On the canonical connection of a 3-web

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Dedicated to Professor A. Rapcsák on his 70th birthday

1. Introduction

Let M be a smooth manifold of dimension $2r$. We say that a 3-web is defined on M if there is given three foliations $\lambda_1, \lambda_2, \lambda_3$ of r-dimensional leaves in M such that every point of M belongs to exactly one leaf of each foliation and the tangent spaces of the leaves through a point of M are in general position.

The notion of a 3-web on a 2-manifold was introduced by W. BLASCHKE [4]. The generalization to higher dimension was studied by S.-S. CHERN [5]. A systematic investigation of this structure has been made by M. A. AKIVIS and his followers in the last 15 years [1].

We shall call the leaves of the first foliation λ_1 as horizontal leaves and of the second family λ_2 as vertical leaves, their tangent vectors horizontal and vertical tangent vectors.

Let $P(M)$ be the subbundle of the linear frame bundle $L(M)$ over M consisting of the frames $\{\bar{e}_1, \dots, \bar{e}_r, \tilde{e}_1, \dots, \tilde{e}_r\}$, where $\bar{e}_1, \dots, \bar{e}_r$ are horizontal, $\tilde{e}_1, \dots, \tilde{e}_r$ are vertical tangent vectors. We can define a tensor field on M with help of the third foliation by the following way:

If v is a horizontal tangent vector at $x \in M$, let $\mathscr{P}_x v$ be the vertical tangent vector at x such that $v - \mathscr{P}_x v$ is a tangent vector to the leaf of the third foliation λ_3 through $x \in M$. The inverse map of $\mathscr{P}_x$ is denoted by Q_x. The matrix

$$\begin{pmatrix} 0 & \mathscr{P}_x \\ Q_x & 0 \end{pmatrix}$$

with respect to the frame $\{\bar{e}_1, \dots, \bar{e}_r, \ \tilde{e}_1, \dots, \tilde{e}_r\}$ defines a tensor field $\mathscr{T}_x$ on M having the property $\mathscr{T}_x^2 = \mathrm{id}$.

The adapted frame bundle $Q(M)$ of the three-web is defined as the subbundle of $P(M)$ consisting of the frames $\{\bar{e}_1, \dots, \bar{e}_r, \tilde{e}_1, \dots, \tilde{e}_r\}$ such that $\tilde{e}_i = \mathscr{P}\bar{e}_i$, $i = 1, \dots, r$. S.-S. CHERN defined the canonical connection on the bundle $Q(M)$ uniquely determined by the structure equations [5]

$$d\overline{\omega}^i = -\omega^i_j \wedge \overline{\omega}^j + a_j{}^i{}_k \overline{\omega}^j \wedge \overline{\omega}^k,$$
$$d\tilde{\omega}^i = -\omega^i_j \wedge \tilde{\omega}^j - a_j{}^i{}_k \tilde{\omega}^j \wedge \tilde{\omega}^k,$$

where ω^i_j are the components of the connection form, $\bar{\omega}^1, \dots, \bar{\omega}^r, \tilde{\omega}^1, \dots, \tilde{\omega}^r$ are the components of the basic form over $Q(M)$, and the tensor field $a_j{}^i{}_k$ is called as the torsion tensor of the three-web. The invariants of this connection are the invariants of the 3-web.

In the following we shall investigate the canonical connection, particularly we shall give its description expressed by the Lie-loop structure on the horizontal leaves induced by the 3-web.

2. The calculation of the connection form

The foliations of horizontal and vertical leaves of a 3-web give a local direct product decomposition of the manifold $M=U\times V$. If $x^1, \dots, x^r$ and $y^1, \dots, y^r$ are the local coordinates on U and V, the collection $x^1, \dots, x^r, y^1, \dots, y^r$ will be a local coordinate system on the manifold M. We suppose that the leaves of the third foliation λ_3 are the level surfaces of the functions $f^i(x^1, \dots, x^r, y^1, \dots, y^r)$, $i=1, \dots, r$. It means that the 3-web is given by the equations:

$$\lambda_1\colon \{(x^1, \dots, x^r, y^1, \dots, y^r)\in U\times V, y^i = c^i\},$$
$$\lambda_2\colon \{(x^1, \dots, x^r, y^1, \dots, y^r)\in U\times V, x^i = c^i\},$$
$$\lambda_3\colon \{(x^1, \dots, x^r, y^1, \dots, y^r)\in U\times V, f^i(x, y) = c^i\},$$

where $c^1, \dots, c^r$ are constant.

We denote $\bar{f}^i_j=\dfrac{\partial f^i}{\partial x^j}$, $\tilde{f}^i_j=\dfrac{\partial f^i}{\partial y^j}$. We can suppose that $\det(\bar{f}^i_j)\neq 0$, $\det(\tilde{f}^i_j)\neq 0$ and we denote by $\bar{g}^i_j$ and $\tilde{g}^i_j$ the inverses of the matrices $\bar{f}^i_j$ and $\tilde{f}^i_j$, respectively. We shall consider the frame field $\left\{\dfrac{\partial}{\partial x^1}, \dots, \dfrac{\partial}{\partial x^r}, \dfrac{\partial}{\partial y^1}, \dots, \dfrac{\partial}{\partial y^r}\right\}$ on $U\times V$. A vector $X^i\dfrac{\partial}{\partial x^i}-Y^i\dfrac{\partial}{\partial y^i}$ is tangent to the leaf $\in\lambda_3$ if and only if

$$(\bar{f}^i_j\, dx^j+\tilde{f}^i_j\, dy^j)\left(X^i\frac{\partial}{\partial x^i}-Y^i\frac{\partial}{\partial y^i}\right) = 0.$$

It follows that the map $P_{(x,y)}\colon T_xU\to T_yV$ can be written in the form

$$\mathscr{P}^i_j dx^j\otimes\frac{\partial}{\partial y^i} = \tilde{g}^i_k\bar{f}^k_j\, dx^j\otimes\frac{\partial}{\partial y^i} \tag{1}$$

and its inverse is

$$Q^i_j\, dy^j\otimes\frac{\partial}{\partial x^i} = \bar{g}^i_k\tilde{f}^k_j\, dy^j\otimes\frac{\partial}{\partial x^i}. \tag{2}$$

The frame $\left\{\bar{g}^i_j\dfrac{\partial}{\partial x^i}, \tilde{g}^i_j\dfrac{\partial}{\partial y^i}, j=1, \dots, r\right\}$ is an adapted one, its dual coframe is

$$\bar{\omega}^i = \bar{f}^i_j\, dx^j, \quad \tilde{\omega}^i = \tilde{f}^i_j\, dy^j.$$

We extend the connection ω^i_j from the adapted frame bundle $Q(M)$ to the larger bundle $P(M)$ containing $Q(M)$ as subbundle. We denote by $\bar\omega^i_j$, $\tilde\omega^i_j$ the components of the extended connection form, and we compute their expressions in the frame

$$\left\{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^r}, \frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^r}\right\},$$

which is a section of the bundle $P(M)$. Our calculation is a modificated version of the calculations in [2].

By standard formulas we have

$$\bar\omega^i_j = \bar g^i_l(d\bar f^l_j + \omega^l_k \bar f^k_j), \quad \tilde\omega^i_j = \tilde g^i_l(d\tilde f^l_j + \omega^l_k \tilde f^k_j). \tag{3}$$

These forms satisfy the structure equations

$$0 = d(dx^i) = -\bar\omega^i_j \wedge dx^j + \bar g^i_l a_j{}^l{}_k \bar f^j_h \bar f^k_m\, dx^h \wedge dx^m,$$

$$0 = d(dy^i) = -\tilde\omega^i_j \wedge dy^j - \tilde g^i_l a_j{}^l{}_k \tilde f^j_h \tilde f^k_m\, dy^h \wedge dy^m.$$

Using Cartan's lemma we get

$$\bar\omega^i_j = \bar g^i_l\left(\frac{\partial^2 f^l}{\partial x^h \partial x^j}\, dx^h + \frac{\partial^2 f^l}{\partial y^h \partial x^j}\, dy^h + \omega^l_k \bar f^k_j\right) = \tag{4}$$

$$= \bar g^i_l a_m{}^l{}_k \bar f^m_h \bar f^k_j\, dx^h + \bar\lambda_j{}^i{}_h\, dx^h,$$

$$\tilde\omega^i_j = \tilde g^i_l\left(\frac{\partial^2 f^l}{\partial x^h \partial y^j}\, dx^h + \frac{\partial^2 f^l}{\partial y^h \partial y^j}\, dy^h + \omega^l_k \tilde f^k_j\right) = \tag{5}$$

$$= -\tilde g^i_l a_m{}^l{}_k \tilde f^m_h \tilde f^k_j\, dy^h + \tilde\lambda_j{}^i{}_h\, dy^h,$$

where $\bar\lambda_j{}^i{}_h$ and $\tilde\lambda_j{}^i{}_h$ are symmetric in the indices j and h. Excluding the forms ω^l_k from the equations (4) and (5) we get

$$-\frac{\partial^2 f^l}{\partial x^h \partial x^j}\, \bar g^j_k + \frac{\partial^2 f^l}{\partial x^h \partial y^j}\, \tilde g^j_k + a_m{}^l{}_k \bar f^m_h + \bar f^l_i \bar\lambda_j{}^i{}_h \bar g^j_k = 0$$

and

$$-\frac{\partial^2 f^l}{\partial y^h \partial x^j}\, \bar g^j_k + \frac{\partial^2 f^l}{\partial y^h \partial y^j}\, \tilde g^j_k + a_m{}^l{}_k \tilde f^m_h - \tilde f^l_i \tilde\lambda_j{}^i{}_h \tilde g^j_k = 0.$$

Using $a_m{}^l{}_k = -a_k{}^l{}_m$ we get from these equations

$$a_m{}^l{}_k = -\frac{\partial^2 f^l}{\partial x^h \partial y^j}\, \tilde g^j_{[k} \bar g^h_{m]},$$

$$\bar\lambda_j{}^i{}_h = \bar g^i_l\left(\frac{\partial^2 f^l}{\partial x^h \partial x^j} - \tilde g^m_k \bar f^k_{(j} \frac{\partial^2 f^l}{\partial x^{h)} \partial y^m}\right),$$

$$\tilde\lambda_j{}^i{}_h = \tilde g^i_l\left(\frac{\partial^2 f^l}{\partial y^h \partial y^j} - \bar g^m_k \tilde f^k_{(j} \frac{\partial^2 f^l}{\partial y^{h)} \partial x^m}\right),$$

where $[..]$ and $(..)$ denote the antisymmetric part and the symmetric part of an expression. We substitute these into (4) and (5), then we get

$$\bar{\omega}^i_j = \bar{g}^i_l\left(\frac{\partial^2 f^l}{\partial x^j \partial x^h} - \frac{\partial^2 f^l}{\partial x^h \partial y^m}\tilde{g}^m_k \bar{f}^k_j\right)dx^h, \tag{6}$$

$$\tilde{\omega}^i_j = \tilde{g}^i_l\left(\frac{\partial^2 f^l}{\partial y^j \partial y^h} - \frac{\partial^2 f^l}{\partial y^h \partial x^m}\bar{g}^m_k \tilde{f}^k_j\right)dy^h. \tag{7}$$

Proposition 1. *The components $\bar{\omega}^i_j$ and $\tilde{\omega}^i_j$ of the extended canonical connection form can be written in the form*

$$\bar{\omega}^i_j = Q^i_m \frac{\partial \mathscr{P}^m_j}{\partial x^h} dx^h, \quad \tilde{\omega}^i_j = \mathscr{P}^i_m \frac{\partial Q^m_j}{\partial y^h} dy^h.$$

PROOF. Using the relations $\tilde{f}^l_m \tilde{g}^m_k = \delta^l_k$ and $\bar{f}^l_m \bar{g}^m_k = \delta^l_k$ we have

$$\frac{\partial \tilde{f}^l_m}{\partial x^h}\tilde{g}^m_k = -\tilde{f}^l_m \frac{\partial \tilde{g}^m_k}{\partial x^h}, \quad \frac{\partial \bar{f}^l_m}{\partial y^h}\bar{g}^m_k = -\bar{f}^l_m \frac{\partial \bar{g}^m_k}{\partial y^h}.$$

Thus from the equations (6) and (7) follows the Proposition.

3. Parallel vector fields along horizontal and vertical surfaces

Let π and ϱ denote the projection maps $\pi\colon M = U \times V \to U$, $\varrho\colon M = U \times V \to V$. Let be given a horizontal curve $(x(t), y_0)$ and a vectorfield $X(t)$ along the curve $(x(t), y_0)$. The vectorfield $X(t) = \bar{X}^i \frac{\partial}{\partial x^i} + \bar{X}^i \frac{\partial}{\partial y^i}$ is parallel along $(x(t), y_0)$ if and only if it satisfies the equations

$$\frac{d\bar{X}^i}{dt} + \bar{\omega}^i_j \bar{X}^j = 0, \quad \frac{d\tilde{X}^i}{dt} = 0.$$

Using Proposition 1 we get

$$\frac{d\bar{X}^i}{dt} + Q^i_m \frac{\partial \mathscr{P}^m_j}{\partial x^h}\bar{X}^j \dot{x}^h = Q^i_m \left(\frac{\partial(\mathscr{P}^m_j \bar{X}^j)}{\partial x^h}\dot{x}^h\right) = 0. \tag{9}$$

Similarly a vector field $X(t) = \bar{X}^i \frac{\partial}{\partial x^i} + \tilde{X}^i \frac{\partial}{\partial y^i}$ is parallel along a vertical curve $(x_0, y(t))$ if and only if

$$\frac{d\bar{X}^i}{dt} = 0 \quad \text{and} \quad \frac{d\tilde{X}^i}{dt} + \tilde{\omega}^i_j \tilde{X}^j = \mathscr{P}^i_m \left(\frac{\partial(Q^m_j \tilde{X}^j)}{\partial y^h}\dot{y}^h\right) = 0. \tag{10}$$

From the relations (9) and (10) we get the

Theorem 1. *The vector field $X(t) = \bar{X}(t) + \tilde{X}(t) = \bar{X}^i \frac{\partial}{\partial x^i} + \tilde{X}^i \frac{\partial}{\partial y^i}$ is parallel along a horizontal curve $(x(t), y_0)$ in M if and only if*

(i) *its vertical part* $\tilde{X}=\tilde{X}^i\frac{\partial}{\partial y^i}$ *is a constant vector, that is*

$$\frac{d}{dt}(\varrho_*\tilde{X})=0,$$

(ii) *the vertical image* $\mathscr{P}\bar{X}=\mathscr{P}^i_j\bar{X}^j\frac{\partial}{\partial y^i}$ *of its horizontal part* $\bar{X}=\bar{X}^i\frac{\partial}{\partial x^i}$ *is a constant vector, that is*

$$\frac{d}{dt}(\varrho_*\mathscr{P}\bar{X})=0.$$

The vector field $X=\bar{X}+\tilde{X}=\bar{X}^i\frac{\partial}{\partial x^i}+\tilde{X}^i\frac{\partial}{\partial y^i}$ is parellel along a vertical curve $(x_0, y(t))$ in M if and only if

(i) its horizontal part $\bar{X}=\bar{X}^i\frac{\partial}{\partial x^i}$ is a constant vector, that is

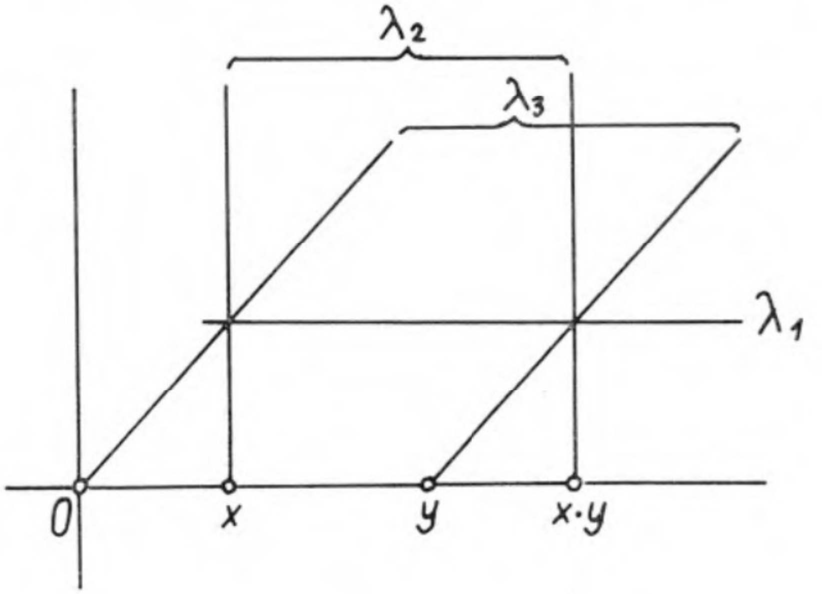
$$\frac{d}{dt}(\pi_*\bar{X})=0,$$

(ii) the horizontal image $Q\tilde{X}=Q^i_j\tilde{X}^j\frac{\partial}{\partial x^i}$ of its vertical part $\tilde{X}=\tilde{X}^i\frac{\partial}{\partial y^i}$ is a constant vector, that is

$$\frac{d}{dt}(\pi_*Q\tilde{X})=0.$$

The proof is an immediate consequence of the equations (9) and (10).

Now we fix an origin $0\in M$ and we suppose that its coordinates are zero. On the horizontal leaf U through 0 we can introduce a loop operation (non-associative but invertible multiplication with unit) by the following schema



It is well known that if we consider on U the introduced loop structure, than we can define a 3-web on the direct product $U\times U$ such that the horizontal leaves $(\in\lambda_1)$ are $U\times\{x_0\}$, $x_0\in U$, the vertical leaves $(\in\lambda_2)$ are $\{x_0\}\times U$, and the leaves of λ_3 are

$$\{(x, y)\in U\times U\colon x\cdot y = z_0, \text{ where } z_0\in U \text{ is constant}\}.$$

This 3-web is isomorphic to the original one [6].

Theorem 2. *Let $\mathcal{L}_g\colon U\to U$ and $\mathcal{R}_g\colon U\to U$, $g\in U$ denote the left and the right translations on the loop U. A horizontal vector field $\overline{X}$ is parallel along the horizontal leaf $U\times\{e\}$ where e is the unit in U, if and only if*

$$\overline{X}(g, e) = (\mathcal{L}_g)_*\overline{X}(e, e).$$

A vertical vector field $\tilde{X}$ is parallel along the vertical leaf $\{e\}\times U$ if and only if

$$\tilde{X}(e, g) = (\mathcal{R}_g)_*\tilde{X}(e, e).$$

Especially if U is a Lie *group then the connection induced on the horizontal leaf $U\times\{e\}$ is defined by the left invariant vector fields, and the connection induced on the vertical leaf $\{e\}\times U$ is defined by the right invariant vector field on the Lie group U.*

Proof. We know by Theorem 1. that the horizontal vector field $\overline{X}^i\dfrac{\partial}{\partial x^i}$ is parallel along a horizontal leaf if and only if $\mathcal{P}^i_j\overline{X}^j=c^i$ (constant). We get by (2) $\overline{X}^j=Q^j_h c^h=\bar{g}^j_k\tilde{f}^k_h c^h$, where the function $f^i(x^1, \dots, x^r, y^1, \dots, y^r)$ can be interpreted as the coordinate functions of the multiplication, that is

$$(x\cdot y)^i = f^i(x^1, \dots, x^r, y^1, \dots, y^r).$$

We suppose that the coordinates of the unit $e\in U$ are zero. We consider the horizontal leaf $U\times\{e\}$ through the origin $0=(e, e)$. The parallel horizontal vector field $\overline{X}$ has to satisfy

$$\overline{X}^j(x, 0) = Q^j_h(x, 0)\,c^h = \bar{g}^j_k(x, 0)\tilde{f}^k_h(x, 0)\,c^h.$$

Since $x\cdot e=x$ we have $f^i(x, 0)=x^i$, $\tilde{f}^i_k(x, 0)=\delta^i_k$ and $\bar{g}^i_k(x, 0)=\delta^i_k$. Thus we get

$$Q^i_h(x, 0) = \tilde{f}^i_h(x, 0).$$

Since $(\mathcal{L}_x y)^i=f^i(x, y)$, the tangent map of $\mathcal{L}_g\colon U\to U$ can be written in the form

$$(\mathcal{L}_g)_*\left(\overline{X}^i_0\frac{\partial}{\partial x^i}\right) = \overline{X}^i_0\frac{\partial f^k}{\partial y^i}\frac{\partial}{\partial x^k} = \tilde{f}^k_h\overline{X}^h_0\frac{\partial}{\partial x^k} = Q^k_h\overline{X}^h_0\frac{\partial}{\partial x^k},$$

which proves the first assertion.

For vertical vector fields similar arguments prove the statement.

References

[1] M. A. Акивис, О три-тканях многомерных поверхностей, Труды Теометрического Семинара, ВИНИТИ Москва, **2** (1961), 7—31.
[2] M. A. Акивис—А. М. Шелехов, О вычислении тензоров кривизны и кручения многометрой три-ткани, Сибирский Мат. Журнал **12** (1971), 953—960.
[3] M. A. Акивис—А. М. Шелехов, Основы теори тканей, Калининский Государственный Университет, Калинин (1981).
[4] W. Blaschke—G. Bol, Geometrie der Gewebe, Springer, *Berlin* (1938).
[5] S.-S. Chern, Eine Invariantentheorie der 3-Gewebe aus r-dimensionalen Mannigfaltigkeiten in R_{2n}, *Abhandl. Math. Semin. Hamburg. Univ.*, **11** (1936), 335—358.
[6] G. Pickert, Projektive Ebene, Springer, *Berlin—Göttingen—Heidelberg* (1955).

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