Some identifiability problems involving generalized Waring distributions

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ABSTRACT

In this paper it is shown that generalized Waring distributions (univariate and bivariate) can be determined uniquely from knowledge on the form of certain conditional distributions and some appropriately chosen regression functions.
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1. Introduction

Regressions of the form $E(X|Y=y)=\varphi(y)$ are of interest in many areas mainly for prediction purposes. In the study of accidents, for instance, JOHNSON [3] obtained, in the context of his accident proneness model, a linear form for $\varphi(\cdot)$; then he considered the resulting linear regression model for predicting the number X of accidents to be sustained by an individual in a given time period conditional on the number Y of accidents sustained by the same individual in a preceding time period. In some of the problems considered in the literature which involve conditional expectations of the above form, one of the random variables ($r.v.$'s) involved (Y) is less than a equal to the other (X) and the mechanism through which such a relationship between X and Y is effected has been represented by the conditional distribution $\{P(Y=y|X=x),\ y=0, 1, 2, \dots, x;\ x>0\}$. In this connection, the binomial probability function (p.f.) has been considered for $P(Y=y|X=x)$ and on the assumption of a linear regression of X on Y, the Poisson, binomial or negative binomial distribution has been arrived at as the distribution of X or Y (see e.g. KORWAR [5], XEKALAKI [6] and DAHIYA and KORWAR [1]. Other forms for $\varphi(\cdot)$ or for the distribution of $Y|(X=x)$ have also been considered leading to other forms of distributions (e.g. XEKALAKI [8], [10], [11], IRWIN [2] studied a three parameter univariate distribution, the univariate generalized Waring ditribution (UGWD), which is more general in structure than the previously mentioned three distributions. In point of fact, the Poisson, binomial and negative binomial distributions are limiting forms of the UGWD. (For more details concerning the structure and properties of this distribution see XEKALAKI [9], [11]). It would therefore be interesting to examine whether a similar result holds for this more general distribution. This paper deals with this problem. It is shown in section 2 that if the distribution of Y conditional on $(X=x)$ is binomial $(x; p)$, p fixed and independent of x and the regression of X on Y is of a given form, the probability distribution of X is identified as UGWD. (Identifiability problems for the UGWD and other discrete distributions have also been examined by XEKALAKI and PANARETOS [12]). Similar results are obtained in section 3 for the two-dimensional case. There, the bivariate generalized Waring distribution (BGWD) defined by XEKALAKI [7] is involved. Finally, in section 4, a bivariate generalized Waring distribution with independent components is obtained on the assumption of linear regression for X_i on (Y_1, Y_2), $i=1, 2$ and of a negative hypergeometric distribution for $Y_i|(X_i=x_i)$, $i=1, 2$.

Before obtaining the main results, we provide the definitions of the univariate and bivariate generalized Waring distributions for ease of reference.

A non-negative and integer-valued r.v. X is said to have the univariate generalized Waring distribution with parameters a, k, and ϱ $(\text{UGWD}(a, k; \varrho))$ if its probability generating function (p.g.f.) is given by

$$G_X(s) = \frac{\varrho_{(k)}}{(a+\varrho)_{(k)}}\, {}_2F_1(a, k;\, a+k+\varrho;\, s), \quad a > 0,\ k > 0,\ \varrho > 0\ \ |s| \leqq 1$$

where $a_{(\beta)}=\Gamma(a+\beta)/\Gamma(a)$, $a>0$, $\beta\in R$ and ${}_2F_1$ is the Gauss hypergeometric function obtained from

$${}_\mu F_\nu(\underset{\sim}{a};\, \underset{\sim}{b};\, z) \equiv {}_\mu F_\nu(a_1, a_2, \ldots, a_\mu;\, b_1, b_2, \ldots, b_\nu;\, z) = \sum_{r=0}^{\infty} \frac{(a_1)_{(r)}(a_2)_{(r)} \ldots (a_\mu)_{(r)}}{(b_1)_{(r)}(b_2)_{(r)} \ldots (b_\nu)_{(r)}} \frac{z^r}{r!}$$

for $\mu=2$, $\nu=1$.

A random vector (X, Y) of non-negative and integer-valued components is said to have the bivariate generalized Waring distribution with parameters a, k, m and ϱ $(\text{BGWD}(a; k, m; \varrho))$ if its p.g.f. is

$$G_{X,Y}(s, t) = \frac{\varrho_{(k+m)}}{(a+\varrho)_{(k+m)}} F_1(a;\, k, m;\, a+k+m+\varrho;\, s, t),\ \ a, k, m, \varrho > 0$$

$$(s, t)\in[-1, 1]\times[-1, 1]$$

where F_1 is the Appell hypergeometric function defined by

$$F_1(a;\, b, b';\, c;\, z, w) = \sum_{r=0}^{\infty} \sum_{l=0}^{\infty} \frac{a_{(r+l)}\, b_{(r)}\, b'_{(l)}}{c_{(r+l)}} \frac{z^r}{r!} \frac{w^l}{l!}.$$

2. Identifiability of the UGWD

Theorem 2.1. *Let X and Y be two non-negative integer-valued r.v.s' such that the conditional distribution of Y given $(X=x)$ is the binomial with parameters x and π, that is*

$$P[Y = y|X = x] \equiv g_{y|x} = \binom{x}{y} \pi^y \varphi^{x-y}, \quad 0 < \pi < 1,\ \varphi = 1-\pi. \tag{2.1}$$

Then

$$(2.2)\quad E(X|Y=y)=y+\varphi\frac{(a+y)(k+y)}{(a+k+\varrho+y)}\frac{{}_2F_1(a+y+1,\,k+y+1;\,a+k+y+1;\,\varphi)}{{}_2F_1(a+y,\,k+y;\,a+k+\varrho+y;\,\varphi)}$$

$$a>0,\ k>0,\ \varrho>0,$$

if and only if the distribution of X *is the* UGWD $(a, k; \varrho)$.

PROOF. *Necessity*. Denote by q_y and p_x the p.f.'s of Y and X respectively and let the distribution of X be the UGWD $(a, k; \varrho)$. Then from (2.1) we have

$$(2.3)\qquad G_{X|Y=y}(t)=\frac{1}{q_y}\sum_{x=y}^{\infty}g_{y|x}p_x t^x=\frac{\sum_{x=y}^{\infty}\binom{x}{y}\pi^y\varphi^{x-y}p_x t^x}{\sum_{x=y}^{\infty}\binom{x}{y}\pi^y\varphi^{x-y}p_x}.$$

But

$$\sum_{x=y}^{\infty}\binom{x}{y}\pi^y\varphi^{x-y}p_x t^x=\frac{\varrho_{(k)}}{(a+\varrho)_{(k)}}\sum_{x=y}^{\infty}\frac{\pi^y\varphi^{x-y}}{y!(x-y)!}\frac{a_{(x)}k_{(x)}}{(a+k+\varrho)_{(x)}}t^x=$$

$$=\frac{\varrho_{(k)}}{(a+\varrho)_{(k)}}\frac{(\pi t)^y}{y!}\frac{a_{(y)}k_{(y)}}{(a+k+\varrho)_{(y)}}\sum_{x=0}^{\infty}\frac{(a+y)_{(x)}(k+y)_{(x)}}{(a+k+\varrho+y)_{(x)}}\frac{(\varphi t)^x}{x!}=$$

$$=\frac{\varrho_{(k)}}{(a+\varrho)_{(k)}}\frac{(\pi t)^y}{y!}\frac{a_{(y)}k_{(y)}}{(a+k+\varrho)_{(y)}}\,{}_2F_1(a+y,\,k+y;\,a+k+\varrho+y;\,\varphi t).$$

Substituting in (2.3) we get

$$G_{X|Y=y}(t)=\frac{t^y\,{}_2F_1(a+y,\,k+y;\,a+k+\varrho+y;\,\varphi t)}{{}_2F_1(a+y,\,k+y;\,a+k+\varrho+y;\,\varphi)}.$$

Differentiating with respect to t and then letting $t=1$ (2.2) follows.

Sufficiency. Consider (2.2) and let the distribution of $Y|(X=x)$ be defined by (2.1). We will show that the distribution of X is the UGWD$(a, k; \varrho)$. We have

$$E(X|Y=y)=\frac{1}{q_y}\sum_{x=y}^{\infty}xg_{y|x}p_x.$$

Substituting for $g_{y|x}$ and making use of the identity

$$x\binom{x}{y}=(y+1)\binom{x}{y+1}+y\binom{x}{y}$$

we obtain

$$(2.4)\quad E(X|Y=y)=\frac{y+1}{q_y}\sum_{x=y+1}^{\infty}\binom{x}{y+1}\pi^y\varphi^{x-y}p_x+y=\frac{\varphi}{\pi}(y+1)\frac{q_{y+1}}{q_y}+y.$$

Combining (2.2) and (2.4) we get

$$\frac{q_{y+1}}{q_y}(y+1)=\pi\frac{(a+y)(k+y)}{(a+k+\varrho+y)}\frac{{}_2F_1(a+y+1,\,k+y+1;\,a+k+\varrho+y+1;\,\varphi)}{{}_2F_1(a+y,\,k+y;\,a+k+\varrho+y;\,\varphi)}$$

or equivalently

(4.1.7)

$$q_{y+1}-\frac{\pi}{y+1}\frac{(a+y)(k+y)}{(a+k+\varrho+y)}\frac{{}_2F_1(a+y+1,\,k+y+1;\,a+k+\varrho+y+1;\,\varphi)}{{}_2F_1(a+y,\,k+y;\,a+k+\varrho+y;\,\varphi)}\,q_y=0.$$

This is a first order difference equation in q_y with a solution which is unique under the condition $\sum_{y=0}^{\infty} q_y=1$. Solving, we get

$$q_y = q_0\prod_{i=0}^{y-1}\pi\frac{(a+i)(k+i)}{(a+k+\varrho+i)(i+1)}\frac{{}_2F_1(a+i+1,\,k+i+1;\,a+k+\varrho+i+1;\,\varphi)}{{}_2F_1(a+i,\,k+i;\,a+k+\varrho+i;\,\varphi)}=$$

(2.5)

$$= q_0\pi^y\frac{a_{(y)}k_{(y)}}{(a+k+\varrho)_{(y)}y!}\frac{{}_2F_1(a+y,\,k+y;\,a+k+\varrho+y;\,\varphi)}{{}_2F_1(a,\,k;\,a+k+\varrho;\,\varphi)}.$$

Summing both sides over y we obtain

$$1 = q_0\sum_{y=0}^{\infty}\sum_{r=0}^{\infty}\frac{a_{(y+r)}k_{(y+r)}}{(a+k+\varrho)_{(y+r)}}\frac{\pi^y}{y!}\frac{\varphi^r}{r!}\{{}_2F_1(a,\,k;\,a+k+\varrho;\,\varphi\}^{-1}$$

or equivalently

$$1 = q_0\frac{{}_2F_1(a,\,k;\,a+k+\varrho;\,\pi+\varphi)}{{}_2F_1(a,\,k;\,a+k+\varrho;\,\varphi)} = q_0\frac{(a+\varrho)_{(k)}}{\varrho_{(k)}\,{}_2F_1(a,\,k;\,a+k+\varrho;\,\varphi)}.$$

Therefore

$$q_0 = \frac{\varrho_{(k)}}{(a+\varrho)_{(k)}}\,{}_2F_1(a,\,k;\,a+k+\varrho;\,\varphi).$$

Substituting in (2.5) yields

$$q_y = \frac{\varrho_{(k)}}{(a+\varrho)_{(k)}}\frac{\pi^y}{y!}\frac{a_{(y)}k_{(y)}}{(a+k+\varrho)_{(y)}}\,{}_2F_1(a+y,\,k+y;\,a+k+\varrho+y;\,\varphi).$$

This implies that the p.g.f. of Y is given by

$$G_Y(t) = \frac{\varrho_{(k)}}{(a+\varrho)_{(k)}}\,{}_2F_1(a,\,k;\,a+k+\varrho;\,\pi t+\varphi). \tag{2.6}$$

But

$$G_Y(t) = \sum_{y=0}^{\infty}\sum_{x=y}^{\infty} g_{y|x}\,p_x t^y = \sum_{y=0}^{\infty}\sum_{x=y}^{\infty}\binom{x}{y}\pi^y\varphi^{x-y}p_x t^y = \tag{2.7}$$

$$= \sum_{x=0}^{\infty}p_x\sum_{y=0}^{x}\binom{x}{y}(\pi t)^y\varphi^{x-y} = \sum_{x=0}^{\infty}p_x(\varphi+\pi t)^x = G_X(\pi t+\varphi).$$

Comparing (2.6) to (2.7) we deduce that the distribution of X is the UGWD$(a, k; \varrho)$. Hence the theorem is established.

Notice that the UGWD$(a, k; \varrho)$ belongs to Kemp's [3] family of distributions defined by

$$G(t) = \frac{{}_\mu F_\nu(\underset{\sim}{\alpha};\,\underset{\sim}{\beta};\,\lambda t)}{{}_\mu F_\nu(\underset{\sim}{\alpha};\,\underset{\sim}{\beta};\,\lambda)}, \tag{2.8}$$

for $\lambda=\nu=\mu/2=1$. It is interesting, therefore, to observe that by an argument similar to that used to prove Theorem 2.1 we can show the following more general theorem involving the family in (2.8).

Theorem 2.2. *Let X and Y be two non-negative integer-valued r.v.'s such that the conditional distribution of Y given $(X=x)$ is given by* (2.1). *Then the distribution of X has p.g.f. given by* (2.8) *if and only if*

$$E(X|Y=y)=y+\lambda\varphi\left(\frac{\prod_{i=1}^{\mu}(\alpha_i+y)}{\prod_{i=1}^{\nu}(\beta_i+y)}\right)\frac{{}_\mu F_\nu(\underset{\sim}{\alpha}+(y+1)\underset{\sim}{1};\ \underset{\sim}{\beta}+(y+1)\underset{\sim}{1};\ \lambda\varphi)}{{}_\mu F_\nu(\underset{\sim}{\alpha}+y\underset{\sim}{1};\ \underset{\sim}{\beta}+y\underset{\sim}{1};\ \lambda\varphi)}.$$

Note. The particular cases

(i) $\mu=\nu=0,\ \lambda>0$
(ii) $\mu=1,\ \nu=0,\ \alpha=-n,\ n$ positive integer, $\lambda<0$
(iii) $\mu=1,\ \nu=0,\ \alpha>0,\ \lambda>0$

provide characterizations for the Poisson (λ), the binomial $\left(n;\ \frac{\lambda}{\lambda-1}\right)$ and the negative binomial $\left(a;\ \frac{1}{1+\lambda}\right)$ respectively. Thus, Korwar's [5] results are special cases of Theorem 2.2. Moreover, the result of Theorem 2.1 can be obtained from Theorem 2.2 for $\mu/2=\nu=\lambda=1,\ \underset{\sim}{\alpha}=(a,k)$ and $\underset{\sim}{\beta}=a+k+\varrho$.

3. The two-dimensional case

Theorem 3.1. *Let X_1, X_2, Y_1, Y_2 be non-negative integer-valued r.v.'s such that the conditional distribution of (Y_1, Y_2) given $(X_1=x_1,\ X_2=x_2)$ is the double binomial with probability function*

$$g_{y_1,y_2|x_1,x_2}=\binom{x_1}{y_1}\binom{x_2}{y_2}\pi_1^{y_1}\pi_2^{y_2}\varphi_1^{x_1-y_1}\varphi_2^{x_2-y_2} \tag{3.1}$$

where $0<\pi_1, \pi_2<1,\ \varphi=1-\pi_1,\ \varphi_2=1-\pi_2,\ y_1, y_2>0$.
Then

$$E(X_1|Y_1=y_1, Y_2=y_2)=y_1+\frac{(a+y_1+y_2)(k+y_1)}{a+k+m+\varrho+y_1+y_2}\varphi_1 F(y_1, y_2) \tag{3.2}$$

and

$$E(X_2|Y_1=y_1, Y_2=y_2)=y_2+\frac{(a+y_1+y_2)(m+y_2)}{a+k+m+\varrho+y_1+y_2}\varphi_2 F(y_1, y_2) \tag{3.3}$$

where

$$F(y_1, y_2) =$$

$$= \frac{F_1(a+y_1+y_2+1;\ k+y_1+1,\ m+y_2+1;\ a+k+m+\varrho+y_1+y_2+1;\ \varphi_1, \varphi_2)}{F_1(a+y_1+y_2;\ k+y_1,\ m+y_2;\ a+k+m+\varrho+y_1+y_2;\ \varphi_1, \varphi_2)}$$

$$a, k, m, \varrho > 0$$

if and only if the distribution of (X_1, X_2) *is the* $BGWD(a; k, m; \varrho)$.

PROOF. *Necessity*. Let the distribution of (X_1, X_2) be the BGWD$(a; k, m; \varrho)$ and let that of $(Y_1, Y_2)|(X_1=x_1, X_2=x_2)$ be given by (3.1). Denote by q_{y_1, y_2} and p_{x_1, x_2} the p.f.'s of (Y_1, Y_2) and (X_1, X_2) respectively. Then, it is easily shown that

$$(3.4)\qquad G_{X_1, X_2|Y_1, Y_2}(s, t) = \frac{\varrho_{(k+m)}}{(a+\varrho)_{(k+m)}} \frac{a_{(y_1+y_2)} k_{(y_1)} m_{(y_2)}}{(a+k+m+\varrho)_{(y_1+y_2)}} \frac{(\pi_1 s)^{y_1}}{y_1!} \frac{(\pi_2 t)^{y_2}}{y_2!} \times$$

$$\times F_1(a+y_1+y_2;\ k+y_1,\ m+y_2;\ a+k+m+\varrho+y_1+y_2;\ \varphi_1 s,\ \varphi_2 t)\, q_{y_1, y_2}{}^{-1}.$$

But

$$(3.5)\qquad G_{Y_1, Y_2}(t) = \sum_{y_1, y_2} q_{y_1, y_2} s^{y_1} t^{y_2} = \sum_{x_1, x_2} \sum_{y_1, y_2} q_{y_1, y_2|x_1, x_2} p_{x_1, x_2} s^{y_1} t^{y_2} =$$

$$= \sum_{y_1, y_2} \sum_{x_1 \geqq y_1} \sum_{x_2 \geqq y_2} \binom{x_1}{y_1} \binom{x_2}{y_2} (\pi_1 s)^{y_1} (\pi_2 t)^{y_2} \varphi_1{}^{x_1 - y_1} \varphi_2{}^{x_2 - y_2} p_{x_1, x_2} =$$

$$= \sum_{x_1, x_2} \sum_{y_1=0}^{x_1} \sum_{y_2=0}^{x_2} \binom{x_1}{y_1} \binom{x_2}{y_2} (\pi_1 s)^{y_1} (\pi_2 t)^{y_2} \varphi_1{}^{x_1 - y_1} \varphi_2{}^{x_2 - y_2} p_{x_1, x_2} =$$

$$= \sum_{x_1, x_2} p_{x_1, x_2} (\pi_1 s + \varphi_1)^{x_1} (\pi_2 t + \varphi_2)^{x_2} = G_{X_1, X_2}(\pi_1 s + \varphi_1,\ \pi_2 t + \varphi_2).$$

Here $\sum_{r, l}$ stands for the double summation $\sum_{r=0}^{\infty} \sum_{l=0}^{\infty}$. Therefore, the p.f. of (Y_1, Y_2) is

$$q_{y_1, y_2} = \frac{\varrho_{(k+m)}}{(a+\varrho)_{(k+m)}} \frac{a_{(y_1+y_2)} k_{(y_1)} m_{(y_2)}}{(a+k+m+\varrho)_{(y_1+y_2)}} \frac{\pi_1{}^{y_1}}{y_1!} \frac{\pi_2{}^{y_2}}{y_2!} \times$$

$$\times F_1(a+y_1+y_2;\ k+y_1,\ m+y_2;\ a+k+m+\varrho;\ \varphi_1, \varphi_2).$$

Substituting for q_{y_1, y_2} in (3.4) we obtain

$$G_{X_1, X_2|Y_1, Y_2}(s, t) =$$

$$= \frac{s^{y_1} t^{y_2} F_1(a+y_1+y_2+1;\ k+y_1+1,\ m+y_2+1;\ a+k+m+\varrho+y_1+y_2+1;\ \varphi_1, \varphi_2)}{F_1(a+y_1+y_2;\ k+y_1,\ m+y_2;\ a+k+m+\varrho+y_1+y_2;\ \varphi_1, \varphi_2)}$$

from which (3.2) and (3.3) follow.

Sufficiency. Assume that hypotheses (3.1), (3.2) and (3.3) hold. Using an argument similar to that used in proving theorem 2.1 we can show that

$$E(X_1|Y_1=y_1,\,Y_2=y_2)=y_1+(y_1+1)\frac{\pi_1}{\varphi_1}\frac{q_{y_1+1,y_2}}{q_{y_1,y_2}},\tag{3.6}$$

$$E(X_2|Y_1=y_1,\,Y_2=y_2)=y_2+(y_2+1)\frac{\pi_2}{\varphi_2}\frac{q_{y_1,y_2+1}}{q_{y_1,y_2}}.\tag{3.7}$$

Then, combining (3.6) with (3.2) and (3.7) with (3.3) we obtain

$$\begin{aligned}\frac{q_{y_1+1,y_2}}{q_{y_1,y_2}}&=\frac{\pi_1}{y_1+1}\frac{(a+y_1+y_2)(k+y_1)}{(a+k+m+\varrho+y_1+y_2)}\cdot F,\\ \frac{q_{y_1,y_2+1}}{q_{y_1,y_2}}&=\frac{\pi_2}{y_2+1}\frac{(a+y_1+y_2)(m+y_2)}{(a+k+m+\varrho+y_1+y_2)}\cdot F,\end{aligned}\tag{3.8}$$

where

$$F=\frac{F_1(a+y_1+y_2+1;\;k+y_1+1,\,m+y_2+1;\;a+k+m+\varrho+y_1+y_2+1;\;\varphi_1,\,\varphi_2)}{F_1(a+y_1+y_2;\;k+y_1,\,m+y_2;\;a+k+m+\varrho+y_1+y_2;\;\varphi_1,\,\varphi_2)}.$$

The unique solution to this system under the condition $\sum_{y_1,y_2} q_{y_1,y_2}=1$ is given by

$$q_{y_1,y_2}=q_{0,0}\prod_{i=0}^{y_1-1}h_1(i,0)\prod_{j=0}^{y_2-1}h_2(y_1,j)\tag{3.9}$$

where $h_i(y_1,y_2)$ is the right hand side of the i^{th} equation of system (3.8) $i=1,2$ and $q_{0,0}$ is obtained from

$$q_{0,0}=\Big\{\sum_{y_1,y_2}\Big(\prod_{i=0}^{y_1-1}h_1(i,0)\prod_{j=0}^{y_2-1}h_2(y_1,j)\Big)\Big\}^{-1}.$$

Substituting for $h_i(y_1,y_2)$ and $q_{0,0}$ in (3.9) and taking the p.g.f. we obtain

$$G_{Y_1,Y_2}(s,t)=\frac{\varrho_{(k+m)}}{(a+\varrho)_{(k+m)}}F_1(a;\,k,\,m;\,a+k+m+\varrho;\,\pi_1s+\varphi_1,\,\pi_2t+\varphi_2).\tag{3.10}$$

Comparison of (3.10) to (3.5) shows that the distribution of (X_1,X_2) is the BGWD$(a;k,m;\varrho)$. Hence the theorem is established.

4. A result concerning bivariate generalized Waring distributions with independent components

Consider (X_1,X_2) and (Y_1,Y_2) to be two random vectors with non-negative, integer valued and independent components. Assume that

$$P(Y_1=y_1,Y_2=y_2|X_1=x_1,X_2=x_2)=\prod_{i=1}^{2}\binom{-m_i}{y_i}\binom{-n_i}{x_i-y_i}\Big/\binom{-m_i-n_i}{x_i},\tag{4.1}$$

$$m_i>0,\;n_i>0,\;y_i=0,1,\dots,x_i;\;i=1,2.$$

One may observe that if the distributions of X_1, X_2 are the UGWD$(a_1, m_1+n_1; \varrho_1)$ and UGWD$(a_2, m_2+n_2; \varrho_2)$ respectively then the regressions $E(X_i|Y_1=y_1, Y_2=y_2)$, $i=1, 2$ are linear. Specifically one can show that

$$(4.2)\quad E(X_i|Y_1 = y_1, Y_2 = y_2) = \frac{(\varrho_i+m_i+n_i-1)\,y_i+a_i+n_i}{\varrho_i+m_i-1}, \quad y_i = 0, 1, \ldots;\ i = 1, 2.$$

The intent of this section is to examine whether the converse of the above result is also true; i.e., whether starting with (4.1) and (4.2) one can deduce that the distribution of X_i is the UGWD$(a_i, m_i+n_i; \varrho_i)$, $i=1, 2$. Before answering this question we prove the following lemma.

Lemma. *Let (X_1, X_2) and (Y_1, Y_2) be two random vectors with non-negative and integer-valued components. Suppose that* (4.1) *is true and that* $P(X_i=0)<1$, $i=1, 2$. *Suppose further that*

$$(4.3)\qquad E(X_i|Y_1 = y_1,\ Y_2 = y_2) = a_i y_i+b_i,\ y_i = 0, 1, 2, \ldots \quad i = 1, 2$$

for some constants a_i, b_i, $i=1, 2$. *Then,*

(i) $$b_i > 0, \quad i = 1, 2.$$

(ii) $$a_i > 1, \quad i = 1, 2.$$

Proof.

(i) From (4.3) we have (since $X_i \geqq Y_i$, $i=1, 2$) $0 \leqq E(X_i|Y_1=Y_2=0)=b_i$. Hence $b_i \geqq 0$. But, if $b_i=0$ it follows that

$$\sum_{x_i=1}^{\infty} x_i P(X_i = x_i)\,(n_1)_{(x_1)}(n_2)_{(x_2)}/(m_1+n_1)_{(x_1)}(m_2+n_2)_{(x_2)} = 0.$$

This implies that $P(X_i=x_i)=0$, $x_i=1, 2, \ldots$; $i=1, 2$ which contradicts the assumption that $P(X_i=0)<1$, $i=1, 2$. Hence $b_i>0$, $i=1, 2$.

(ii) Using (4.3) and the fact that $X_i \geqq Y_i$, $i=1, 2$ we have $y_i < E(X_i|Y_1=y_1, Y_2=y_2)=a_i y_i+b_i$ for every y_i, $i=1, 2$, i.e., $b_i>(1-a_i)\,y_i$, $y_i=0, 1, 2, \ldots$; $i=1, 2$. Since $b_i>0$ the latter inequality holds for all the values of y_i only when $a_i>1$. This completes the proof of the lemma.

The theorem that will be proved in the sequel provides a positive answer to the question posed at the beginning of this section, for the case $n_1=n_2=1$.

Theorem. *Let (X_1, X_2) and (Y_1, Y_2) be two random vectors on* $\{0, 1, 2, \ldots\}\times\times\{0, 1, 2, \ldots\}$ *such that* $P(X_i=0)<1$, $i=1, 2$ *and*

$$(4.4)\qquad P(Y_1 = y_1, Y_2 = y_2|X_1 = x_1, X_2 = x_2) = \prod_{i=1}^{2} \binom{m_i+y_i-1}{y_i}\Big/\binom{m_i+x_i}{x_i}$$

$$m_i > 0;\ y_i = 0, 1, 2, \ldots, x_i;\ i = 1, 2$$

(i.e. $(Y_1, Y_2)|(X_1=x_1, X_2=x_2)$ *follows the joint distribution of two independent negative hypergeometric r.v.'s with parameters,* $m_1>0$, $n_1=1$, $m_2>0$, $n_2=1$). *Then* (4.3) *is true for* $a_i<1+m_i^{-1}$, $i=1, 2$ *if and only if* (X_1, X_2) *has the joint distribution* *of two independent generalized Waring r.v.'s with parameters* $\frac{b_i}{a_i-1}$, m_i+1 *and* $\frac{a_i}{a_i-1}-m_i$, $i=1, 2$.

PROOF. The "if" part follows immediately from (4.2) for $n_i=1$, $i=1,2$.
"*Only if*" *part*. From (4.4) and (4.3) we have

$$\sum_{x_1=y_1}^{\infty}\sum_{x_2=y_2}^{\infty} x_i g(x_1, x_2) = (a_i y_i+b_i)\sum_{x_1=y_1}^{\infty}\sum_{x_2=y_2}^{\infty} g(x_1, x_2) \tag{4.5}$$

$$y_i = 0, 1, 2, \dots;\ i = 1, 2$$

where $g(x_1, x_2)=x_1!\,x_2!\;P(X_1=x_1,\ X_2=x_2)/(m_1+1)_{(x_1)}\,(m_2+1)_{(x_2)}$. Consider relation (4.5) for $i=1$ and specialize it for $y_1=r$ and $y_1=r+1$. Subtracting the resulting equations by parts we obtain

$$((a_1-1)r+b_1)G(r, y_2)-\alpha_1 \sum_{x_1=r+1}^{\infty} G(x_1, y_2)=0 \quad \text{where}$$

$G(r, l)=\sum_{x_2=l}^{\infty} g(r, x_2)$. Applying the same technique to the above equation we get

$$[(a_1-1)r+b_1]G(r, y_2)-[(a_1-1)(r+1)+b_1+a_1]G(r+1, y_2)=0.$$

Specializing this equation for $y_2=l$ and $y_2=l+1$ and subtracting the two resulting equations we obtain

$$[(a_1-1)r+b_1]g(r+1, l)-[(a_1-1)(r+1)+b_1+a_1]g(r+1, l)=0$$

which, since from the lemma $a_1>1$, becomes

$$g(r+1, l)-\frac{r+\dfrac{b_1}{a_1-1}}{r+\dfrac{2a_1+b_1-1}{a_1-1}}\, g(r, l)=0, \quad r=0, 1, 2; \dots, l=0, 1, 2, \dots . \tag{4.6}$$

In a similar manner we obtain from (4.5), for $i=2$

$$g(r, l+1)-\frac{l+\dfrac{b_2}{a_2-1}}{l+\dfrac{2a_2+b_2-1}{a_2-1}}\, g(r, l)=0, \quad r=0, 1, 2, \dots, l=0, 1, 2, \dots . \tag{4.7}$$

Solving the system of equations (4.6) and (4.7) we obtain

$$g(r, l)=g(0, 0)\left(\frac{b_1}{a_1-1}\right)_{(r)}\left(\frac{b_2}{a_2-1}\right)_{(l)}\Bigg/\left(\frac{2a_1+b_1-1}{a_1-1}\right)_{(r)}\left(\frac{2a_2+b_2-1}{a_2-1}\right)_{(l)}$$

$$r=0, 1, 2, \dots, l=0, 1, 2, \dots .$$

Therefore

$$P(X_1 = r, X_2 = l) = P(X_1 = 0, X_2 = 0)\left(\frac{b_1}{a_1-1}\right)_{(r)}(m_1+1)_{(r)}\left(\frac{b_2}{a_2-1}\right)_{(l)}(m_2+1)_{(l)} \div$$

$$\div\left(\frac{2a_1+b_1-1}{a_1-1}\right)_{(r)}\left(\frac{2a_2+b_2-1}{a_2-1}\right)_{(l)} r!\,l!.$$

From the fact that $a_i < 1 + m_i^{-1}$, $i=1, 2$ and the condition $\sum_{r=0}^{\infty}\sum_{l=0}^{\infty} P(X_1=r, X_2=l) = 1$ we obtain

$$P(X_1 = 0, X_2 = 0) \prod_{i=1}^{2}\left(\frac{a_i}{a_i-1} - m_i\right)_{(m_i+1)}\left(\frac{a_i+b_i}{a_i-1} - m_i\right)_{(m_i+1)}.$$

Hence, (X_1, X_2) has a bivariate distribution whose marginals are independent UGWD's with parameters $\frac{b_i}{a_i-1}, m_i+1, \frac{a_i}{a_i-1} - m_i$, $i=1, 2$. (The positivity of $b_i/(a_i-1)$ is ensured by the lemma). Therefore the theorem has been established.

Note that Xekalaki [11] has obtained an analogous result for the univariate case.

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