x, i fixed) to get

$$(D_2^s D_1^r F_{ij})(x, y)-\sum_\alpha r_\alpha (D_2^s D_1^r F_{ij})(h_\alpha, y) = \bar{b}_j(x, y) \quad (x, y\in I) \tag{24}$$

where $\bar{b}_j: I\times R\to R$ is additive in the second variable. Define $a_{ij}: R\times I\to R$ by the equation

$$a_{ij}(x, y) = \sum_\alpha r_\alpha (D_2^s D_1^r F_{ij})(h_\alpha, y) \tag{25}$$

if $x=\sum_\alpha r_\alpha h_\alpha\in R$, $y\in I$ then a_{ij} is additive in the first variable and by (24), (25)

$$(D_2^s D_1^r F_{ij})(x, y) = a_{ij}(x, y)+\bar{b}_j(x, y) \quad (x, y\in I). \tag{26}$$

Interchanging the roles of the variables p_i, q_j in the above argument we get

$$(D_1^r D_2^s F_{ij})(x, y) = \bar{a}_i(x, y) + b_{ij}(x, y) \quad (x, y \in I) \tag{27}$$

where $\bar{a}_i$, b_{ij} are additive in the first, second variable respectively.

A comparison of (26), (27) shows that

$$a_{ij}(x, y) - \bar{a}_i(x, y) = b_{ij}(x, y) - \bar{b}_j(x, y) \quad (x, y \in I). \tag{28}$$

Defining $c_{ij}(x, y)$ by the right hand side of this equation we see that $c_{ij}: I \times R \to R$ is additive in the second variable and by (28) it is also additive in the first variable on the triangle $\{(u, v) | u, v, u+v \in I\}$. Applying theorem 4 of [9] we conclude that there exists a biadditive extension $\bar{c}_{ij}: R \times R \to R$ of c_{ij}. Since $b_{ij}(x, y) = \bar{b}_j(x, y) + \bar{c}_{ij}(x, y)$ we obtain (22) from (27).

3. The solution of equation (3)

First we determine the *measurable solutions of* (3).

Theorem 1. *Suppose that* $k, l \geqq 3$ *are fixed integers,* $F_{ij}: I^2 \to R$ *are functions such that* $F_{1j}(\cdot, y)$ $(j=1, \dots, l)$, $F_{i1}(x, \cdot)$ $(i=1, \dots, k)$ *are measurable for every fixed value of* $y, x \in I$ *respectively. Then* (3) *holds if and only if*

$$\begin{aligned} F_{ij}(x, y) = {} & F_{ij}(x, s) + F_{ij}(r, y) - F_{ij}(r, s) + \\ & + \frac{x-r}{kr-1} \sum_{u=1}^{k} [F_{uj}(r, y) - F_{uj}(r, s)] + \frac{y-s}{ls-1} \sum_{v=1}^{l} [F_{iv}(x, s) - F_{iv}(r, s)] + \\ & + \frac{(x-r)(y-s)}{(kr-1)(ls-1)} \sum_{u=1}^{k} \sum_{v=1}^{l} F_{uv}(r, s) \quad (x, y \in I,\ i = 1, \dots, k;\ j = 1, \dots, l) \end{aligned} \tag{29}$$

holds where r, s *are fixed numbers in* I *with* $(kr-1)(ls-1) \neq 0$.

In other words, if $F_{i1}(x, \cdot)$, $F_{ij}(\cdot, y)$ are measurable then the general solution of (3) is given by (29) where $x \to F_{ij}(x, s)$, $y \to F_{ij}(r, y)$ are arbitrary functions (measurable if $i=1$, $j=1$ respectively) having the same value $F_{ij}(r, s)$ at $x=r$, $y=s$ respectively.

Proof. Writing (3) in the form (19) and applying corollary 2 we obtain

$$\sum_{j=1}^{l} \left[F_{ij}(x, q_j) - F_{ij}(r, q_j) - \frac{x-r}{kr-1} \sum_{u=1}^{k} F_{uj}(r, q_j) \right] = 0.$$

Using corollary 2 again we get

$$\begin{aligned} F_{ij}(x, y) - F_{ij}(r, y) - \frac{x-r}{kr-1} \sum_{u=1}^{k} F_{uj}(r, y) - F_{ij}(x, s) + F_{ij}(r, s) + \frac{x-r}{kr-1} \sum_{u=1}^{k} F_{uj}(r, s) - \\ - \frac{y-s}{ls-1} \sum_{v=1}^{l} \left[F_{iv}(x, s) - F_{ir}(r, s) - \frac{x-r}{kr-1} \sum_{u=1}^{k} F_{uv}(r, s) \right] = 0 \end{aligned}$$

which is exactly (29).

The next theorem gives the *general solution* of (3).

Theorem 2. *Let* $k, l\geqq 3$ *be fixed integers,* $F_{ij}: I^2\to R$ $(i=1, \dots, k;\ j=1, \dots, l)$ *functions satisfying* (3). *Then*

$$(D_2^s D_1^r F_{ij})(x, y) = A_j(x, y)+B_i(x, y)+C(x, y) \tag{30}$$

or, in detailed form,

$$\begin{aligned}F_{ij}(x, y) = {} & 2F_{ij}(x, s)-F_{ij}(x, 2s)+2F_{ij}(r, y)-F_{ij}(2r, y)+\\ & +2F_{ij}(r, 2s)+2F_{ij}(2r, s)-F_{ij}(2r, 2s)-4F_{ij}(r, s)+\\ & +A_j(x, y)+B_i(x, y)+C(x, y)\\ & (x, y\in I,\ i = 1, \dots, k;\ j = 1, \dots, l)\end{aligned} \tag{31}$$

where r, s *are constants with* $r, s, 2r, 2s\in I$ $A_j: R\times I\to R$ $(j=1, \dots, l)$ *and* $B_i: I\times\times R\to R$ $(i=1, \dots, k)$ *are additive functions in the first and second variable respectively,* $C: R\times R\to R$ *is a biadditive function such that there are additive functions* $a, b: R\to R$ *with the properties*

$$a(y) = (D_2^s A_j)(1, y)+\sum_{i=1}^{k}[2(D_2^s F_{ij})(r, y)-(D_2^s F_{ij})(2r, y)],\quad (y\in I,\ j = 1, \dots, l) \tag{32}$$

$$b(x) = (D_1^r B_i)(x, 1)+\sum_{j=1}^{l}[2(D_1^r F_{ij})(x, s)-(D_1^r F_{ij})(x, 2s)],\quad (x\in I,\ i = 1, \dots, k) \tag{33}$$

$$\begin{aligned}a(1)+b(1)+C(1, 1) = {} & \sum_{j=1}^{l}[A_j(1, 2s)-2A_j(1, s)]+\sum_{i=1}^{k}[B_i(2r, 1)-2B_i(r, 1)]+\\ & +\sum_{i=1}^{k}\sum_{j=1}^{l}[2F_{ij}(r, 2s)+2F_{ij}(2r, s)-F_{ij}(2r, 2s)-4F_{ij}(r, s)].\end{aligned} \tag{34}$$

Conversely, functions of the form (31) *satisfy* (3) *if* (32), (33), (34) *hold.*

Proof. Suppose that (3) holds. By Lemma 2 and Lemma 3 (formulae (18), (22)) we get (30) where $A_j=a_j+\bar{a}_1$, $B_i=b_i+\bar{b}_1$, $C=\bar{c}_{11}$ are functions with the additivity properties stated in theorem 2. (31) is just a reformulation of (30).

Now we show that (31) is a solution of (3) if and only if (32), (33), (34) hold. Substituting (31) into (3) and using the additivity properties of A_j, B_i, C we get

$$\begin{aligned}& \sum_{j=1}^{l}\Big[A_j(1, q_j)+\sum_{i=1}^{k}\big(2F_{ij}(r, q_j)-F_{ij}(2r, q_j)\big)\Big] =\\ & = -d-C(1, 1)-\sum_{i=1}^{k}\Big[B_i(p_i, 1)+\sum_{j=1}^{l}\big(2F_{ij}(p_i, s)-F_{ij}(p_i, 2s)\big)\Big]\end{aligned} \tag{35}$$

where $d=\sum_{i=1}^{k}\sum_{j=1}^{l}D_{ij}$ and

$$D_{ij} = 2F_{ij}(r, 2s)+2F_{ij}(2r, s)-F_{ij}(2r, 2s)-4F_{ij}(r, s) \tag{36}$$

is the "constant part" of F_{ij}. By corollary 1, applied for the function

$$\Phi_j(y) = A_j(1, y) + \sum_{i=1}^{k} [2F_{ij}(r, y) - F_{ij}(2r, y)],$$

(35) holds if and only if there exists an additive function $a\colon R \to R$ such that (32) and

$$(37) \qquad a(1) = -d - C(1, 1) - \sum_{i=1}^{k} \big[B_i(p_i, 1) + \sum_{j=1}^{l} \big(2F_{ij}(p_i, s) - F_{ij}(p_i, 2s)\big)\big] +$$

$$+ \sum_{j=1}^{l} \big[A_j(1, 2s) - 2A_j(1, s) + \sum_{i=1}^{k} \big(2F_{ij}(r, 2s) - F_{ij}(2r, 2s) - 4F_{ij}(r, s) + 2F_{ij}(2r, s)\big)\big]$$

hold. (37) is again an equation of type (6). Applying corollary 1 for (37) we see that it holds if and only if there is an additive function $b\colon R \to R$ such that (33) and

$$(38) \qquad b(1) = -d - C(1, 1) - a(1) +$$

$$+ \sum_{j=1}^{l} \big[A_j(1, 2s) - 2A_j(1, s) + \sum_{i=1}^{k} \big(2F_{ij}(r, 2s) - F_{ij}(2r, 2s) - 4F_{ij}(r, s) + 2F_{ij}(2r, s)\big)\big] +$$

$$+ \sum_{i=1}^{k} \big[B_i(2r, 1) - 2B_i(r, 1) + \sum_{j=1}^{l} \big(2F_{ij}(2r, s) - F_{ij}(2r, 2s) - 4F_{ij}(r, s) + 2F_{ij}(r, 2s)\big)\big]$$

are valid. Using the definition of D_{ij}, d (38) reduces to (34). Thus we have proved that the functions (31) satisfy (3) if and only if (32), (33), (34) hold.

Remark. *If* $I = [0, 1]$ *we may choose* $r = s = 0$ *in* (31) *and obtain*

$$(31)^* \quad F_{ij}(x, y) = F_{ij}(x, 0) + F_{ij}(0, y) - F_{ij}(0, 0) + A_j(x, y) + B_i(x, y) + C(x, y)$$

as the general solution of (3). ((32), (33), (34) have much simpler forms too).

The following theorem shows how equation (3) can be reduced to the equation

$$\sum_{i=1}^{k} \sum_{j=1}^{l} F(p_i, q_j) = 0 \quad (p \in \Delta_k,\ q \in \Delta_l).$$

Theorem 3. *Suppose that* $k, l \geqq 3$ *are fixed integers,* $F_{ij}\colon I^2 \to R$ *are functions satisfying* (3) *and* r, s *are constants with* $r, s, 2r, 2s \in I$. *Let*

$$(39) \qquad \bar{F}_{ij}(x, y) = (D_2^s D_1^r F_{ij})(x, y) +$$

$$+ x\big[2F_{ij}(r, 2s) - 2F_{ij}(r, s) + \sum_{u=1}^{k} \big(2(D_2^s F_{uj})(r, y) - (D_2^s F_{uj})(2r, y)\big)\big] +$$

$$+ y\big[2F_{ij}(2s, r) - 2F_{ij}(r, s) + \sum_{v=1}^{l} \big(2(D_1^r F_{iv})(x, s) - (D_1^r F_{iv})(x, 2s)\big)\big] -$$

$$- xy\big[\sum_{u=1}^{k} \big(2F_{uj}(2r, s) - 2F_{uj}(r, s)\big) + \sum_{v=1}^{l} \big(2F_{iv}(r, 2s) - 2F_{iv}(r, s)\big) +$$

$$+ \sum_{u=1}^{k} \sum_{v=1}^{l} \big(2F_{uv}(r, 2s) + 2F_{uv}(2r, s) - F_{uv}(2r, 2s) - 4F_{uv}(r, s)\big)\big]$$

if $x, y \in I$, $i=1, \dots, k$; $j=1, \dots, l$. *Then for all possible fixed indices* i, j *we have*

$$\sum_{m=1}^{k} \sum_{n=1}^{l} \bar{F}_{ij}(p_m, q_n) = 0 \quad (p \in \Delta_k,\ q \in \Delta_l). \tag{40}$$

PROOF. By theorem 2

$$(D_2^s D_1^r F_{ij})(x, y) = A_j(x, y) + B_i(x, y) + C(x, y) \tag{30}$$

and (32), (33), (34) hold. Using the additivity properties of A_j, B_i, C we easily obtain from (30) that

$$\begin{aligned} A_j(1, 2s) - 2A_j(1, s) &= (D_2^s D_1^r F_{ij})(1, 2s) - 2(D_2^s D_1^r F_{ij})(r, s) = \\ &= 2F_{ij}(r, 2s) - 2F_{ij}(r, s) \end{aligned} \tag{41}$$

and

$$B_i(2r, 1) - 2B_i(r, 1) = 2F_{ij}(2s, r) - 2F_{ij}(r, s). \tag{42}$$

By (32) and (41) the coefficient of x in $\bar{F}_{ij}$ is exactly $a(y) - A_j(1, y)$ while that of y is $b(x) - B_i(x, 1)$. Thus $\bar{F}_{ij}$ can be rewritten as

$$\begin{aligned} \bar{F}_{ij}(x, y) &= A_j(x, y) + B_i(x, y) + C(x, y) + x[a(y) - A_j(1, y)] + y[b(x) - B_i(x, 1)] - \\ &- xy\Big[\sum_{u=1}^{k} \big(B_u(2r, 1) - 2B_u(r, 1)\big) + \sum_{v=1}^{l} \big(A_v(1, 2s) - 2A_v(1, s)\big) + \sum_{u=1}^{k} \sum_{v=1}^{l} D_{uv}\Big] \end{aligned}$$

where D_{uv} is the expression defined by (36). If we use the additivity properties of A_j, B_i, C, a, b we get

$$\begin{aligned} \sum_{m=1}^{k} \sum_{n=1}^{l} \bar{F}_{ij}(p_m, q_n) &= \sum_{n=1}^{l} A_j(1, q_n) + \sum_{m=1}^{k} B_i(p_m, 1) + C(1, 1) + \\ &+ a(1) - \sum_{n=1}^{l} A_j(1, q_n) + b(1) - \sum_{m=1}^{k} B_i(p_m, 1) - \\ - \sum_{u=1}^{k} \big(B_u(2r, 1) - 2B_u(r, 1)\big) &- \sum_{v=1}^{l} \big(A_v(1, 2s) - 2A_v(1, s)\big) - \sum_{u=1}^{k} \sum_{v=1}^{l} D_{uv} = 0 \end{aligned}$$

which completes the proof.

4. The investigation of equation (4)

Equation (4) can be obtained from (3) by choosing

$$F_{ij}(x, y) = f_{ij}(xy) - \sum_{t=1}^{N} g_{it}(x) h_{jt}(y) \quad (x, y \in I). \tag{43}$$

Theorem 4. *Let* $k, l \geqq 3$ *be fixed integers and assume that the functions* f_{1j}, f_{i1}, g_{it}, h_{jt} $(i=1, \dots, k;\ j=1, \dots, l;\ t=1, \dots, N)$ *are measurable. Then*

$$\sum_{i=1}^{k} \sum_{j=1}^{l} \Big[f_{ij}(p_i q_j) - \sum_{t=1}^{N} g_{it}(p_i) h_{jt}(q_j)\Big] = 0 \quad (p \in \Delta_k,\ q \in \Delta_l) \tag{4}$$

holds if and only if

$$\tilde{F}_{ij}(x, y) - \sum_{t=1}^{N} \bar{g}_{it}(x)\bar{h}_{jt}(y) = 0 \quad (x, y \in I;\ i = 1, \dots, k;\ j = 1, \dots, l) \tag{44}$$

where

$$\tilde{F}_{ij}(x, y) = f_{ij}(xy) - f_{ij}(xs) - f_{ij}(ry) + f_{ij}(rs) + \tag{45}$$

$$+\frac{x-r}{1-kr}\sum_{u=1}^{k}\big(f_{uj}(ry)-f_{uj}(rs)\big)+\frac{y-s}{1-ls}\sum_{v=1}^{l}\big(f_{iv}(xs)-f_{iv}(rs)\big)+$$

$$+\frac{(x-r)(y-s)}{(1-kr)(1-lr)}\sum_{u=1}^{k}\sum_{v=1}^{l}f_{uv}(rs),$$

$$\bar{g}_{it}(x) = g_{it}(x) - g_{it}(r) + \frac{x-r}{1-kr}\sum_{u=1}^{k} g_{ut}(r), \tag{46}$$

$$\bar{h}_{jt}(x) = h_{jt}(x) - h_{jt}(s) + \frac{x-s}{1-ls}\sum_{v=1}^{l} h_{vt}(s) \tag{47}$$

$$(i = 1, \dots, k;\ j = 1, \dots, l;\ t = 1, \dots, N;\ x, y \in I)$$

r, s are being arbitrary elements of I with $kr \neq 1$, $ls \neq 1$.

Proof. Applying theorem 1 for the function (43) we immediately obtain (44).

Remarks 1. If $I=[0, 1]$, $\Delta_n=\Gamma_n$ we may choose $r=s=0$. Then (44) reduces to

$$\bar{f}_{ij}(xy) - \sum_{t=1}^{N} \bar{g}_{it}(x)\bar{h}_{jt}(y) = 0 \quad (x, y \in I;\ i = 1, \dots, k;\ j = 1, \dots, l) \tag{48}$$

where

$$\bar{f}_{ij}(x) = f_{ij}(x) - f_{ij}(0) + x\sum_{u=1}^{k}\sum_{v=1}^{l} f_{uv}(0) \tag{49}$$

and $\bar{g}_{it}$, $\bar{h}_{jt}$ are obtained from (46), (47) with $r=0$, $s=0$.

2. Since (44) holds for every fixed pair (i, j) of indices it is also an equation of type

$$f(xy) - \sum_{t=1}^{N} g_t(x)h_t(y) = 0 \quad (x, y \in I).$$

For the sake of simplicity we specialize theorem 3 only in the case $I=[0, 1]$.

Theorem 5. *If $k, l \geqq 3$ are fixed integers and* (4) *holds for all $p \in \Gamma_k$, $q \in \Gamma_l$ then for all possible fixed pair of indices (i, j) we have*

$$\sum_{m=1}^{k}\sum_{n=1}^{l}\Big[\bar{f}_{ij}(p_m q_n) - \sum_{t=1}^{N}\bar{g}_{it}(p_m)\bar{h}_{jt}(q_n)\Big] = 0 \quad (p \in \Gamma_k,\ q \in \Gamma_l) \tag{50}$$

where $\bar{f}_{ij}$ is given by (49), *and $\bar{g}_{it}$, $\bar{h}_{jt}$ are obtained from* (40), (47) *by substituting $r=s=0$.*

PROOF. We may choose $r=s=0$ hence (39) goes over into

$$\bar{F}_{ij}(x, y) = F_{ij}(x, y) - F_{ij}(x, 0) - F_{ij}(0, y) + F_{ij}(0, 0) +$$

$$+ x \sum_{u=1}^{k} [F_{uj}(0, y) - F_{uj}(0, 0)] + y \sum_{v=1}^{l} [F_{iv}(x, 0) - F_{iv}(0, 0)] + xy \sum_{u=1}^{k} \sum_{v=1}^{l} F_{uv}(0, 0).$$

Now F_{ij} has the form (43) thus

$$\bar{F}_{ij}(x, y) = \bar{f}_{ij}(xy) - \sum_{t=1}^{N} \bar{g}_{it}(x) \bar{h}_{jt}(y).$$

The proof is completed by applying theorem 3.

5. Applications

First we find the *measurable solutions of* (2). Introducing the functions

$$(51) \qquad F_{ij}(x) = x + \lambda f_{ij}(x), \quad G_i(x) = x + \lambda g_i(x), \quad H_j(x) = x + \lambda h_j(x)$$

$x \in [0, 1]$; $i=1, \dots, k$; $j=1, \dots, l$ equation (2) can be written as

$$(52) \qquad \sum_{i=1}^{k} \sum_{j=1}^{l} [F_{ij}(p_i q_j) - G_i(p_i) H_j(q_j)] = 0 \quad (p \in \Gamma_k, \ q \in \Gamma_l)$$

provided that $\lambda \neq 0$. If f_{i1}, f_{1j}, g_i, h_j are measurable then by theorem 4 (52) holds if and only if

$$(53) \qquad \bar{F}_{ij}(xy) = \bar{G}_i(x) \bar{H}_j(y) \quad (x, y \in [0, 1]; \ i = 1, \dots, k; \ j = 1, \dots, l)$$

where by (46), (47) (with $r=s=0$), (49), (51)

$$\bar{F}_{ij}(x) = x + \lambda f_{ij}(x) - \lambda f_{ij}(0) + \lambda x \sum_{u=1}^{k} \sum_{v=1}^{l} f_{uv}(0),$$

$$(54) \qquad \bar{G}_i(x) = x + \lambda g_i(x) - \lambda g_i(0) + \lambda x \sum_{u=1}^{k} g_u(0),$$

$$\bar{H}_j(x) = x + \lambda h_j(x) - \lambda h_j(0) + \lambda x \sum_{v=1}^{l} h_v(0).$$

Lemma 4. *If $f, g, h \colon [0, 1] \to R$ satisfy the functional equation*

$$(55) \qquad f(xy) = g(x) h(y) \quad (x, y \in [0, 1])$$

and g, h are measurable, $g(0)=h(0)=0$ then

$$(56) \qquad f(x) = 0 \quad x \in [0, 1], \quad g(x) = \begin{cases} 0 & x \in E \\ \vartheta(x) & x \in F, \end{cases} \quad h(x) = \begin{cases} \eta(x) & x \in E \\ 0 & x \in F \end{cases}$$

or

$$(57) \qquad f(x) = \begin{cases} 0 & x \in [0, 1) \\ ab & x = 1, \end{cases} \quad g(x) = \begin{cases} 0 & x \in [0, 1) \\ a & x = 1, \end{cases} \quad h(x) = \begin{cases} 0 & x \in [0, 1) \\ b & x = 1 \end{cases}$$

or

$$(58)\quad f(x)=\begin{cases} abx^{\delta} & x\in(0,1] \\ 0 & x=0, \end{cases}\quad g(x)=\begin{cases} ax^{\delta} & x\in(0,1] \\ 0 & x=0, \end{cases}\quad h(x)=\begin{cases} bx^{\delta} & x\in(0,1] \\ 0 & x=0 \end{cases}$$

where E, F *are disjoint measurable sets whose union is* $[0, 1]$, $0\in E$, ϑ, η *are arbitrary measurable functions on* E, F *respectively such that* $\vartheta(x)\neq 0$ $x\in E$, $\eta(0)=0$, $\delta, a\neq 0$, $b\neq 0$ *are arbitrary constants.*

PROOF. *If* $g(1)=0$ *or* $h(1)=0$ then (55) shows that $f(x)=0$ on $[0, 1]$. Let $E=\{x\in[0,1]\,|g(x)=0\}$ then $0\in E$, E is measurable. h is arbitrary on E and zero on $F=[0,1]-E$ since on F $g(x)\neq 0$. Then $h(0)=0$ and g is arbitrary on F. This gives solution (56).

If $g(1)h(1)\neq 0$ then the substitutions $x=1$ and $y=1$ give $h(y)=f(y)/g(1)$ and $g(x)=f(x)/h(1)$ thus $\bar{f}(x)=f(x)/g(1)h(1)$ satisfies

$$(59)\qquad \bar{f}(xy)=\bar{f}(x)\bar{f}(y)\quad (x, y\in[0,1]).$$

We have $\bar{f}(1)=f(1)/g(1)h(1)=1$. We claim that *either* $\bar{f}(x)=0$ *if* $x\in(0,1)$ *or* $\bar{f}(x)\neq 0$ *if* $x\in(0,1)$. Namely if there exists an $x_0\in(0,1)$ with $\bar{f}(x_0)=0$ then by (59) $\bar{f}(x)=0$ for $x\in(0,x_0]$. For any $y\in(0,1)$ we can find an $x\in(0,x_0]$ and a positive integer n such that $y^n=x$. Hence $0=\bar{f}(x)=\bar{f}(y^n)=[\bar{f}(y)]^n$ and $\bar{f}(y)=0$.

If $\bar{f}(x)=0$ $x\in[0,1)$ then we obtain solution (57) with $a=g(1)$, $b=h(1)$.

If $\bar{f}(x)\neq 0$ $x\in(0,1]$ then $\bar{f}(x)=\bar{f}(\sqrt{x})^2>0$ for $x\in(0,1]$. By (59) the function $F(t)=\ln\bar{f}(e^{-t})$ $t\in[0,\infty)$ is measurable and additive in the first quadrant:

$$F(t+s)=F(t)+F(s)\quad t, s\geqq 0.$$

Thus (see ACZÉL [1], ACZÉL—ERDŐS [4]) F is linear and $\bar{f}(x)=x^{\delta}$ if $x\in(0,1]$, $\bar{f}(1)=1$. This gives solution (58) with $a=g(1)$, $b=h(1)$.

Theorem 6. *Let* $k, l\geqq 3$ *be fixed integers* f_{i1}, f_{1j}, g_i, h_j $(i=1,\dots,k;\ j=1,\dots,l)$ *be measurable functions and suppose that* (2) *holds for all* $p\in\Gamma_k$, $q\in\Gamma_l$ *and* $\lambda\neq 0$. *Then*

$$f_{ij}(x)=-\frac{x}{\lambda}+\alpha_{ij}-x\sum_{u=1}^{k}\sum_{v=1}^{l}\alpha_{uv}+\frac{1}{\lambda}\bar{F}_{ij}(x),$$

$$g_i(x)=-\frac{x}{\lambda}+\beta_i-x\sum_{u=1}^{k}\beta_u+\frac{1}{\lambda}\bar{G}_i(x),$$

$$h_j(x)=-\frac{x}{\lambda}+\gamma_j-x\sum_{v=1}^{l}\gamma_v+\frac{1}{\lambda}\bar{H}_j(x)$$

$x\in[0,1]$; $i=1,\dots,k$; $j=1,\dots,l$ *where* $\alpha_{ij}, \beta_i, \gamma_j$ *are arbitrary constants and* $\bar{F}_{ij}, \bar{G}_i, \bar{H}_j$ *are given in the following way. All possible pairs* (i,j) *of indices can be divided in three disjoint sets* A, B, C *such that if* $(i,j)\in A$ *then*

$$\bar{F}_{ij}(x)=0\quad x\in[0,1],\quad \bar{G}_i(x)=\begin{cases} 0 & x\in E_i \\ \vartheta_i(x) & x\in F_i, \end{cases}\quad \bar{H}_j(x)=\begin{cases} \eta_j(x) & x\in E_i \\ 0 & x\in F_i \end{cases}$$

if $(i,j)\in B$ *then*

$$\bar{F}_{ij}(x)=\begin{cases}0 & x\in[0,1)\\ a_ib_j & x=1,\end{cases}\quad \bar{G}_i(x)=\begin{cases}0 & x\in[0,1)\\ a_i & x=1,\end{cases}\quad \bar{H}_j(x)=\begin{cases}0 & x\in[0,1)\\ b_j & x=1\end{cases}$$

if $(i,j)\in C$ *then*

$$\bar{F}_{ij}(x)=\begin{cases}a_ib_jx^{\delta_{ij}} & x\in(0,1]\\ 0 & x=0,\end{cases}\quad \bar{G}_i(x)=\begin{cases}a_ix^{\delta_{ij}} & x\in(0,1]\\ 0 & x=0,\end{cases}\quad \bar{H}_j(x)=\begin{cases}b_jx^{\delta_{ij}} & x\in(0,1]\\ 0 & x=0.\end{cases}$$

Here E_i, F_i *are disjoint measurable sets whose union is* $[0,1]$, $0\in E_i$, ϑ_i, η_j *are arbitrary measurable functions on* E_i, F_i *respectively such that* $\vartheta_i(x)\neq 0$ *on* E_i, $\eta_j(0)=0$. *Depending on* A *the some sets* E_i *may have to be equal.* $a_i\neq 0$, $b_j\neq 0$, δ_{ij} *are arbitrary constants. Depending on* C *some constants* δ_{ij} *may have to be equal.*

The *proof is* immediate if we apply lemma 4 for equation (53).

As *second example* we solve the equation

$$\sum_{i=1}^{k}\sum_{j=1}^{l} f(p_iq_j)=0\quad (p\in\Gamma_k^0,\ q\in\Gamma_l^0) \tag{60}$$

assuming the measurability of f.

Theorem 7. *If* $f\colon(0,1)\to R$ *is measurable,* $k,l\geqq 3$ *and* (60) *holds then*

$$f(x)=a\left(x-\frac{1}{kl}\right)\quad (x\in(0,1)) \tag{61}$$

where a *is a constant.*

PROOF. By theorem 4 (60) holds if and only if

$$f(xy)=f(xs)\frac{ly-1}{ls-1}+f(ry)\frac{kx-1}{kr-1}-f(rs)\frac{kx-1}{kr-1}\,\frac{ly-1}{ls-1} \tag{62}$$

for all $x,y\in[0,1)$, $r,s\in(0,1)$, $kr\neq 1$, $ls\neq 1$. With $x=\frac{1}{k}$ we obtain from (62)

$$f(y/k)=\frac{f(s/k)}{ls-1}(ly-1)=a_1y+b_1\quad (y\in(0,1))$$

or $f(y)=ay+b$ if $y\in\left(0,\frac{1}{k}\right)$. If $r,s<\min\left\{\frac{1}{k},\frac{1}{l}\right\}$ then $xs, ry\in\left(0,\frac{1}{k}\right)$ thus (62) gives

$$f(xy)=Axy+Bx+Cy+D\quad (x,y\in(0,1))$$

where A,B,C,D are constants. Since f is linear on $\left(0,\frac{1}{k}\right)$ we get $A=a$, $B=C=0$, $D=b$ and

$$f(x)=ax+b\quad x\in(0,1).$$

This is a solution of (60) if and only if $b=-\frac{1}{kl}a$.

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