Exponential polynomials and differential equations

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Introduction. In [1] we introduced the Fourier transform of exponential polynomials on Abelian topological groups, which is a polynomial-valued function on the set of all exponentials. We have shown some interesting properties of this Fourier transform and we have pointed out that it can be used to determine all exponential polynomial solutions of some types of linear functional equations. In this note we show that it can be used to determine all solutions of inhomogeneous linear differential equations with constant coefficients if the right hand side is an exponential polynomial. The interest of the method is that we obtain all solutions without integration and it is much simpler than the classical method of "variation of the constants". Further, the procedure obviously extends to linear systems of ordinary differential equations, and even in the case of polynomial coefficients we can simply determine all exponential polynomial solutions — if they exist at all.

We remark that the method extends to Cauchy-problems for linear partial differential equations of evolution type, as the heat equation, Schrödinger-equation, etc. In particular, we get simple explicit formulas for the solution if the initial data are exponential polynomials (see [1]).

We need here some basic notations and results concerning exponential polynomials and their Fourier transform (see [1]).

If G is a topological Abelian group, then functions of the form $f=\sum_{i=1}^{n} p_i m_i$ are called exponential polynomials, if the functions p_i are continuous complex polynomials and the functions m_i are continuous exponentials on G. If in this representation the exponentials m_i are different, then the representation is unique, and we define the Fourier transform of f by denoting $\hat{f}(m)$ the polynomial coefficient of the exponential m in the above representation. Hence, $\hat{f}$ is a polynomial-valued function defined on the set of all exponentials and having the following properties:

(i) $$(pf)^{\wedge}(m) = p\cdot\hat{f}(m);$$

(ii) $$(f_y)^{\wedge}(m) = m(y)[\hat{f}(m)]_y;$$

(iii) $$f\to\hat{f} \quad \text{is linear.}$$

Here f is an exponential polynomial, p is a polynomial, m is an exponential, and f_y denotes the translate of f by y.

In the case $G=\mathbf{R}$ all exponentials have the form $m(x)=e^{\lambda x}$ $(x\in\mathbf{R})$ where λ is a complex number, and hence $\hat{f}(m)$ is denoted by $\hat{f}(\lambda)$. In this case one more important property concerning differentiation is the following:

(iv) $$\big(P(D)f\big)^{\wedge}(\lambda) = P(D+\lambda)\hat{f}$$

where P is a complex polynomial, and D is the operator of differentiation.

All these properties are introduced and proved in [1].

Linear differential equations. We consider the inhomogeneous linear differential equation

(1) $$P(D)\,y = f$$

where P is a complex polynomial of degree n and f is an exponential polynomial. We show, that all solutions of this equation are exponential polynomials and we show how to compute them without integration.

It is enough to show that equation (1) has a solution which is an exponential polynomial, because all solutions of the homogeneous equation are exponential polynomials.

Suppose, that y is an exponential polynomial, which is a solution of (1). Then by Fourier transformation we have

(2) $$P(D+\lambda)\,\hat{y}(\lambda) = \hat{f}(\lambda)$$

for all $\lambda\in\mathbf{C}$. Here $\hat{y}(\lambda)$, $\hat{f}(\lambda)$ are polynomials and $\hat{f}(\lambda)=0$ with the exception of a finite set. Hence the problem of solving (1) is reduced to the problem of finding all polynomial solutions q of an equation of the form

(3) $$P(D+\lambda)\,q = p$$

where p is a given polynomial.

First we suppose that $p(x)=\sum_{k=0}^{N} c_k x^k$, $c_N\neq 0$ then (3) is equivalent to the equation

(4) $$\sum_{j=0}^{n} \frac{P^{(j)}(\lambda)}{j!} D^j q(x) = \sum_{i=0}^{N} c_i x^i.$$

Suppose, that λ is a zero of k-th order of P $(0\leqq k\leqq n)$, that is $P^{(j)}(\lambda)=0$ $(j=0, 1, \dots, k-1)$, and $P^{(k)}(\lambda)\neq 0$. Then we put $r=D^k q$ and we obtain

(5) $$\sum_{j=0}^{n-k} \frac{P^{(j+k)}(\lambda)}{(j+k)!} D^j r(x) = \sum_{i=0}^{N} c_i x^i.$$

It follows, that $\deg r\leqq N$ and let $r(x)=\sum_{i=0}^{N} b_i x^i$. Substitution into (5) yields

(6) $$\sum_{j=0}^{n-k} \sum_{i=j}^{N} \frac{P^{(j+k)}(\lambda)}{(j+k)!} j! \binom{i}{j} b_i x^{i-j} = \sum_{i=0}^{N} c_i x^i.$$

Comparing the coefficients of x^i we have

(7) $$\sum_{j=0}^{\min(n-k,\,N-i)} \frac{P^{(j+k)}(\lambda)}{(j+k)!} j! \binom{i+j}{j} b_{i+j} = c_i \quad (i = 0, 1, \dots, N).$$

This is a very simple system of linear equations for the b_i's and it is easily solvable as it has a triangular form. If $n-k>N$, then this system has the form:

$$
\begin{aligned}
c_N &= \frac{P^{(k)}(\lambda)}{k!}\, b_N \\
c_{N-1} &= \frac{P^{(k)}(\lambda)}{k!}\, b_{N-1} + \frac{P^{(k+1)}(\lambda)}{(k+1)!}\, N \cdot b_N \qquad (8)\\
&\;\vdots \\
c_0 &= \frac{P^{(k)}(\lambda)}{k!}\, b_0 + \frac{P^{(k+1)}(\lambda)}{(k+1)!}\, b_1 + \frac{P^{(k+2)}(\lambda)}{(k+2)!}\, 2!\, b_2 + \ldots + \frac{P^{(k+N)}(\lambda)}{(k+N)!}\, N!\, b_N
\end{aligned}
$$

which is obviously uniquely solvable.

If $n-k \leqq N$, then we have from (7)

$$
\begin{aligned}
c_N &= \frac{P^{(k)}(\lambda)}{k!}\, b_N \\
c_{N-1} &= \frac{P^{(k)}(\lambda)}{k!}\, b_{N-1} + \frac{P^{(k+1)}(\lambda)}{(k+1)!}\, N \cdot b_N \\
&\;\vdots \\
c_{N-(n-k)} &= \frac{P^{(k)}(\lambda)}{k!}\, b_{N-(n-k)} + \frac{P^{(k+1)}(\lambda)}{(k+1)!} \binom{N-(n-k)+1}{1} b_{N-(n-k)+1} + \ldots \\
&\quad \ldots + \frac{P^{(n)}(\lambda)}{n!}\, (n-k)! \binom{N}{n-k} b_N \qquad (9)\\
c_{N-(n-k)-1} &= \frac{P^{(k)}(\lambda)}{k!}\, b_{N-(n-k)-1} + \frac{P^{(k+1)}(\lambda)}{(k+1)!} \binom{N-(n-k)}{1} b_{N-(n-k)} + \ldots \\
&\quad \ldots + \frac{P^{(n)}(\lambda)}{n!}\, (n-k)! \binom{N-1}{n-k} b_{N-1} \\
&\;\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
c_0 &= \frac{P^{(k)}(\lambda)}{k!}\, b_0 + \frac{P^{(k+1)}(\lambda)}{(k+1)!}\, b_1 + \ldots + \frac{P^{(n)}(\lambda)}{n!}\, (n-k)!\, b_{n-k}
\end{aligned}
$$

which has a single solution too.

In the case $p=0$ we see from (4) that $q \neq 0$ implies $P(\lambda)=0$, and hence λ is a characteristic value of (1) with multiplicity $k \geqq 1$. Then by (5) it follows $r=0$, and hence q is an arbitrary polynomial of degree at most $k-1$.

It is obvious, that the exponential polynomial y for which $\hat{y}(\lambda)=q$, where q is the polynomial whose coefficients are determined by the system of equations (8) or (9) respectively, is a solution of (1). Hence we have proved the following theorem:

Theorem. *Let $f\colon \mathbf{R} \to \mathbf{C}$ be an exponential polynomial. Then all solutions of* (1) *are exponential polynomials. Further, if y is a solution, then for any $\lambda \in \mathbf{C}$ which is* *a zero of k-th order of P, the cofficients b_j of the polynomial $D^k\hat{y}(\lambda)$ satisfy the system of equations* (8) *or* (9) *respectively, where the c_i's are the coefficients of $\hat{f}(\lambda)$.*

Summarizing the results we can solve (1) as follows. First we determine the zeros of P with their multiplicities. Suppose that the support of $\hat{f}$ is the finite set $\lambda_1, \dots, \lambda_l$. Then we determine the value of k for which $P^{(j)}(\lambda_i)=0$ ($j=0, 1, \dots, k-1$) and $P^{(k)}(\lambda_i)\neq 0$, and we solve the system of equations (8) or (9), respectively, with $\lambda=\lambda_i$. The constants c_j ($j=0, 1, \dots, N$) are the coefficients of $\hat{f}(\lambda_i)$. From the solution we have the polynomial $q_i=\hat{y}(\lambda_i)$. Then the general solution of (1) is the following:

$$y(x) = \sum_{j=1}^{s} p_j(x)e^{\mu_j x} + \sum_{i=1}^{l} q_i(x)e^{\lambda_i x}$$

where $\mu_1, \dots, \mu_s$ are the zeros of P with multiplicities $n_1, \dots, n_s$ ($n_1+\dots+n_s=n$), and p_j is an arbitrary polynomial of degree at most n_j-1 ($j=1, \dots, s$).

We see that the same method can be applied to determine all exponential solutions of an inhomogeneous linear differential equation with polynomial coefficients.

Now we present two simple examples.

Example 1. We solve the equation

$$y''-y = x^2e^x - x\cos x + 1.$$

The characteristic polynomial is $P(\lambda)=\lambda^2-1$, the characteristic values are $\mu_1=1$, $\mu_2=-1$. By Fourier transformation we have

$$q''+2\lambda q'+(\lambda^2-1)q = \begin{cases} x^2 & \text{if } \lambda=1 \\ -\dfrac{x}{2} & \text{if } \lambda=i \\ -\dfrac{x}{2} & \text{if } \lambda=-i \\ 1 & \text{if } \lambda=0 \\ 0 & \text{otherwise.} \end{cases}$$

Here we used the notation $q=\hat{y}(\lambda)$. For $\lambda=1$ we have from (9)

$$1 = 2b_2$$
$$0 = 2b_1+2b_2$$
$$0 = 2b_0+b_1$$

and it means that $b_2=\frac{1}{2}$, $b_1=-\frac{1}{2}$, $b_0=\frac{1}{4}$. Hence

$$\hat{y}(1)(x) = \frac{1}{6}x^3-\frac{1}{4}x^2+\frac{1}{4}x+c.$$

Similarly, for $\lambda=i$ we have from (8)

$$-\frac{1}{2} = -2b_1$$

$$0 = -2b_0 + 2ib_1.$$

We obtain

$$\hat{y}(i)(x) = \frac{1}{4}x + \frac{i}{4}.$$

In the case $\lambda = -i$ we have similarly

$$\hat{y}(-i)(x) = \frac{1}{4}x - \frac{i}{4}.$$

Now let $\lambda = 0$, then $\hat{y}(0)(x) = -1$. Finally, if $\lambda \neq 1$, $\lambda \neq \pm i$, $\lambda \neq 0$, then $\hat{y}(\lambda) \neq 0$ implies $\lambda = -1$ and $\hat{y}(\lambda)(x) = d$, a constant.

Thus the general solution of the equation is

$$y(x) = ce^x + de^{-x} + \left(\frac{1}{6}x^3 - \frac{1}{4}x^2 + \frac{1}{4}x\right)e^x + \frac{1}{2}x\cos x - \frac{1}{2}\sin x - 1.$$

Example 2. Determine all exponential polynomial solutions of the equation

$$(x^2-1)y'' - (3x+1)y' - (x^2-x)y = 0.$$

By Fourier transformation we have

$$(x^2-1)q'' + (2\lambda x^2 - 3x - 2\lambda - 1)q' + \big(x^2(\lambda^2-1) - x(3\lambda-1) - \lambda^2 - \lambda\big)q = 0$$

where $q = \hat{y}(\lambda)$. From this equation by comparing the leading coefficients we have that $q \neq 0$ implies $\lambda^2 = 1$. If $\lambda = 1$, then

$$(x^2-1)q'' + (2x^2 - 3x - 3)q' - (2x+2)q = 0$$

and it follows, that $q \neq 0$ implies $\deg q = 1$, which is a contradiction. Hence $\hat{y}(1) = 0$. If $\lambda = -1$, then

$$(x^2-1)q'' + (-2x^2 - 3x + 1)q' + 4xq = 0$$

and it follows, that $q \neq 0$ implies $\deg q = 2$. Substituting $q(x) = x^2 + ax + b$ into the equation we have $a = 2$, $b = 1$, and hence $q(x) = (x+1)^2$. Hence an exponential polynomial solution of the equation is

$$y(x) = (x+1)^2 e^{-x}.$$

Using this solution the equation can be reduced to a linear equation of first order by a standard method.

Finally, we remark that the same method can be used in the case of systems of linear differential equations with polynomial coefficients.

References

[1] L. SZÉKELYHIDI, The Fourier transform of exponential polynomials, *(submitted to Publ. Math. Debrecen)*

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