Hypoelliptic convolution equations in the space $\mathscr{H}'\{M_p\}$

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0. Introduction

In the papers [5] and [6] we introduced and analysed the space of distributions $\mathscr{H}'\{M_p\}$ and gave necessary and sufficient conditions for the solvability of the convolution equation

$$(1) \qquad S*U=V, \quad S\in\mathcal{O}_c'(\mathscr{H}'\{M_p\})$$

in the space $\mathscr{H}'\{M_p\}$. $\mathcal{O}_c'(\mathscr{H}'\{M_p\})$ denotes the space of convolutors on $\mathscr{H}'\{M_p\}$.

For the convenience of the reader we repeat some definitions from [5] and [6]. The space $\mathscr{H}\{M_p\}$ is the space of smooth functions $\varphi(x)$ on $\mathbf{R}$ such that for every $p\in\mathbf{N}$

$$\gamma_p(\varphi):=\sup\left\{|\varphi^{(j)}(x)|\exp\left(M_p(x)\right);\ x\in\mathbf{R},\ 0\leqq j\leqq n\right\}<\infty.$$

The space $\mathscr{H}\{M_p\}$ equipped with the sequence of norms becomes a Fréchet space. The dual of $\mathscr{H}\{M_p\}$, denoted by $\mathscr{H}'\{M_p\}$, can be regarded as a subspace of $\mathscr{D}_F'$, the space of finite-order distributions. The space $\mathcal{O}_c'(\mathscr{H}'\{M_p\})$ is the space of convolutors on $\mathscr{H}'\{M_p\}$. A distribution S is in $\mathcal{O}_c'(\mathscr{H}'\{M_p\})$ iff for every $\varphi\in\mathscr{H}\{M_p\}$ $S*\varphi$ is again in $\mathscr{H}\{M_p\}$; one proves then that this implies that for every $T\in\mathscr{H}'\{M_p\}$ $S*T\in\mathscr{H}'\{M_p\}$.

In this paper we shall give necessary and sufficient conditions for the hypoellipticity of (1) and of the convolutor S. These conditions will be given on the Fourier transform of S, which by [6] is an entire analytic function.

Throughout the paper we suppose that a sequence of even, convex continuously differentiable functions $\{M_p(x)\}$ is given such that, for every $p\in\mathbf{N}$, the function $M_p(x)$ is increasing for $x>0$; $M_p(0)=0$, $\lim_{|x|\to\infty} M_p(x)/|x|=\infty$. We assume that the following condition is satisfied:

(A) For every $p\in\mathbf{N}$ there exist $p'\in\mathbf{N}$ and X_p, $p'\geqq X_p>p$, so that $M_p(px)\leqq$ $\leqq M_{p'}(x)$ for $|x|\geqq X_p$.

It was shown in [5] that this condition implies the condition (N) ([1], p. 111), in general for a greater $p'\in\mathbf{N}$. We denote by $r(p)$ the smallest natural number p'

Mathematical subject Classifications (1980) AMS (MOS) 46 FXX.

for which the inequalities in (A) hold, and the function $\exp(M_p(x)-M_{p'}(x))$ is summable on the set $\mathbf{R}$.

It is important to note that we do not assume the smoothness of the functions $M_p(x)$, $p\in\mathbf{N}$. However, since smoothness is necessary in Section 5., we have to define smooth functions which behave at infinity as (for instance) $M_p(x)$ (see Section 4.).

We are now ready to define the space $\mathcal{E}\mathcal{H}'\{M_p\}$.

1. The Space $\mathcal{E}\mathcal{H}'\{M_p\}$

Definition 1. $\mathcal{E}\mathcal{H}'\{M_p\}$ is the space of smooth functions $U(x)$ defined on $\mathbf{R}$ which satisfy the inequalities

$$|U^{(j)}(x)| \leqq C_j \cdot \exp\left(M_{p_0}(x)\right), \quad j=0,1,2,\ldots \tag{2}$$

for some $C_j>0$ and $p_0\in\mathbf{N}$ (which depends only on $U(x)$).

The space $\mathcal{E}\mathcal{H}'\{M_p\}$ can be obtained as the dual space of the space of convolutors $\mathcal{O}'_c(\mathcal{H}'\{M_p\})$, but this is not essential for us. Rather, let us return to the convolution equation (1). If $S\in\mathcal{O}'_c(\mathcal{H}'\{M_p\})$ and $U\in\mathcal{E}\mathcal{H}'\{M_p\}$, then the following Lemma shows that $V:=S*U$ must be in $\mathcal{E}\mathcal{H}'\{M_p\}$.

Lemma 1. *Let* $S\in\mathcal{O}'_c(\mathcal{H}'\{M_p\})$ *and* $U\in\mathcal{E}\mathcal{H}'\{M_p\}$. *Then* $V:=S*U$ *is in* $\mathcal{E}\mathcal{H}'\{M_p\}$.

Proof. Let us suppose that $U^{(j)}(x)$, $j=0,1,2,\ldots$, satisfy (2), and let us put $p_1:=r(r(p_0))$. By Theorem 3 from [5] for given $p_1\in\mathbf{N}$ there exists $m\in\mathbf{N}_0$ and a continuous function $F(x)$ with the property $|F(x)|\leqq C\exp\left(-M_{p_1}(x)\right)$ for some $C>0$, such that $S=D^mF$. (D denotes the distributional derivative.) Hence

$$|(S*U)^{(j)}(x)| = \left|\int_{\mathbf{R}} F(y)\,U^{(m+j)}(x-y)\,dy\right| \leqq$$

$$\leqq \int_{\mathbf{R}} C\exp\left(-M_{p_1}(y)\right) C_{m+j}\exp\left(M_{p_0}(x-y)\right)dy \leqq$$

$$\leqq \int_{\mathbf{R}} C'_j\exp\left(-M_{p_1}(x-y)+M_{p_0}(y)\right)dy.$$

From the inequality

$$M_{p_0}(y) \leqq M_{r(p_0)}(y/2)+K \leqq M_{r(p_0)}(x)+M_{r(p_0)}(x-y)+K$$

for $x,y\in\mathbf{R}$ and some $K=K(p_0)>0$ we obtain at last

$$|(S*U)^{(j)}(x)| \leqq C''_j\exp\left(M_{r(p_0)}(x)\right. \quad \square$$

The converse statement is not true in general; if, however, every solution of (1) is in $\mathcal{E}\mathcal{H}'\{M_p\}$ when V is, then the equation (1) and the convolutor S are called hypoelliptic in $\mathcal{H}'\{M_p\}$.

We shall introduce two new conditions for the functions $M_p(x)$, $p\in\mathbf{N}$; in the following, we suppose that they are satisfied:

(B) For every pair $(p, q)\in\mathbf{N}^2$ there exist $k\in\mathbf{N}$ and $X(p, q)>0$ such that

$$M_p(qx) \leqq M_p(kx)/k \quad \text{for} \quad |x| \geqq X(p, q).$$

(Observe that this condition is not satisfied for an arbitrary convex function which increases in infinity faster than any linear function. On the other hand, every function from the sequence $\{M_p(x)\}$, where $M_p(x):=p\cdot|x|^s$, $s>1$ fixed natural number, satisfies this condition. We know that this sequence leads to the space of exponential distributions $\mathscr{K}_s'$. If $M_p(x):=M(px)$, where $M(x)$ is a convex function which grows faster than any linear function when $|x|\to\infty$ ([8], [3]) then this condition becomes:

For $q\in\mathbf{N}$ there exist $k\in\mathbf{N}$ and $X(q)>0$ such that

$$M(qx) \leqq M(kx)/k \quad \text{for} \quad x \geqq X(q).) \tag{3}$$

The other condition is:

(C) For every $p\in\mathbf{N}$ there exist $p'\in\mathbf{N}$, $k\in\mathbf{N}$ and $K_p>0$ such that

$$M_p(x+y) \leqq kM_p(x)+M_{p'}(y)+K_p \quad \text{for} \quad x, y\in\mathbf{R}.$$

(This condition is a "new" one only for $0\leqq|y|<|x|$, otherwise it follows from (A).)

The main result of this paper is the following

Theorem. *Suppose $S\in\mathscr{O}_c'(\mathscr{H}'\{M_p\})$ and let $\hat{S}(w)$, $w=u+iv\in\mathbf{C}$ be its Fourier transform. The following conditions are equivalent:*

(H_1) *S is hypoelliptic in $\mathscr{H}'\{M_p\}$;*

(H_2) a) *For each $p\in\mathbf{N}$ there exist $B, M>0$ such that*

$$|\hat{S}(u)| \geqq |u|^{-B} \quad for \quad |u| \geqq M;$$

(b) $\lim\,(\tilde{M}_p(v)/\log|w|)=\infty$ *when $|w|\to\infty$ statying on the surface $\hat{S}(w)=0$;*

(H_3) *For every $p\in\mathbf{N}$ and $d>0$ there exists a constant $\bar{B}>0$ so that for every $m\in\mathbf{N}$ there exists $C_m>0$ with the property*

$$|1/\hat{S}(w)| \leqq |w|^{\bar{B}} \exp\left(d\cdot\tilde{M}_p(v)\right)$$

for

$$|\tilde{M}_p(v)| \leqq m\log|w| \quad and \quad |w| \geqq C_m.$$

$\tilde{M}_p(v)$ denotes the function dual in the sense of Young to the function $M_p(x)$ ([2] p. 18).

2. The Necessity of (H_2)

The implication $(H_1)\Rightarrow(H_2a)$ can be proved in the same way as in [7], because the proof in [7] runs for the space of tempered distributions $\mathscr{S}'$, and obviously $\mathscr{S}'\subset\subset\mathscr{H}'\{M_p\}$. In order to prove $(H_1)\Rightarrow(H_2b)$ we follow the lines of the proof of Theorem 7 in [7], though in our case this proof needs certain modifications. The following statement implies at once the necessity of the condition (H_2b).

Proposition 1. *If each continuous solution of the homogenous equation*

$$S*U=0 \tag{4}$$

which satisfies the estimate

$$U(x) = O\left(\exp\left(M_p(x)\right)\right) \quad when \quad |x| \to \infty \tag{5}$$

for some $p \in \mathbf{N}$, *is in fact a continuously differentiable function in some neighbourhood of zero, then the condition* (H_2b) *holds.*

Proof. Let us fix p and let $\mathscr{L}=\{x\in\mathbf{R};\ |x|\leqq 1\}$ be the mentioned neighbourhood of zero. Let H_p denote the space of continuous solutions of (4) which satisfy (5) and let $H_p^*=H_p\cap C^1(\mathscr{L})$, where $C^1(\mathscr{L})$ denotes the space of continuously differentiable functions on the interval $\mathscr{L}$. The spaces H_p and H_p^* endowed with the norms

$$\|U\|_p := \sup\left\{|U(x)|\exp\left(-M_p(x)\right);\ x\in\mathbf{R}\right\} \quad \text{and}$$
$$\|U\|_p^* := \sup\{|U'(x)|;\ x\in\mathscr{L}\}+\|U\|_p$$

respectively are Banach spaces. Since these spaces have the same elements, from the closed graph theorem it follows that

$$\|U\|_p^* \leqq C_p\|U\|_p \quad \text{for every } U\in H_p \text{ and some } C_p>1. \tag{6}$$

If $w=u+iv$ is such that $\hat{S}(w)=0$, then $U_0(x):=\exp(iwx)\in H_p$ and $\|U_0\|_p = \sup\left\{\exp\left(-vx-M_p(x)\right);\ x\in\mathbf{R}\right\}=\exp\left(\tilde{M}_p(v)\right)$. The relation (6) implies now

$$(C_p-1)\exp\left(\tilde{M}_p(v)\right) \geqq |w| \quad \text{or}$$

(7) $\tilde{M}_p(v)/\log|w|\geqq 1$ for sufficiently large $|w|$.

Let us take $p<p'<p''$ such that $M_p(p'x)\leqq M_{p''}(x)$ for sufficiently large $|x|$ (condition (A) implies the existence of such natural numbers p' and p'' for given $p\in\mathbf{N}$). So, we obtain

$$\tilde{M}_p(x/p') \geqq \tilde{M}_{p''}(x) \quad \text{or} \quad \tilde{M}_p(x) \geqq \tilde{M}_{p''}(p'x) \tag{8}$$

for sufficiently large $|x|$. From (7) (p replaced with p'') and (8) we get

$$\tilde{M}_p(v)/\log|w| \geqq \tilde{M}_p(v)/\tilde{M}_{p''}(v) \geqq \tilde{M}_{p''}(p'v)/\tilde{M}_{p''}(v) \geqq p'$$

for sufficiently large $|w|$ and $\hat{S}(w)=0$. This implies $\lim \tilde{M}_p(v)/\log|w|=\infty$ when $|w|\to\infty$ on the surface $\hat{S}(w)=0$. □

3. The Implication $(\mathrm{H}_2)\Rightarrow(\mathrm{H}_3)$

In order to prove this implication, we shall state two Lemmas. Let $\{d_p\}$ be a sequence of positive numbers which monotonically tends to zero. Let us denote by $\{\underline{m}_p(x)\}$ the sequence of continuous functions such that $\underline{m}_p(x):=M_p'(x)$ for $x>d_p>0$ and $\underline{m}_p(x):=M_p'(d_p)x/d_p$ for $0\leqq x\leqq d_p$. By $\{\underline{M}_p(x)\}$ we denote the corresponding sequence of convex functions, i.e.

$$\underline{M}_p(x) := \int_0^{|x|} \underline{m}_p(t)\,dt, \quad x\in\mathbf{R}, \quad p=1,2,\ldots.$$

It is clear that the functions $M_p(x)$ (resp. $\tilde{M}_p(x)$) and $\underline{M}_p(x)$ (resp. $\underline{\tilde{M}}_p(x)$) have the same behaviour in infinity. Obviously, condition (A) holds for the sequence $\{\underline{\tilde{M}}_p(x)\}$ provided that it holds for $\{M_p(x)\}$. The following lemma is obvious, so we omit the proof.

Lemma 2. $\lim_{y\to 0+} (\underline{\tilde{M}}_p(xy)/\underline{\tilde{M}}_p(y))=x^2$ *for every* $x\in\mathbf{R}$.

The next lemma is a version of Lemma 1 from [4].

Lemma 3. *For given* $A, \bar{B}, b>0$ *and* $p\in\mathbf{N}$ *there exists a positive constant* N *such that if* $U=U(x, y)$ *is a harmonic function for* $x^2+y^2<T^2$ *which satisfies the inequalities*

$$U(x, 0) \leqq 0, \quad U(x, y) \geqq -a\underline{\tilde{M}}_p(y)-\bar{B}\underline{\tilde{M}}_p(R) \tag{9}$$

for $x^2+y^2<T^2$ *then*

$$U(x, y) \leqq a\underline{\tilde{M}}_p(y)+(\bar{B}+b)\,\underline{\tilde{M}}_p(R) \tag{10}$$

for $x^2+y^2<R^2$ *provided that* $0<a<A$, $0<R<T/N$.

Proof. If we suppose that this lemma were false, then in the same way as in [4] we can show that there exist sequences $\{a_n\}$, $\{T_n\}$ and $\{R_n\}$ such that $0<a_n<A$, $T_n>n\cdot R_n$ and a sequence of harmonic functions $\{U_n(x, y)\}$ such that

$$U_n(x, 0) \leqq 0,\ U_n(x, y) \geqq -a_n\,\underline{\tilde{M}}_p(y)-\bar{B}\underline{\tilde{M}}_p(R_n) \quad \text{for} \quad x^2+y^2 < T_n^2$$

and $U_n(x_n, y_n)\geqq a_n\cdot\underline{\tilde{M}}_p(y_n)+(\bar{B}+b)\cdot\underline{\tilde{M}}_p(R_n)$ for some (x_n, y_n) which satisfies $x_n^2+y_n^2\leqq R_n^2$.

Let us put

$$V_n(x, y) := U_n(R_n x, R_n y)/\underline{\tilde{M}}_p(R_n),\ x_n' = x_n/R_n,\ y_n' = y_n/R_n.$$

We obtain

$$V_n(x, 0) \leqq 0,\ V_n(x, y) \geqq -a_n\,\underline{\tilde{M}}_p(R_n\,y)/\underline{\tilde{M}}_p(R_n)-\bar{B}$$

for $x_n'^2+y_n'^2\leqq n^2$ and

$$V_n(x_n', y_n') \geqq a_n\,\underline{\tilde{M}}_p(R_n\,y_n')/\underline{\tilde{M}}_p(R_n)+(\bar{B}+b)$$

for $x_n'^2+y_n'^2<1$.

As in [4], using Lemma 2, we can show that there exists a harmonic function $V(x, y)$ and real numbers x_0, y_0 and a_0 such that

$$V(x, 0) \leqq 0,\ V(x, y) \geqq -a_0\,y^2/2-\bar{B},\ V(x_0, y_0) \geqq a_0\,y_0^2/2+(\bar{B}+b). \tag{11}$$

From Harnack's inequality it follows that $V(x, y)$ is of the form $V(x, y)=cy+d$; (11) implies $d\leqq 0$. Putting $y:=-y_0$ in the second inequality in (11) we obtain $-cy_0+d\geqq a_0y_0^2/z-\bar{B}$, and this with the last inequality in (11) yields $2d\geqq b>0$, contradicting $d\leqq 0$. □

The proof of the implication $(\mathrm{H}_2)\Rightarrow(\mathrm{H}_3)$

Let $p, m\in\mathbf{N}$ and $b>0$ be given. Using the method from [6] one can prove the following

Proposition 2. *For given $p\in\mathbf{N}$ and $d>0$ there exists a positive constant n with the property that the Fourier transform $\hat{S}(w)$ of the convolutor S satisfies the inequality*

$$|\hat{S}(w)| \leqq |u|^n \exp\left(d\underline{\tilde{M}}_p(v)\right) \tag{12}$$

for sufficiently large $|w|$.

Let $w=u+iv$ be a complex number such that $0\leqq\underline{\tilde{M}}_p(v)<m\cdot\log|w|$ and let L be a positive constant which we fix later. The function

$$F_w(z):=\hat{S}\left(u+z\frac{v}{|v|}\right),\quad z=x+iy\in\mathbf{C}$$

is an entire analytic function of the complex variable z. Condition (H_2) implies

$$|F_w(x)|\geqq(2|u|)^{-B}\quad\text{for}\quad \underline{\tilde{M}}_p(x)<L\log|u| \tag{13}$$

and using (12)

$$|F_w(z)|\leqq(2|u|)^n\exp\left(d\underline{\tilde{M}}_p(y)\right) \tag{14}$$

for sufficiently large $|w|$ and $\underline{\tilde{M}}_p(|z|)\leqq L\cdot\log|u|$.

Let us analyse the function

$$\left(U_w(z):=-\log(2|u|)^B|F_w(z)|\right).$$

It is harmonic in z, if $\underline{\tilde{M}}_p(|z|)<L\cdot\log|u|$, for sufficiently large $|w|$. Let us put $A:=1+d,\ \ a:=d,\ \bar{B}:=(B+n+1)/(m+1),\ \ b:=1/(m+1),\ \ R:=\underline{\tilde{M}}_p^{-1}((m+1)\cdot\log|u|)$, where "$-1$" stands for the inverse function. Relations (13) and (14) show that we can apply Lemma 3 for these A, $\bar{B}$, b and the given $p\in\mathbf{N}$. Now, Lemma 3 implies the existence of a natural number N, so that for $|z|<\underline{\tilde{M}}_p^{-1}(L\cdot\log|u|)/N$ we have

$$U_w(z)\leqq d\underline{\tilde{M}}_p(y)+(B+n+2)\log|u|. \tag{15}$$

The moment has come to use condition (B). For a given $p\in\mathbf{N}$ and given $q:=N$ (N from Lemma 3) there exists $k\in\mathbf{N}$ such that (B) holds for sufficiently large values of the variable. Putting $L:=k(m+1)$ we see that $R\cdot N<T:=k(m+1)\cdot\log|u|$. The monotonicity of $\underline{\tilde{M}}_p(v)$ implies $|v|<R$, since $\underline{\tilde{M}}_p(v)<m\cdot\log|w|\leqq(m+1)\times\times\log|u|=\underline{\tilde{M}}_p(R)$. So, we can put $z:=i|v|$ in (15) which gives at last

$$\left(1/|\hat{S}(u+iv)|\right)\leqq|u|^{2B+2n+3}\exp\left(d\underline{\tilde{M}}_p(v)\right)$$

for sufficiently large $|w|$, and this is the inequality in condition (H_3) with $\underline{\tilde{M}}_p$ instead of $\tilde{M}_p$. But $\underline{\tilde{M}}_p(v)=\tilde{M}_p(v)+C_p'$ for sufficiently large $|v|$, and this proves $(H_2)\Rightarrow(H_3)$.

4. The functions $\underline{N}_p(x)$ and $Q_p(x)$, $p\in\mathbf{N}$

In order to prove the sufficiency of condition (H_3), we shall introduce some sequences of smooth functions.

Let $\omega:\mathbf{R}\to\mathbf{R}$ be a smooth nonnegative function which satisfies the following conditions: $\operatorname{supp}\omega\subset[0,1]$ and $\int_{\mathbf{R}}\omega(x)\,dx=1$. For $x>1$ and $p\in\mathbf{N}$ we define

$$\bar{N}_p(x):=\int_{\mathbf{R}}\tilde{M}_p(t)\,\omega(x-t)\,dt.$$

By $\underline{N}_p(x)$ we denote the smooth even function on $\mathbf{R}$ which equals $\overline{N}_p(x)$ for $x>1$, and is increasing for $x>0$. The next Lemma shows that the sequences $\{N_p(x)\}$ and $\{\tilde{M}_p(x)\}$ are equivalent in the sense that for every function from the first sequence there exists a greater one (at least, for $|x|$ sufficiently large) from the second and vice versa.

Now, we write down the following condition, which is by Lemma 4 equivalent to (H_3) from the Theorem.

(H_3') For every $p\in\mathbf{N}$ and $d>0$ there exists a constant $\overline{B}>0$ so that for every $m\in\mathbf{N}$ there exists $C_m>0$ such that

$$|1/\hat{S}(w)| \leqq |w|^{\overline{B}} \exp\left(d\underline{N}_p(v)\right) \quad \text{for} \quad \underline{N}_p(v) < m\log|w|$$

and $|w|\geqq C_m$.

Remark. One checks easily that this condition implies that there exists an $\overline{L}_1>\overline{B}$ which fits for any d_1, $0<d_1<d$, ($\overline{B}$ and d as in (H_3')), though in general for greater C_m.

Lemma 4. *Let $p\in\mathbf{N}$, $p>r(2)$ be given. Then:*

a) $\underline{N}_p(x)\leqq\tilde{M}_p(x)$ *for* $|x|\geqq 1$;

b) *There exists $p'\in\mathbf{N}$, $1<p'<p$ such that $\tilde{M}_p(x)\leqq\underline{N}_{p'}(x)+K_{pp'}$ for each $x\in\mathbf{R}$ and some $K_{pp'}>0$.*

c) *The condition (H_3') is equivalent to the condition (H_3).*

Proof. a) Let $x>1$. Then

$$\underline{N}_p(x)=\int_{\mathbf{R}}\tilde{M}_p(x-t)\,\omega(t)\,dt=\int_0^1\tilde{M}_p(x-t)\,\omega(t)\,dt\leqq\tilde{M}_p(x)\int_0^1\omega(t)\,dt=\tilde{M}_p(x).$$

b) Let p' be the largest natural number such that $r(p')\leqq p$. Let us show first

$$\tilde{M}_p(x)\leqq\tilde{M}_{p'}(x-1) \tag{16}$$

for sufficiently large $|x|$.

In fact, condition (A) implies $M_{p'}(p'x)\leqq M_p(x)$, hence $\tilde{M}_{p'}(x/p')\geqq\tilde{M}_p(x)$ for sufficiently large $|x|$. Since $\tilde{M}_{p'}(x)$ increases with $|x|$, the relation (16) follows. This implies

$$\tilde{M}_p(x)\leqq\tilde{M}_{p'}(x-1)\int_0^1\omega(t)\,dt\leqq\int_{\mathbf{R}}\tilde{M}_p(x-t)\,\omega(t)\,dt=\underline{N}_{p'}(x)$$

for sufficiently large $|x|$, and since $\tilde{M}_p(x)$ and $\underline{N}_p(x)$ are even b) follows.

c) Let us suppose that (H_3) holds. From part b) it follows that for every p_0' there exists $p_0>p_0'$ such that for sufficiently large $|x|$ the inequality:

$$\tilde{M}_{p_0}(x)<\underline{N}_{p_0'}(x) \quad \text{holds. If} \quad \underline{N}_{p_0'}(v)<m\log|w| \quad \text{and}$$

$|w|\geqq C_m$ then $\tilde{M}_{p_0}(v)<m\cdot\log|w|$. This implies that for $p=p_0$ and sufficiently large $|w|$ (H_3') holds, and hence (H_3') holds for $p=p_0'$. The opposite implication follows in a similar way from part a) of the Lemma. □

We prove now an important property of the functions $\underline{N}_p(x)$, namely: for given $(p, q)\in\mathbf{N}^2$ there exists $k=k(p, q)\geqq q$ such that

$$\underline{N}_p(qx) \leqq k\underline{N}_p(x) \quad \text{for } |x| \text{ sufficiently large.} \tag{17}$$

In fact, from (B) follows that for given $(p, q_1)\in\mathbf{N}^2$ there exists $k=k(p, q_1)\geqq q_1$, such that $\tilde{M}_p(q_1 y)\leqq k\cdot\tilde{M}_p(y)$ for $|y|$ sufficiently large. We choose $q_1\geqq q$ such that $q_1|y|\geqq q|y|+q-1$ (for large $|y|$), so $\tilde{M}_p(qy+q-1)\leqq k\cdot\tilde{M}_p(y)$. This implies

$$\tilde{M}_p(qy+qt-t) \geqq k\cdot\tilde{M}_p(y) \quad \text{for} \quad t\in[0, 1] \quad \text{and} \quad |y| \text{ large.}$$

Putting $y=x-t$ in the last inequality, multiplying it with $\omega(t)$ and integrating by t on $[0, 1]$ we obtain (17).

Next, we shall need the sequence of smooth functions on $\mathbf{R}$, $\{Q_p(x)\}_{p\in\mathbf{N}}$, which are defined in the following way:

$$Q_p(x) = \int_0^1 \underline{N}_p^{-1}(x-t)\,\omega(t)\,dt \quad \text{for} \quad x \geqq 1; \tag{18}$$

$(\underline{N}_p^{-1}(x)$ is the inverse function of $\underline{N}_p(x)$, $x>0)$. We define $Q_p(x)$ on $(-\infty, 1]$ so that each $Q_p(x)$ becomes smooth and even on $\mathbf{R}$ and increasing on $(0, \infty)$. We need later the obvious inequality

$$Q_p(x) \leqq \underline{N}_p^{-1}(x), \quad |x| \geqq 1,\ p\in\mathbf{N}. \tag{19}$$

5. The (p, q)-parametrix for S

Definition 2. Let $S\in\mathcal{O}_c'(\mathcal{H}'\{M_p\})$ and $p, q\in\mathbf{N}$. The distribution $P\in\mathcal{H}'\{M_p\}$ is called the (p, q) — parametrix for S if it satisfies the following two conditions:

(P_1) There exists $n\in\mathbf{N}$ and a continuous function $F(x)$ on $\mathbf{R}$ with the property

$$F(x) = O\left(\exp\left(-N_p(x)\right)\right) \quad \text{when } |x| \to \infty \text{ such that } P(x) = D^n F(x), \tag{20}$$

where $N_p(x)$ is an even, smooth nonnegative function such that for $x>1$

$$N_p(x) = \int_{\mathbf{R}} M_p(t)\,\omega(x-t)\,dt \tag{21}$$

and the sequence $\{N_p(x)\}$ is monotonically increasing (see [5]).

(P_2) If $W(x):=\delta(x)-(S*P)(x)$, then $W\in C^q(\mathbf{R})$ and for every j, $0\leqq j\leqq q$,

$$W^{(j)}(x) = O\left(\exp\left(-N_p(x)\right)\right) \quad \text{when} \quad |x| \to \infty.$$

We prove now

Proposition 3. *Let $S\in\mathcal{O}_c'(\mathcal{H}'\{M_p\})$ and $(p, q)\in\mathbf{N}^2$ be given. There exists a (p, q) — parametrix P for S if $\hat{S}(w)$ satisfies the condition (H_3').*

Proof. In the proof of this proposition we use the idea of an analogous statement from [7]. However, we give the complete proof since many details on the functions $M_p(x)$ occur.

Let b be an even number such that

$$b > (\bar{B}+1)/2; \tag{22}$$

$\bar{B}$ is the constant from (H_3') for given $p\in\mathbf{N}$ and $d>0$. By the assumption, for given $m\in\mathbf{N}$ (which we fix later), there exists $C_m>0$ such that the function

$$F(x, w) := \frac{1}{2\pi}\frac{\exp(ixw)}{\hat{S}(w)w^{2b}}, \quad w = u+iv\in\mathbf{C}, \tag{23}$$

is analytic for $\underline{N}_p(v)\leqq m\cdot\log|w|$ and $|w|\geqq C_m>1$. Condition (22) implies that the function $F(x, u)$ is integrable for each $x\in\mathbf{R}$ on the set $I_m:=\{u\in\mathbf{R};\ |u|\geqq C_m\}$. If we put

$$h(x) := \int_{I_m} F(x, u)\,du \quad\text{and}\quad g(x) := D^{2b}h(x) \quad\text{then}$$

$$S * g = \delta-\frac{1}{2\pi}\int_{-C_m}^{C_m}\exp(ixu)\,du.$$

Let $a>0$, $p\in\mathbf{N}$ and $c\in\mathbf{N}$ be given; take $q=2c$ and the constant $k>2c$ from (17) such that

$$c < k < ma/2 \tag{24}$$

(as wee see later, this can be achieved by letting m increase. Let $s(t)$ be an odd smooth function on $\mathbf{R}$ such that $s(t)=C_m$ for $0<t\leqq C_m$ and $s(t)=C_m\cdot\exp(aN_p(t))$ for $|t|\geqq 2C_m$. We suppose also that $s(t)$ increases for $t>C_m$. With $s^*(t)$ we denote the increasing smooth function on $\mathbf{R}$, which equals $s(t)$ for $|t|\geqq C_m$, with the property $s^{*(j)}(0)=0$ for $j=0, 1, 2, \dots$,. Let $z(t)$ be an even smooth function on $\mathbf{R}$ such that $z(t)=0$ for $|t|\leqq C_m$ and $z(t)=c\cdot Q_p(N_p(t))$ for $|t|\geqq 2C_m$ (see (18) and (21)); we suppose $z(t)$ to be increasing for $t>C_m$. From (17), (19) and (24) follows that

$$\underline{N}_p(2z(t)) \leqq m\log|s(t)| \quad\text{for}\quad |t|\geqq C_m. \tag{25}$$

We take the curve $L(x)\equiv w(t):=s(t)+i\cdot z(t)\cdot sgn\,x$, $t\in I_m$. Condition (24) shows that the function $h(x)$ can be written as

$$h(x) = \int_{L(x)} F(x, u)\,du = \int_{I_m} F(x, w(t))\,w'(t)\,dt$$

provided that $2b>\bar{B}+dm+2$.

Let us put

$$h_1(x) := \frac{1}{2\pi}\int_{L_1(x)}\frac{\exp(ixw)\,dw}{\hat{S}(w)w^{2b}}, \quad |x|\geqq C_m, \tag{26}$$

$$h_2(x) := \frac{1}{2\pi}\int_{L_2(x)}\frac{\exp(ixw)\,dw}{\hat{S}(w)w^{2b}}, \quad |x|\geqq C_m \tag{27}$$

where $L_1(x)$ and $L_2(x)$ are parts of the curve $L(x)$ for the values of the parameter t such that $|t|\geqq|x|$, respectively $C_m\leqq|t|\leqq|x|$.

We put now

$$W(x):=(S*D^{2b}h_2)(x)+\frac{1}{2\pi}\int_{-C_m}^{C_m}\exp(ixu)\,du \tag{28}$$

and finally $P(x):=D^{2b}h_1(x)$.

The distribution P is the (p,q)-parametrix for S, and $W(x)$ from (28) is just the function from the condition (P_2). In fact, from the following two lemmas conditions (P_1) and (P_2) follow.

Lemma 5. *The function $h_1(x)$ from (26) is $O(\exp(-c\cdot N_p(x)))$ when $|x|\to\infty$.*

PROOF. We write $h_1(x)$ as

$$h_1(x)=\frac{1}{2\pi}\int_{\{t:|t|\geqq|x|\}}\frac{\exp(ixw(t))\,w'(t)\,dt}{\hat{S}(w(t))\,w^{2b}(t)}.$$

First of all let us prove

$$|\exp(ixw(t))|\leqq K_0\exp(-cN_p(x)) \tag{29}$$

for sufficiently large $|x|$, $|t|\geqq|x|$ and some $K_0>0$. Since $\underline{N}_p^{-1}(x-1)\leqq Q_p(x)\leqq\underline{N}_p^{-1}(x)$ and (see [3]) $\dfrac{|x|}{M_p^{-1}(x)}\leqq\tilde{M}_p^{-1}(x)$ we have by Lemma 4 and [5]

$$Q_p(x)\geqq\underline{N}_p^{-1}(|x|-1)\geqq\tilde{M}_p^{-1}(|x|-1)\geqq\frac{|x|-1}{N_p^{-1}(|x|-1)}\geqq\frac{|x|-1}{N_p^{-1}(x)}.$$

This implies

$$Q_p(N_p(t))\geqq Q_p(N_p(x))\geqq\frac{N_p(x)-1}{N_p^{-1}(N_p(x))}=\frac{N_p(x)-1}{x},\quad |t|\geqq|x|,$$

hence

$$c|x|Q_p(N_p(t))\geqq cN_p(x)-c$$

and so (29) follows.

Let us observe now that

(30′) $|Q_p^{(j)}(x)|\leqq K_j'\cdot Q_p(x)\leqq K_j'\cdot|x|$,

(30″) $|N_p^{(j)}(x)|\leqq K_j''\cdot N_p(x)$ and (see [5]),

(30‴) $N_p(x)\leqq K_j'''\cdot\exp(aN_p(x))$, $|x|$ sufficiently large, for some K_j', $\underline{K}_j''$ and K''' (which depend also on $p\in\mathbf{N}$). To simplify the notations, we shall use the same letter K_j for these constants; moreover, when some constant depends only on $j\in\mathbf{N}_0$ and a fixed $p\in\mathbf{N}$ we denote it also by K_j.

Let us estimate $|z^{(j)}(t)|$. For $j=1$ the relations (30) imply

$$|z'(t)|\leqq K_1\exp(2adN_p(t)) \tag{31}$$

and by induction

$$|z^{(j)}(t)|\leqq K_j\exp(a(j+1)\,dN_p(t)),\quad j\in\mathbf{N}_0.$$

Now, it is easy to prove

$$|w^{(j)}(t)|\leqq K_j\exp(a(1+(j+1)\,d))N_p(t)) \tag{32}$$

for $|t|$ sufficiently large and some $K_j>0$, $j\in\mathbf{N}_0$.

At last, we have

$$|w(t)| \geqq K \exp\left(aN_p(t)\right) \quad \text{for } |t| \text{ sufficiently large} \tag{33}$$

and some $K>0$.

Combining (H_3'), (29), (32) (for $j=1$) and (33) we obtain

$$|h_1(x)| \leqq K_0' \exp\left(-cN_p(x)\right) \int_{\mathbf{R}} \exp\left(\left(a\bar{B}+\frac{dam}{2}+a(1+2d)-2ab\right)N_p(t)\right)dt =$$
$$= O\left(\exp\left(-cN_p(x)\right)\right) \quad \text{when} \quad |x| \to \infty$$

provided that

$$b > 1+\left(\bar{B}+\frac{dm}{2}+1+2d\right)\Big/2, \tag{34}$$

where b is an even natural number. So we obtain Lemma 5. □

Lemma 6. *For given $(p,q)\in\mathbf{N}^2$ there exist sufficiently small constants d, a and sufficiently large constants m, c such that*

$$W^{(j)}(x) = O\left(\exp\left(-N_p(x)\right)\right) \quad \textit{when} \quad |x| \to \infty$$

for every j, $0\leqq j\leqq q$. ($W(x)$ from (28)).

Proof. It is clear that the function $h_2(x)$ defined by (27) is smooth and equals zero for $|x|\leqq C_m$. We suppose that $x>0$ (the case $x<0$ is analogous).

Let us prove now

$$|h_2^{(j)}(x)| \leqq K_j \exp\left(ak_1 N_p(x)\right) \tag{35}$$

for some $k_1=k_1(j,d)\geqq 1$ (one can compute it exactly) and some K_j. In fact we have

$$2\pi h_2^{(j)}(x) = \int_{\{t:\, C_m\leqq|t|\leqq x\}} \frac{\exp\left(iw(t)x\right)w'(t)}{\hat{S}\left(w(t)\right)w(t)^{2b}}\,dt + R_j(x) \tag{35'}$$

where

$$R_0(x)=0,\;\; R_1(x)=\left(y_1(x)\exp\left(ixw(x)\right)-y_1(-x)\exp\left(ixw(-x)\right)w'(x),\;\; R_{j+1}(x)=\right.$$
$$=R_j'(x)+\left(y_1(x\exp\left(ixw(x)\right)\left(iw(x)\right)^{j+1}-y_1(-x)\exp\left(ixw(-x)\right)\left(iw(-x)\right)^{j+1}\right)w'(x),$$
$$y_1(x)=y_2\left(w(x)\right)=1/\left(\hat{S}(w(x))\,w(x)^{2b}\right),\; j=0,1,\ldots.$$

Our next goal is to prove that the function $y_1(w)=1/\left(\hat{S}(w)\cdot w^{2b}\right)$ is bounded together with its derivatives on the set $\Omega_1=\{w=u+iv\in\mathbf{C},\;\; \underline{N}_p(2v)\leqq m\cdot\log|u|,\; |w|\geqq T>C_m\}$ for sufficiently large T. In order to prove this, we shall use a version of Theorem 1.2.4 from [9], p. 17; also, we use the notations from [9].

First of all, the integral

$$\iint_{\Omega} |y_1(w)|\,dw\wedge d\bar{w}$$

converges in view of the condition $2b>\bar{B}+dm+2$, where $\Omega=\{w=u+iv\in\mathbf{C},\; \underline{N}_p(v)\leqq m\cdot\log|u|,\; |w|\geqq C_m\}$ (see (25)).

Let $\psi(w)\in C_0^\infty(\mathbf{C})$ be such that $\psi(w)=1$ on $B(0,1)$ and $\psi(w)=0$ for $w\notin B(0,2)$, where $B(w_0,R)=\{w\in\mathbf{C},\; |w-w_0|\leqq R\}$ $w_0\in\mathbf{C}$, $R>0$.

If $w\in\Omega_1$, then for T sufficiently large there exists $w_0\in\Omega$ such that $w\in B(w_0, 1)$ and $B(w_0, 2)\subset\Omega$. The function $y_1(w)$ is analytic on Ω_1, so $\frac{\partial}{\partial\bar{w}}(y_1(w)\cdot\psi(w-w_0))=$ $=y_1(w)\cdot\frac{\partial}{\partial\bar{w}}\psi(w-w_0)$, $w\in\Omega_1$. By the Cauchy formula we get

$$y_1(w)\psi(w-w_0)=\frac{1}{2\pi i}\Big(\int\limits_{\{\zeta:|\zeta-w_0|=2\}}\frac{y_1(\zeta)\psi(\zeta-w_0)}{\zeta-w}d\zeta+$$

$$+\iint\limits_{B(w_0,2)}y_1(\zeta)\frac{\frac{\partial}{\partial\bar{\zeta}}\psi(\zeta-w_0)}{\zeta-w}d\zeta\wedge d\bar{\zeta}\Big).$$

Observing that the first integral is zero, differentiating this equality j — times and using the following facts: $\frac{\partial}{\partial\bar{\zeta}}\psi(\zeta-w_0)$ is zero on $B(w_0, 1)$ and $\sup\{1/|\zeta-w|;$ $w\in B(w_0,1),\ \zeta\in\operatorname{supp}\frac{\partial}{\partial\bar{\zeta}}\psi(\zeta-w_0)\}<\infty$, we get the following inequality:

$$|y_1^{(j)}(w)|\leqq K_j'\iint\limits_{\Omega}|y_1(\zeta)|\,d\zeta\wedge d\bar{\zeta}=:K_j,\quad j=0,1,\dots$$

for any $w\in\Omega_1$ and some $K_j>0$. Observe that this K_j does not depend on w or w_0.

In view of (32) this implies that the last integral in (35′) can be estimated with

$$2x\cdot K_j\exp\big(ak_1'N_p(x)\big)\leqq K_j\exp\big(a(k_1'+d)N_p(x)\big)$$

and the remainders $R_j(x)$ with $K_j\cdot(\exp(ak_1''\cdot N_p(x))$ for some $k_1', k_1''\geqq 1$ which depend on $j\in\mathbf{N}_0$ and $d>0$ ($|x|$ sufficiently large). So we obtain (35) for $k_1=$ $=\max(k_1'+d, k_1'')$.

Let us observe that Lemma 6. implicitely states that $W(x)$ is in fact a function from the class C^q; from the smoothness of $h_2(x)$ and (35) this follows at once. We have

$$W^{(j)}(x)=(S*h_2{}^{(j+2b)})(x)+\Big(\frac{1}{2\pi}\int\limits_{-C_m}^{C_m}\exp(ixu)\,du\Big)_x^{(j)}.$$

Since S is a convolutor on $\mathscr{H}'\{M_p\}$, for given p_2 (which we fix later) there exists a continuous bounded function on $\mathbf{R}$ and a non-negative integer n such that $S=D^n\big(G(x)\exp(-N_{p_2}(x))\big)$. This implies

$$W^{(j)}(x)=$$

(36)

$$=\int\limits_{-\infty}^{\infty}G(y)\exp\big(-N_{p_2}(y)\big)\Big(\int\limits_{\{t:\,{}_mC\leqq|t|\leqq|x-y|\}}F\big(x-y,w(t)\big)w'(t)\,dt\Big)_y^{(j+2b+n)}dy\Big)+$$

$$+\frac{1}{2\pi}\Big(\int\limits_{-C_m}^{C_m}\exp(iux)\,du\Big)_x^{(j)}$$

$(F(x,w)$ from (23) and $w(t)=s(t)+iz(t)\cdot sgn(x-y))$.

We suppose that $x > C_m$, write the inner integral in (36) as

$$\int\limits_{\{t:\, C_m \leqq |t| \leqq |x-y|\}} = \int\limits_{\{t:\, C_m \leqq |t| \leqq x\}} + \int\limits_{\{t:\, x \leqq |t| \leqq |x-y|\}}$$

and estimate the obtained integrals separately. We shall show that they are both $O(\exp(-N_p(x)))$ when $x \to \infty$. The first integral can be written as

$$\text{(37)}\quad \int\limits_{-\infty}^{\infty} G(y)\exp(-N_{p_2}(y))\Big(\int\limits_{\{t:\, C_m \leqq |t| \leqq x\}} \big(F(x-y,\, w(t))\, w'(t)\, dt\big)_y^{(j+2b+n)}\Big)\, dy =$$

$$= \int\limits_{-\infty}^{x} + \int\limits_{x}^{\infty} \pm \int\limits_{x}^{\infty} \big(G(y)\exp(-N_{p_2}(y))\Big(\int\limits_{\{t:\, C_m \leqq |t| \leqq x\}} \frac{\exp\big(i(x-y)\,w(t)\big)\big(-iw(t)\big)^{j+n}}{\hat{S}(w(t))}\, dt\Big)\, dy$$

where in the second integral $w(t) = s(t) - iz(t)$, and in the others $w(t) = s(t) + iz(t)$.)

Now the first integral and the third integral with the "plus" sign in (37) give

$$(-1)^{j+n}\frac{2\pi}{2\pi}\int\limits_{\{t:\, C_m \leqq |t| \leqq x\}} \frac{\exp\big(ixw(t)\big)\, w^{j+n}(t)\, w'(t)\, dt}{\hat{S}(w(t))} \times$$

$$\times \int\limits_{-\infty}^{\infty} G(y)\exp(-N_{p_2}(y))\exp(-yiw(t))\, dy = \Big(\frac{1}{2\pi}\int\limits_{L_1(x)} \exp(ixw)\, dw\Big)_x^{(j)}.$$

Take now $p_1 \geqq r(p)$ such that the integral $\int\limits_{-\infty}^{\infty} \exp\big(-N_{p_1}(y) + c|y|Q_p(N_p(y))\big)\, dy$ converges and (for a given k_2) $p_2 \geqq r(p_1)$ such that $N_{p_2}(y) \geqq N_{p_1}(y) + k_2 N_p(y)$, $y \in \mathbf{R}$. (p_1 and p_2 with such a property exist by (A)). Then the fourth integral in (37) (with the "—" sign) can be estimated with

$$\text{const}\, 2x\, |w^{j+2b+n}(x)|\, |w'(x)|\, \big|y_2(w(x))\big| \exp(-k_2 N_{p_1}(x)) \times$$

$$\times \int\limits_{x}^{\infty} \exp\big(-N_{p_1}(y) + cyQ_p(N_p(y))\big)\, dy = O\big(\exp((k_3 a - k_2)\, N_p(x))\big)$$

when $x \to \infty$, where $k_3 > 0$ can be computed exactly; it depends on j, n, b and d. We take $k_2 > k_3 + 1 > k_3 a + 1$ if $0 < a < 1$.

At last, we can estimate the second integral in (37) as $O\big(\exp((k_3 a - k_2) N_p(x))\big)$ when $x \to \infty$, k_3 and k_2 as before.

The integral

$$\int\limits_{-\infty}^{\infty} G(y)\exp(-N_{p_2}(y))\Big(\int\limits_{\{t:\, x \leqq |t| \leqq |x-y|\}} F\big(x-y,\, w(t)\, w'(t)\, dt\big)_y^{(j+n+2b)}\Big)\, dy$$

can be written as

(38)

$$\int\limits_{-\infty}^{\infty} G(y)\exp(-N_{p_2}(y))\Big(\int\limits_{\{t:\, x \leqq |t| \leqq |x-y|\}} \frac{\exp\big(i(x-y)\,w(t)\big)\big(-iw(t)\big)^{j+n}}{\hat{S}(w(t))}\, w'(t)\, dt\Big)\, dy + $$

$$+ S_j(x)$$

where the S_j are the "remainders" (we estimate them at the end).

The integral in (38) we write as

$$\int_{-\infty}^{\infty} = \int_{-\infty}^{0} + \int_{0}^{x} + \int_{x}^{2x} + \int_{2x}^{\infty} = I_1 + I_2 + I_3 + I_4. \tag{39}$$

Now, if necessary, we increase the previous p_1 so that the inequality in the condition (C) holds for some $k=k_4$; this implies:

$$N_p(x-y) \leqq k_4 N_p(x) + N_{p_1}(y)$$

for x sufficiently large and $y<0$. If necessary, we increase $p_2 \geqq r(p_1)$ so that the integral

$$\int_{-\infty}^{\infty} |yG(y)| \exp\left(k_5 a N_{p_1}(y) - N_{p_2}(y)\right) dy$$

converges and obtain

$$|I_1| \leqq \underline{K}_1 \exp\left(-cN_p(x)\right) \int_{-\infty}^{0} yG(y) \exp\left(k_4 k_5 a N_p(x) \times\right.$$

$$\left.\times \exp\left(k_5 a N_{p_1}(y) - N_{p_2}(y)\right)\right) dy = O\left(\exp\left(k_4 k_5 a - c\right) N_p(x)\right)) \quad \text{when} \quad x \to \infty;$$

here $k_5 = (1+d)(j+2b+n)+(1+2d)$.

Next, since for $0<y<x$ we have $N_p(x-y) \geqq N_p(x) - N_{p_1}(y)$ (for given p there exists $p_1 \geqq r(p)$) and if p_2 is so large that

$$\int_{\pm\infty}^{\infty} |yG(y)| \exp\left(cN_{p_1}(y) - N_{p_2}(y)\right) dy < \infty$$

we have $|I_2| \leqq K_2 \cdot \exp\left((k_5 a - c) N_p(x)\right)$.

In a similar way we get (without condition (C)!):

$$I_3 = O\left(\exp\left((k_5 a - c) N_p(x)\right)\right)$$

and

$$I_4 = O\left(\exp\left(-cN_p(x)\right)\right) \quad \text{when} \quad x \to \infty$$

hence the integral in (38) is $O\left(\exp\left(k_4 k_5 a - c\right) N_p(x)\right))$ when $x \to \infty$.

The remainders $S_j(x)$ are linear combinations of the integrals

$$\int_{-\infty}^{\infty} G(y) \exp\left(-N_{p_2}(y)\right) \left(y_1^{(n_1)}(x-y)\right)^{n_2} \left((xw(x-y))_{(y)}^{n_3}\right)^{n_4} \times$$

$$\times \left((iw(x-y))_y^{(n_5)}\right)^{n_6} w^{(n_7)}(x-y) \exp\left(i(x-y)w(x-y)\right) dy, \quad n_i \in \mathbf{N}_0$$

so by the previously used inequalities we get

$$S_j(x) = O\left(\exp\left((k_6 a - c) N_p(x)\right)\right) \quad \text{when} \quad x \to \infty,$$

for some k_6 which depends on j, n, b and d.

Now we can write

$$W^{(j)}(x) = O\left(\exp\left(k_7 a - c\right) N_p(x)\right)) + \frac{1}{2\pi} \Big(\int_{L^*(x)} \exp(ixw)\, dw \Big)_x^{(j)}$$

where $L^*(x)$ is the contour obtained by joining the segment $[-C_m, C_m]$ on the real axis with the curve $w^*(t)=s^*(t)+iz(t)$ for $C_m \leqq |t| \leqq x$, where $k_7=\max(k_4k_5, k_5, k_6)$. Let us show

$$(40) \qquad \frac{1}{2\pi}\Big(\int_{L^*(x)} \exp(ixw)\,dw\Big)_x^{(j)} = O\left[\exp\left(k_8-\frac{c}{2}\right) N_p(x)\right]$$

when $x\to\infty$ for some $k_8>1$ which depends on j, n, b and d.

From the Cauchy theorem it follows that we can replace $L^*(x)$ with the contour $L_1^*(x)$, which is the complementary contour to $L^*(x)$, i.e. its equation is $w^*(t)= =s^*(t)+iz(t)$ for $o<x<|t|$. So we get that the integral in (40) is equal to

$$(41) \qquad \frac{1}{2\pi}\int_{L_1^*(x)} (iw)^j \exp(ixw)\,dw + S_j^*(x)$$

where $S_j^*(x)$ is the remainder. For sufficiently large x we can estimate the last integral with

$$\exp\left(-\frac{c}{2}N_p(x)\right)\int_{|t|\geqq x} \exp\left(-\frac{c}{2}N_p(t)\right)\exp\left(k_8' a N_p(t)\right)dt =$$
$$= O\left[\exp\left(-\frac{c}{2}N_p(x)\right)\right] \quad \text{when} \quad x\to\infty,\ k_8'=k_8'(j,b,d)$$

provided thet $k_8'a<\frac{c}{2}+1$. The remainders $S_j^*(x)$ are estimated easily as $O\left[\exp\left(\left(k_8''a-\frac{c}{2}\right)N_p(x)\right)\right]$ when $x\to\infty$ for some $k_8''>0$. Taking $k_8=\max(k_8', k_8'')$ we obtain (40). So, for $k_9:=\max_{0\leqq j\leqq q}\max(k_7, k_8)$ we get

$$(42) \qquad W^{(j)}(x) = O\left[\exp\left(\left(k_9 a-\frac{c}{2}\right)N_p(x)\right)\right] \quad \text{when} \quad x\to\infty \text{ and } 0\leqq j\leqq q.$$

Take now $c=2p+2$, choose p_1, p_2 and n as indicated before. Taking now $b>\bar{B}+3$ ($\bar{B}$ from (H_3')), a so small that $k_9a-c/2<-1$, m sufficiently large that $m>2c/a$ and observing that k_9 does not grow by decreasing d, take $d<2/m$, (see the Remark after condition (H_3')) we obtain the Lemma. □

As we said before, proving Lemmas 5 and 6, we finish the proof of Proposition 3. We prove in the sixth section that the existence of a (p, q)-parametrix for the convolutor S implies its hypoellipticity.

6. Sufficiency of (H_3) for hypoellipticity

Proposition 4. Let us suppose that for given $(p, q)\in\mathbf{N}^2$ there exists a (p, q)-parametrix $P\in\mathcal{H}'\{M_p\}$ for the convolutor $S\in\mathcal{O}_c'(\mathcal{H}'\{M_p\})$. Then the equation

$$(1) \qquad S*U=V$$

and S are hypoelliptic in $\mathcal{H}'\{M_p\}$.

PROOF. Let $V \in \mathscr{E}\mathscr{H}'\{M_p\}$ be given and let us suppose that there exists a solution $U \in \mathscr{H}'\{M_p\}$ of (1). We know that there exists $p_1 \in \mathbf{N}$ such that

$$|V^{(j)}(x)| \leqq C_j \exp\left(M_{p_1}(x)\right) \quad \text{for every} \quad j = 0, 1, 2, \dots.$$

Theorem 1 in [5] implies the existence of numbers $k_1 \in \mathbf{N}_0$ and $p_2 \in \mathbf{N}$ and a continuous function $K_1(x)$ on $\mathbf{R}$ such that

$$U(x) = D^{k_1} K_1(x) \quad \text{and} \quad |K_1(x)| \leqq C \exp\left(M_{p_2}(x)\right).$$

Let $j \in \mathbf{N}_0$ be given and put $p := r\left(r(r(p_1+p_2))\right)$, $q := j + k_1$. For this pair (p, q) we take the parametrix P for the convolutor S, so that the conditions (P_1) and (P_2) from Definition 2. hold. This implies

(43) $\quad U = U*\delta = U*(S*P+W) = U*(S*P)+U*W = (U*S)*P+U*W \Rightarrow$

(43′) $\quad U = V*P+U*W.$

The existence and the associtivity of the convolutions in (43) follows from the assumptions on U, V, P and S.

We have yet to show the smoothess of the two summands in (43′), and that they are $O\left(\exp\left(M_{p_0}(x)\right)\right)$ when $|x| \to \infty$ for some $p_0 \in \mathbf{N}$, i.e.

$$U^{(j)}(x) = O\left(\exp\left(M_{p_0}(x)\right)\right) \quad \text{when} \quad |x| \to \infty \tag{44}$$

and p_0 should not depend on $j \in \mathbf{N}_0$.

The function $V*P$ is smooth, since

$$(V*P)(x) = (V*D^n F)(x) = (V^{(n)}*F)(x) = \int_{\mathbf{R}} V^{(n)}(x-y) F(y)\, dy$$

and generally

$$(V*P)^{(j)}(x) = \int_{\mathbf{R}} V^{(n+j)}(x-y) F(y)\, dy.$$

Let us estimate this integral:

$$\left|\int_{\mathbf{R}} V^{(n+j)}(x-y) F(y)\, dy\right| \leqq c_{n,j} \int_{\mathbf{R}} \exp\left(M_{p_1}(x-y)\right) \exp\left(-N_p(y)\right) dy.$$

Using the inequality

$$\begin{aligned} M_{p_1}(x-y) &\leqq M_{r(p_1)}(x) + M_{r(p_1)}(y) \leqq M_{r(p_1)}(x) + N_{r(r(p_1))}(y) + \\ &+ K' \leqq M_{r(p_1)}(x) + N_{r(r(p_1+p_2))}(y) + K' \end{aligned} \tag{45}$$

for some $K' = K(p_2)$ we obtain

$$|V*P)^{(j)}(x)| \leqq c \exp\left(M_{r(p_1)}(x)\right), \quad c > 0. \tag{46}$$

Since W is a C^q function, the choice of p shows that this is true also for $U*W$. So, we have

$$(U*V)^{(j)}(x) = (K_1*W)^{(j+k_1)}(x)$$

and

$$|U*W)^{(j)}(x)| \leqq C' \int_{\mathbf{R}} \exp\left(M_{p_2}(y)\right) \exp\left(-N_p(x-y)\right) dy$$

for some $C'>0$. Hence from a relation like (45) we obtain

$$|(U*W)^{(j)}(x)| \leqq C'' \exp\left(M_{r(p_2)}(x)\right). \tag{47}$$

From (43′), (46) and (47) we see that putting $p_0 := \max\left(r(p_1), r(p_2)\right)$ we get (44). Observe that p_0 does not depend on j. □

References

[1] I. M. GEL'FAND, G. E. ŠILOV, Generalized Functions, Vol. II, *Academic Press, New York, London* 1968.

[2] I. M. GEL'FAND, G. E. ŠILOV, Generalized Function, Vol. III, *Academic Press, New York, London* 1967.

[3] O. V. GRUDZINSKI, Convolutions-Gleichungen in Räumen von Beurling Distributionen endlicher Ordnung, *Habilitationsschrift, Kiel*, 1980.

[4] L. HÖRMANDER, On the Range of Convolution Operators, *Ann. Math.* **76** (1962), pp. 148—169.

[5] S. PILIPOVIĆ, A. TAKAČI, The Space $\mathscr{H}'\{M_p\}$ and Convolutors, Proceedings of the Moscow Conference on Generalized Functions, *Moscow* 1981, pp. 415—426.

[6] S. PILIPOVIĆ, A. TAKAČI, Solvability of Convolution Equations in the Space $\mathscr{H}'\{M_p\}$, *to appear* in Revue Rour. Math. Pur. Appl.

[7] G. SAMPSON, Z. ZIELEZNY, Hypoelliptic Convolution Equations in $\mathscr{K}_p'$ $p>1$, *Trans. Amer. Math. Soc.* **223** (1976), pp. 133—154.

[8] J. WLOKA, Über die Hurewič — Hörmanderschen Distributionsräume, *Math. Ann.*, **160** (1965), pp. 321—362.

[9] L. HÖRMANDER, An Introduction to Complex Analysis in Several Complex Variables, (In Russian), *Mir, Moskva*, 1968.

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