A common fixed point theorem

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We prove the following theorem.

Theorem. *Let S and T be commuting mappings of the complete metric space (X, d) into itself satisfying the inequality*

$$(1) \qquad d(Sx, Ty) \leqq c.\max\{d(x, y), d(x, Sx), d(y, Ty), d(x, Ty), d(y, Sx)\}$$

for all x, y in X, where $0<c<1$. Suppose that

$$(2) \qquad \max\{d(S^i x, S^j x), d(T^i x, T^j x), d(x, S^i T^j x)\colon 0 \leqq i,\ j \leqq n+1\} \leqq$$
$$\leqq b.\max\{d(S^i x, S^j x), d(T^i x, T^j x), d(x, S^i T^j x)\colon 0 \leqq i,\ j \leqq n\}$$

for some x in X and $n=1, 2, \ldots$, where $0<bc<1$. Then S and T have a unique common fixed point z. Further z is the unique fixed point of S and T.

Proof. With an x satisfying inequality (2) let

$$E_n = \{d(S^i x, S^j x), d(T^i x, T^j x), d(x, S^i T^j x)\colon 0 \leqq i,\ j \leqq n\},$$
$$F_n = \{d(S^i T^{i'} x, T^j S^{j'} x)\colon 1 \leqq i,\ j \leqq n;\ 0 \leqq i',\ j' \leqq n\}$$

and let

$$K_n = \max E_n, \quad L_n = \max F_n$$

for $n=1, 2, \ldots$. It follows that inequality (2) can be written in the form

$$(3) \qquad K_{n+1} \leqq bK_n.$$

We now note that inequality (1) can be applied to every term in F_n to give terms in either E_n or F_n and it follows that

$$L_n \leqq c.\max\{K_n, L_n\}.$$

Since $c<1$, we have

$$(4) \qquad L_n \leqq cK_n$$

for $n=1, 2, \ldots$.

Inequality (1) can also be applied n times to the term $d((ST)^n x, T(ST)^{n-1} x)$ and then to the resulting terms, before terms in E_n appear. It follows that

$$d((ST)^n x, T(ST)^{n-1} x) \leqq c^n.\max\{K_n, L_n\} \leqq c^n.\max\{K_n, cK_n\}$$

on using inequality (4) and so since $c<1$

$$d((ST)^n x, T(ST)^{n-1} x) \leqq c^n K_n.$$

Now using inequality (3) it follows that

$$d((ST)^n x, T(ST)^{n-1} x) \leqq c^n b^{n-1} K_1$$

for $n=1, 2, \ldots$.

Similarly we can prove that

$$d(T(ST)^n x, (ST)^n x) \leqq c^n b^{n-1} K_1$$

for $n=1, 2, \ldots$. Since $bc<1$, it follows that the sequence

$$\{x, Tx, STx, \ldots, (ST)^n x, T(ST)^n x, \ldots\}$$

is a Cauchy sequence in the complete metric space X and so has a limit z in X.

Using inequality (1) we have

$$d(Sz, T(ST)^n x) \leqq$$

$$\leqq c.\max\{d(z, (ST)^n x), d(z, Sz), d((ST)^n x, T(ST)^n x), d(z, T(ST)^n x), d((ST)^n x, Sz)\}$$

and on letting n tend to infinity we have

$$d(Sz, z) \leqq cd(Sz, z).$$

It follows that z is a fixed point of S.

Similarly we can prove that z is a fixed point of T.

Now suppose that S has a second fixed point z'. Then

$$d(z', z) = d(Sz', Tz) \leqq$$

$$\leqq c.\max\{d(z', z),\ d(z', Sz'),\ d(z, Tz),\ d(z', Tz),\ d(z, Sz')\} = cd(z', z)$$

and so z is the unique fixed point of S. The proof that z is the unique fixed point of T follows similarly. This completes the proof the theorem.

We note that the condition that inequality (2) holds for some x in X is not necessary if X is bounded, see [2]. It is not known whether or not this condition is necessary if X is unbounded.

The condition that S and T commute is necessary even if X is bounded, see [1].

References

[1] B. Fisher, On a conjecture on common fixed points, *Math. Sem. Notes, Kobe Univ.*, **6** (1978), 153—6.

[2] B. Fisher, Results on common fixed points on bounded metric spaces, *Math. Sem. Notes, Kobe Univ.*, **7** (1979), 73—80.

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