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A regularity theorem in information theory

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Abstract. We show a general regularity theorem for the functional equation of the type

$$f(x)+\sum_{i=1}^{m}(1-x)^{\alpha_i}g_i\left(\frac{y}{1-x}\right)=h(y)+\sum_{i=1}^{m}(1-y)^{\alpha_i}k_i\left(\frac{x}{1-y}\right)$$

and present the regular solution in a special case.

Introduction

In the characterization of symmetric divergences and distance measures the functional equation

$$f(x)+\sum_{i=1}^{m}(1-x)^{\alpha_i}g_i\Big(\frac{y}{1-x}\Big)=h(y)+\sum_{i=1}^{m}(1-y)^{\alpha_i}k_i\Big(\frac{x}{1-y}\Big) \qquad \text{(FE)}$$

arises in the special case $m=2$, $f=h$, $g_1=k_1$ and $g_2=k_2$ (see [12] where $\alpha_2=1-\alpha_1$ and [11]). In this case all tuples (f,h,g_1,g_2,k_1,k_2) satisfying (FE) for $x,y,x+y\in(0,1)$ are determined in [11] under the assumption that f, g_1 and $g_2:(0,1)\mapsto\mathbb{R}$ are three times continuously differentiable. One aim of this paper is to show that this result remains true if f, g_1 and g_2 are Lebesgue measurable or satisfy the Baire property. In a more general setting we assume that $f,h,g_i,k_i:(0,1)^n\to\mathbb{R}$ $(1\le i\le m)$ satisfy (FE) for all

$$(x,y)\in D_n^\circ=\{(u,v):u,v,u+v\in(0,1)^n\}.$$

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Here all operations on vectors like addition, multiplication and division are done componentwise. Thus we put $1 = (1, 1, \dots, 1)$ and $0 = (0, 0, \dots, 0)$ and for instance

$$1 - x = (1 - x_1, 1 - x_2, \dots, 1 - x_n)$$

for $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$. Moreover we agree on writing

$$x^\alpha = (x_1, \dots, x_n)^{(\alpha_1, \dots, \alpha_n)} = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$$

and

$$x \odot y = (x_1, \dots, x_n) \odot (y_1, \dots, y_n) = x_1 y_1 + \dots + x_n y_n$$

for the usual inner product in $\mathbb{R}^n$. Further we denote by

$$\mathcal{U}_n = \{(1, 0, \dots, 0), \dots, (0, 0, \dots, 0, 1)\}$$

the set of the n unit vectors in $\mathbb{R}^n$ with respect to the canonical basis in $\mathbb{R}^n$ and we put $\log x = (\log x_1, \dots, \log x_n)$.

We'll prove that the assumption that f, g_i or h, k_i are Lebesgue-measurable or Baire-functions satisfying (FE) implies that f, h, g_i, k_i are infinitely often differentiable ($1 \le i \le m$). The idea of the proof is to show that — after some transformations — rather general regularity theorems of [4] and [5] work also in the case of the functional equation (FE). If $m > 1$ then the regular solutions of (FE) are known only if $m = 2$, $n = 1$ (and $f = h$, $g_1 = k_1$, $g_2 = h_2$; see [11]) but the general solution of (FE) is unknown. If $m = 1$ then (FE) goes over into the so-called generalized fundamental equation of information (with four unknown functions), which has many applications (see [6] and [10]). The general solution of (FE) for $m = 1$ is known [3] but the proof is rather long and is dependent upon many additional results. Moreover, we cannot always easily deduce from the representation of the general solution of (FE) (for $m = 1$) the regular ones (see [10], p. 162). Therefore we give a brief direct proof for the form of topologically and measure theoretically characterized solutions of (FE) in the case $m = 1$ using functional equations and differential equations. Our theorem generalizes the results in [6] and [9] where the distributional solutions of the generalized entropy equation, which is equivalent to the generalized fundamental equation of information (see [6], p. 144), are determined.

2. A regularity theorem

Before we prove a general regularity theorem for functional equation (FE) we present two results because we think they are worth while to be included for handy reference in the future. We remark that measurability always means Lebesgue-measurability.

Lemma 2.1. *Let T, U be open subsets of $\mathbb{R}^p$ and $\mathbb{R}^q$, respectively, let $f_i, h_i : U \to \mathbb{R}$, let $g_i, k_i : T \to \mathbb{R}$ and let $F : T \times U \to \mathbb{R}$ $1 \leq i \leq m$.*

(a) If $f_1, \dots, f_m$ are linearly independent on U and satisfy

$$\sum_{j=1}^{m} f_j(x) g_j(t) = F(t, x) \tag{2.1}$$

for $(t, x) \in T \times U$ then there exists $x_1, \dots, x_m \in U$ and real-valued functions $a_{i,j}$, $1 \leq i, j \leq m$ depending upon $f_j(x_i)$ such that

$$g_j(t) = \sum_{i=1}^{m} a_{i,j} F(t, x_i), \qquad 1 \leq j \leq m \tag{2.2}$$

for all $t \in T$. Moreover, if F is measurable (or has the Baire property) in the first variable then $g_1, \dots, g_m$ are also measurable (or have the Baire property).

(b) If $f_1, \dots, f_m$ and $g_1, \dots, g_m$ are linearly independent on U and T, respectively, and if they satisfy (2.1) for all $(t, x) \in T \times U$ where F is measurable (or has the Baire property) in both variables then $f_1, \dots, f_m$ and $g_1, \dots, g_m$ are also measurable (or have the Baire property).

(c) If $f_1, \dots, f_m$ and $k_1, \dots, k_m$ are linearly independent on U and T, respectively, and if they satisfy

$$\sum_{j=1}^{m} f_j(x) g_j(t) = \sum_{i=1}^{m} h_i(x) k_i(t) \tag{2.3}$$

for $(t, x) \in T \times U$ then there exists constants $b_{i,j} \in \mathbb{R}$ $(1 \leq i, j \leq m)$ such that

$$g_j(t) = \sum_{i=1}^{m} b_{i,j} k_i(t), \qquad 1 \leq j \leq m \tag{2.4}$$

and

$$h_i(x) = \sum_{j=1}^{m} b_{i,j} f_j(x), \qquad 1 \leq i \leq m \tag{2.5}$$

for all $(t, x) \in T \times U$. Moreover, if f_i, k_i, $1 \leq i \leq m$ are measurable (or have the Baire property) then g_i, h_i, $1 \leq i \leq m$ are also measurable (or have the Baire property).

Proof. (a) By the linear independence of $f_1, \dots, f_m$ there exists elements $x_1, \dots, x_m \in U$ such that $\det M \neq 0$ where the matrix M is given by $M = f_j(x_i)_{i,j=1}^m$ ([1], p. 229). Putting $x = x_i$, $1 \le i \le m$ into (2.1) we get a linear system for $g_1, \dots, g_m$ with coefficient matrix M. By Cramer's rule we obtain

$$g_j(t) = \sum_{i=1}^m a_{i,j} F(t, x_i)$$

where the functions $a_{i,j}$, $1 \le i, j \le m$ are the cofactors of $f_j(x_i)$ divided by $\det M$. The measurability statement in (a) follows from the representation (2.2).

(b) Statement (b) follows from (a) and replacement $x \leftrightarrow t$ and $f_j \leftrightarrow g_j$.

(c) Using part (a) with $F(t,x) = \sum_{i=1}^m h_i(x) k_i(t)$ we get from (2.2) for $1 \le j \le m$

$$g_j(t) = \sum_{l=1}^m \Big(\sum_{i=1}^m h_l(x_i) a_{i,j} \Big) k_l(t) = \sum_{i=1}^m b_{i,j} k_i(t)$$

for all $t \in T$. Substituting (2.4) into (2.3) we arrive at

$$\sum_{i=1}^m \Big(\sum_{j=1}^m b_{i,j} f_j(x) \Big) k_i(t) = \sum_{i=1}^m h_i(x) k_i(t)$$

which implies (2.5) using the linear independence of $k_1, \dots, k_m$. The second statement in (c) follows immediately from (2.4) and (2.5).

The following result is a special case of much more general results in [5] but it is a version which can be used directly rather often in applications.

Theorem 2.2. *Let T, U be open subsets of $\mathbb{R}^n$ and $\mathbb{R}^r$, respectively, let D be an open subset of $T \times U$ and $X_i \subset \mathbb{R}^n$, let $a : T \to \mathbb{R}$, $b : U \to \mathbb{R}$, let $H_i : D \times \mathbb{R} \to \mathbb{R}$ $(0 \le i \le N)$, let $F_i : X_i \to \mathbb{R}$ and let $G_i : D \to X_i$, $1 \le i \le N$. Suppose that*

(1) for each $(t, x) \in D$

$$a(t) = H_0(t, x, b(x)) + \sum_{i=1}^N H_i(t, x, F_i[G_i(t,x)]), \tag{2.6}$$

(2) H_i is $(p+1)$-times continuously differentiable $(0 \le i \le N)$,

(3) G_i is $(p+1)$-times continuously differentiable $(1 \le i \le N)$,

(4) For all $t \in T$ there exists $x \in U$ such that

$$(t, x) \in D, \quad \operatorname{rank} \frac{\partial G_i}{\partial x}(t, x) = n \qquad (1 \le i \le N). \tag{2.7}$$

Then the following statements hold:

(a) *If* (2) *and* (3) *are valid with* $p = 0$ *and if* $F_1, \dots, F_N$ *are measurable or have Baire property then* a *is continuous on* T. *(We remark that no regularity condition for* b *is needed.)*

(b) *If* (2) *and* (3) *are valid with* $p = 1$ *and if* $b, F_1, \dots, F_N$ *are continuous then* a *is continuously differentiable on* T.

(c) *If* (2) *and* (3) *are valid with* $p > 1$, *if* b *is continuously differentiable and* $F_1, \dots, F_N$ *are* p*-times continuously differentiable then* a *is* $(p+1)$*-times continuously differentiable on* T.

Proof. The result follows from Theorem 2.7.2 in [4] and Theorem 4.3, Theorem 5.2 and Theorem 7.2 in [5].

Now we are ready to prove our main result

Theorem 2.3. *Let* $\alpha_i \in \mathbb{R}^n$ *be different vectors, let* $f, h, g_i, k_i : (0,1)^n \to \mathbb{R}^n$ $(1 \leq i \leq m)$ *and suppose that the functional equation (FE) is satisfied for all* $(x, y) \in D_n^\circ$. *If either the functions* $h, k_1, \dots, k_m$ *or the functions* $f, g_1, \dots, g_m$ *are measurable or have the Baire property, then* $f, h, g_1, \dots, g_m, k_1, \dots, k_m$ *are infinitely often differentiable.*

Proof. (1) Let us introduce in (FE) the new variable $t = \dfrac{y}{1-x}$ instead of y so that we obtain

$$\sum_{i=1}^{m} (1-x)^{\alpha_j} g_j(t) = F(t, x) \qquad (t, x) \in T \times T \tag{2.8}$$

where $T = (0,1)^n$ and $F : T \times T \to \mathbb{R}$ is defined by

$$F(t, x) = h(t(1-x)) - f(x) + \sum_{i=1}^{m} (1 - t(1-x))^{\alpha_i} k_i\Big(\frac{x}{1 - t - tx}\Big). \tag{2.9}$$

We remark that the function $\psi : D_n^\circ \to T \times T$ defined by

$$\psi(x, y) = \Big(x, \frac{y}{1-x}\Big) \qquad (x, y) \in D_n^\circ \tag{2.10}$$

is bijective with inverse

$$\psi^{-1}(s, t) = (s, t(1-s)) \qquad (s, t) \in T \times T. \tag{2.11}$$

Moreover, for fixed x the function $t \mapsto F(t, x)$ is measurable (or has the Baire property) since by hypothesis h and $k_1, \dots, k_m$ are measurable (or have the Baire property). Moreover, it is known that the functions $(1-x)^{\alpha_1}, \dots, (1-x)^{\alpha_m}$ are linearly independent on every open interval of $\mathbb{R}^n$ since the α's are different. Thus Lemma 2.1(a) implies that $g_1, \dots, g_m$

and thus f (see (FE)) are measurable (or have Baire property). In the same manner we get that $h, k_1, \dots, k_m$ are measurable (or have Baire property) if $f, g_1, \dots, g_m$ are measurable (or have Baire property). Thus we know that all occuring functions in (FE) are measurable (or have the Baire property).

(2) The idea of the proof is now to apply Theorem 2.2 for all occuring functions in (FE). From (FE) we get

$$h(y) = f(x) + \sum_{i=1}^{m}(1-x)^{\alpha_i} g_i\Big(\frac{y}{1-x}\Big) - \sum_{i=1}^{m}(1-y)^{\alpha_i} k_i\Big(\frac{x}{1-y}\Big), \tag{2.12}$$

for all $(x,y) \in D_n^\circ$. Thus (2.12) is obviously of the form (2.6) with $D = D_n^\circ \subset T \times U = (0,1)^n \times (0,1)^n$. Let us put $a = h$, $b = f$, $N = 2m$, $F_i = g_i$, $G_i(t,x) = \frac{t}{1-x}$, $H_0(t,x,z) = z$, $H_i(t,x,z) = (1-x)^{\alpha_i} z$, $1 \le i \le m$ $F_i = k_i$, $G_i(t,x) = \frac{x}{1-t}$, $H_i(t,x,z) = -(1-t)^{\alpha_i} z$, $m+1 \le i \le 2m$ $((t,x,z) \in D \times \mathbb{R})$. Because of the componentwise definition of division in $\mathbb{R}^n$ condition (4) of Theorem 2.2 is satisfied (since the matrix $\dfrac{\partial G_i}{\partial x}(t,x)$ has elements different from zero only in the diagonal). Thus by Theorem 2.2 (a) f is continuous on $T = (0,1)^n$.

Now we prove the continuity of g_j, $1 \le j \le m$. Using (2.8) and Lemma 2.1 we obtain the representation

$$g_j(t) = \sum_{i=1}^{m} a_{i,j}(x_1, \dots, x_m) F(t, x_i), \qquad 1 \le j \le m \tag{2.13}$$

where F is given by (2.9). By hypothesis and because of the functions $a_{i,j}$ are the quotients of determinants where the elements are $\mathcal{C}^\infty$-functions we get that the $a_{i,j}$ are $\mathcal{C}^\infty$-functions defined on some non-void open set $U_1 \times \dots \times U_m$. Applying once more Theorem 2.2 (a) (with $N = m(m+2)$, $a = g_j$, $D = T \times U_1 \times \dots \times U_m$) we get that g_j, $1 \le j \le m$ is continuous.

In the same manner we can prove the continuity of f and $k_1, \dots, k_m$.

(3) Finally using Theorem 2.2 (b) and Theorem 2.2 (c) we get first that f, h, g_i, k_i $(1 \le i \le m)$ are continuously differentiable and then (by the induction step of Theorem 2.2 (c)) that all occuring functions in (FE) are infinitely often differentiable. (Note that the inner functions of the unknown functions are infinitely often differentiable.)

Remark 2.4. (a) We remark that if $\alpha_i = \alpha_j$ for some $i \neq j$, then the regularity of the functions g_i, g_j and k_i, k_j does not follows from the equation, because by adding an *arbitrary* function to g_i (or to k_i) and subtracting from g_j (or k_j), the equation remains satisfied. But such cases may eliminate reducing such terms.

(b) To prove the continuity of f and h in Theorem 2.3 it was possible to apply Theorem 2.2 (a) directly. But for the proof of the continuity of $g_1, \dots, g_m$ (or $k_1, \dots, k_m$) we needed a trick: After introducing a new variable t we arrived at (2.8) and could express g_i as a linear combination of functions of the form (2.9) (which is of the form (2.6)).

(c) Sometimes the above trick is not necessary. In case of (FE) we can take the bijective transformation $\varphi : D_n^\circ \to T \times T = (0,1)^n \times (0,1)^n$, defined by

$$\varphi(x,y) = \Big(\frac{x}{1-y}, \frac{y}{1-x}\Big), \qquad (x,y) \in D_n^\circ \tag{2.14}$$

with inverse

$$\varphi^{-1}(s,t) = \Big(\frac{s(1-t)}{1-st}, \frac{t(1-s)}{1-st}\Big), \qquad (s,t) \in T \times T. \tag{2.15}$$

Putting $s = \frac{x}{1-y}$ and $t = \frac{y}{1-x}$ into (FE) we get, dividing (FE) by $(1-x)^{\alpha_i}$ for fixed i, $1 \leq i \leq m$:

$$\begin{aligned} g_i(t) &= b(s,t)^{-\alpha_i}\big[h(a(t,s)) - f(a(s,t))\big] \\ &+ \sum_{j=1}^{m} b(t,s)^{\alpha_j} b(s,t)^{-\alpha_i} k_j(s) - \sum_{j=1,\, j\neq i}^{m} b(s,t)^{\alpha_j - \alpha_i} g_j(t), \end{aligned} \tag{2.16}$$

where $a(s,t) = \frac{s(1-t)}{1-st}$, $b(s,t) = 1 - a(s,t) = \frac{1-s}{1-st}$. This equation is of the form (2.6) but Theorem 2.2(a) cannot be applied to prove the continuity of g_i since in the last sum of equation (2.16) the inner functions of g_j $(1 \leq j \leq m, j \neq i)$ are only dependent upon t but not on x so that the rank condition (2.7) is not satisfied. Nevertheless, if $m = 1$ in (FE) (so that the last sum in (2.16) disappears) then the continuity of g_1 follows (similarly we get the continuity of k_1).

3. The generalized fundamental equation of information

Theorem 3.1. *Let $\alpha \in \mathbb{R}^n$ and let $f, h, g, k : (0,1)^n \to \mathbb{R}$ satisfy*

$$f(x) + (1-x)^\alpha g\Big(\frac{y}{1-x}\Big) = h(y) + (1-y)^\alpha k\Big(\frac{x}{1-y}\Big) \tag{3.1}$$

for all $(x,y)\in D_n^\circ$. *Then all functions* f, g, h, k *which are measurable or have the Baire property are given by*

$$\begin{aligned}
f(x) &= B_1\odot\log x+(A_1+A_2)\odot\log(1-x)+b_3+a_4\\
g(x) &= A_1\odot\log x+A_2\odot\log(1-x)+a_3\\
h(x) &= A_1\odot\log x+(B_1+A_2)\odot\log(1-x)+a_3+a_4\\
k(x) &= B_1\odot\log x+A_2\odot\log(1-x)+b_3
\end{aligned}\tag{3.2}$$

for $\alpha=(0,0,\dots,0)$, *or*

$$\begin{aligned}
f(x) &= b_1x^\alpha-a_3(1-x)^\alpha+a_4+\varphi_1(x)\\
g(x) &= a_1x^\alpha+a_2(1-x)^\alpha+a_3+\varphi_2(x)\\
h(x) &= a_1x^\alpha-b_2(1-x)^\alpha+a_4+\varphi_2(x)\\
k(x) &= b_1x^\alpha+a_2(1-x)^\alpha+b_2+\varphi_1(x)
\end{aligned}\tag{3.3}$$

for $\alpha\neq(0,0,\dots,0)$, *where*

$$\begin{aligned}
a_2=0 \quad \text{and} \quad &\varphi_1(x)=\varphi_2(x)=S_{\alpha,A}(x)\\
&:=x^\alpha(A\odot\log x)+(1-x)^\alpha(A\odot\log(1-x))
\end{aligned}\tag{3.4}$$

if $\alpha\in\mathcal{U}_n$, *and where*

$$\varphi_1(x)=a(x_i-x_j),\quad \varphi_2(x)=-\varphi_1(x)\quad \text{if } x^\alpha=x_ix_j \text{ for some fixed } i\neq j\tag{3.5}$$

and where

$$\varphi_1=\varphi_2=0\quad \text{in all other cases;}\tag{3.6}$$

here $A,A_1,A_2,B_1\in\mathbb{R}^n$ *and* $a,a_1,a_2,a_3,a_4,b_1,b_2,b_3\in\mathbb{R}$.

Proof. (a) It is only an easy computation to show that the solutions (3.2)–(3.6) satisfy equation (3.1). Moreover these solutions are measurable and have the Baire property.

(b) To prove the reverse we first prove the theorem in the case $n=1$. Let us differentiate equation (3.1) with respect to x, and the resulting equation with respect to y, to obtain (after some rearrangement)

$$\begin{aligned}
&(1-x)^{\alpha-2}\left[g'\Big(\frac{y}{1-x}\Big)(1-\alpha)+\frac{y}{1-x}g''\Big(\frac{y}{1-x}\Big)\right]\\
&\quad=(1-y)^{\alpha-2}\left[k'\Big(\frac{x}{1-y}\Big)(1-\alpha)+\frac{x}{1-y}k''\Big(\frac{x}{1-y}\Big)\right].
\end{aligned}\tag{3.7}$$

Using the bijective transformation $\varphi : D_1^\circ \to (0,1)^2$ given by (2.14), equation (3.7) goes over into

$$(3.8)\qquad \begin{aligned}&(1-t)^{2-\alpha}\big(g'(t)(1-\alpha)+tg''(t)\big)\\ &= (1-s)^{2-\alpha}(k'(s)(1-\alpha)+sk''(s)) = c \quad \text{(say)}.\end{aligned}$$

Introducing the functions

$$(3.9)\qquad G(t) = t^{1-\alpha}g'(t), \qquad K(t) = t^{1-\alpha}k'(t), \qquad t \in (0,1),$$

we get from equation (3.8)

$$(3.10)\qquad t^\alpha(1-t)^{2-\alpha}G'(t) = s^\alpha(1-s)^{2-\alpha}K'(s) = c$$

for all $s, t \in (0,1)$. Integrating G' in (3.10) we get

$$G(t) = \int \frac{c\,dt}{t^\alpha(1-t)^{1-\alpha}} = -c\int \Big(\frac{t}{1-t}\Big)^\alpha \Big(\frac{t}{1-t}\Big)' dt$$

which implies

$$g'(t) = t^{\alpha-1}G(t) = \begin{cases} -\dfrac{c(1-t)^{\alpha-1}}{\alpha-1} + bt^{\alpha-1} & \alpha \neq 1 \\ -c\ln\dfrac{1-t}{t} + b & \alpha = 1 \end{cases}$$

for some constants $b, c \in \mathbb{R}$. Integration of g' yields $g(x) = L^\alpha_{a_1,a_2,a_3}(x)$ with

$$(3.11)\quad L^\alpha_{a_1,a_2,a_3}(x) = \begin{cases} a_1\log x + a_2\log(1-x) + a_3 & \alpha = 0 \\ a_1(x\log x + (1-x)\log(1-x)) + a_2 x + a_3 & \alpha = 1 \\ a_1 x^\alpha + a_2(1-x)^\alpha + a_3 & \alpha \notin \{0,1\}, \end{cases}$$

where $a_1, a_2, a_3 \in \mathbb{R}$. In the same manner, (3.9) leads to $k(x) = L^\alpha_{b_1,b_2,b_3}(x)$ for some constants $b_1, b_2, b_3 \in \mathbb{R}$. Since g and k satisfy the same differential equation (see (3.8)), substitution of g and k into (3.8) leads to the additional information

$$(3.12)\qquad a_1 = b_1 \quad \text{if } \alpha = 1 \in \mathcal{U}_1, \qquad a_2 = b_2 \quad \text{if } \alpha \neq 1.$$

Thus g and k have exactly the form given in (3.2) to (3.6) (Note that the case (3.5) can only occur if $n \geq 2$). If we now substitute the forms of g and k into (3.1), then we can separate the variables x and y, and obtain the forms of f and h (for example, in the case $\alpha \notin \{0,1\}$ we get

$$f(x) - b_1x^\alpha + a_3(1-x)^\alpha = h(y) - a_1y^\alpha + b_3(1-y)^\alpha = a_4 \quad \text{(say)}.$$

Thus we get exactly (3.3) with $\varphi_1 = \varphi_2 = 0$; note that — for instance — in the case $\alpha = 1$ the function f can be rewritten into $f(x) = A(x \log x + (1-x)\log(1-x)) + (b_1 + a_3)x + a_4 - a_3$.)

(c) To prove Theorem 3.1 in the n-dimensional case we use induction. We will write $x = (x_1, \xi)$, $y = (y_1, \eta)$, $\alpha = (\alpha_1, \beta) \in \mathbb{R}^n$ where $\xi = (x_2, \dots, x_n)$, $\eta = (y_2, \dots, y_n)$, $\beta = (\alpha_2, \dots, \alpha_n) \in \mathbb{R}^{n-1}$ and we assume that Theorem 3.1 is valid for all dimensions k, $1 \le k \le n-1$. Now (3.1) goes over into

$$\begin{aligned} &f(x_1, \xi) + (1-x_1)^{\alpha_1}(1-\xi)^{\beta} g\Big(\frac{y_1}{1-x_1}, \frac{\eta}{1-\xi}\Big) \\ &= h(y_1, \eta) + (1-y_1)^{\alpha_1}(1-\eta)^{\beta} k\Big(\frac{x_1}{1-y_1}, \frac{\xi}{1-\eta}\Big) \end{aligned} \tag{3.13}$$

for all $(x_1, y_1) \in D_1^\circ$ and $(\xi, \eta) \in D_{n-1}^\circ$. Fixing (ξ, η) and defining

$$F(r) := f(r, \xi), \qquad H(r) := h(r, \eta) \tag{3.14}$$

$$G(r) := (1-\xi)^{\beta} g\Big(r, \frac{\eta}{1-\xi}\Big), \quad K(r) := (1-\eta)^{\beta} k\Big(r, \frac{\xi}{1-\eta}\Big) \tag{3.15}$$

for $r \in (0,1)$ we get from (3.13)

$$F(x_1) + (1-x_1)^{\alpha_1} G\Big(\frac{y_1}{1-x_1}\Big) = H(y_1) + (1-y_1)^{\alpha_1} K\Big(\frac{x_1}{1-y_1}\Big) \tag{3.16}$$

for $(x_1, y_1) \in D_1^\circ$. Now we consider three cases:

(d) $\alpha_1 = 1$ in (3.16) with the subcases

- (d1) $\beta = 0$,
- (d2) $\beta \ne 0$, $\beta \in \mathcal{U}_{n-1}$,
- (d3) $\beta \ne 0$, $\beta \notin \mathcal{U}_{n-1}$,

(e) $\alpha_1 = 0$ in (3.16),

(f) $\alpha_1 \notin \{0, 1\}$ in (3.16).

(d) Let $\alpha_1 = 1$ in (3.16). Then by part (b) we get letting ξ, η vary again and putting $S(r) = r \log r + (1-r)\log(1-r)$, $r \in (0,1)$:

$$f(x_1, \xi) = a_1(\xi) S(x_1) + a_2(\xi) x_1 + a_3(\xi), \tag{3.17}$$

$$h(x_1, \xi) = a_1(\xi) S(x_1) + b_2(\xi) x_1 + b_3(\xi), \tag{3.18}$$

$x_1 \in (0,1)$, $\xi \in (0,1)^{n-1}$ (see (3.3) and (3.4)). Fixing temporarily $\xi = \eta \in (0, 1/2)^{n-1}$ in (3.15), equation (3.16) is again valid so that again from the

1-dimensional solution of (3.16) we obtain

$$(3.19)\qquad (1-\xi)^\beta g\Big(x_1, \frac{\xi}{1-\xi}\Big) = a_1(\xi)S(x_1) + c_2(\xi)x_1 + c_3(\xi),$$

$$(3.20)\qquad (1-\xi)^\beta k\Big(x_1, \frac{\xi}{1-\xi}\Big) = a_1(\xi)S(x_1) + d_2(\xi)x_1 + d_3(\xi),$$

for $x_1 \in (0,1)$, $\xi \in (0,1/2)^{n-1}$. Using the bijective transformation ψ : $(0,1/2)^{n-1} \to (0,1)^{n-1}$ given by ($\xi \in (0,1/2)^{n-1}$, $\varrho \in (0,1)^{n-1}$)

$$(3.21)\qquad \psi(\xi) = \frac{\xi}{1-\xi} = \varrho, \qquad \psi^{-1}(\varrho) = \frac{\varrho}{1+\varrho} = \xi,$$

(3.19) and (3.20) go over into

$$(3.22)\qquad g(x_1,\varrho) = A_1(\varrho)S(x_1) + A_2(\varrho)x_1 + A_3(\varrho)$$

$$(3.23)\qquad k(x_1,\varrho) = A_1(\varrho)S(x_1) + B_2(\varrho)x_1 + B_3(\varrho)$$

for all $x_1 \in (0,1)$, $\varrho \in (0,1)^{n-1}$ where

$$(3.24)\quad A_1(\varrho) = (1+\varrho)^\beta a_1\Big(\frac{\varrho}{1+\varrho}\Big), \quad A_i(\varrho) = (1+\varrho)^\beta c_i\Big(\frac{\varrho}{1+\varrho}\Big),$$
$$B_1(\varrho) = (1+\varrho)^\beta d_1\Big(\frac{\varrho}{1+\varrho}\Big)$$

($i = 2,3$). In order to determine f, g, h, k we determine the functions dependent on ξ and $\varrho \in (0,1)^{n-1}$ defined in (3.17), (3.18), (3.22) and (3.23). Note that a_i, b_j, A_i, B_j ($i = 1,2,3$, $j = 1,2$) are measurable or have Baire property because Lemma 2.1(a) can be applied since $\{S(x_1), x_1, 1\}$ are linearly independent. We substitute the forms of f, g, h, k into (3.1) and by comparison of the coefficients of $x_1 \log x_1$, $(1-x_1)\log(1-x_1)$, $(1-x_1-y_1)\log(1-x_1-y_1)$, x_1, y_1, 1 we obtain (in this order) for

$(\xi,\eta)\in D_{n-1}^{\circ}$

$$a_1(\xi)=(1-\eta)^{\beta}A_1\Big(\frac{\xi}{1-\eta}\Big)\quad\text{or}\quad a_1(p\cdot q)=q^{\beta}A_1(p),\tag{3.25}$$

$$a_1(\xi)=(1-\xi)^{\beta}A_1\Big(\frac{\eta}{1-\xi}\Big)\quad\text{or}\quad q^{-\beta}a_1(1-q)=A_1(p),\tag{3.26}$$

$$(1-\xi)^{\beta}A_1\Big(\frac{\eta}{1-\xi}\Big)=(1-\eta)^{\beta}A_1\Big(\frac{\xi}{1-\eta}\Big)\quad\text{or}\quad \frac{A_1(t)}{(1-t)^{\beta}}=\frac{A_1(s)}{(1-s)^{\beta}},\tag{3.27}$$

$$a_2(\xi)-(1-\xi)^{\beta}A_3\Big(\frac{\eta}{1-\xi}\Big)=(1-\eta)^{\beta}B_2\Big(\frac{\xi}{1-\eta}\Big),\tag{3.28}$$

$$(1-\xi)^{\beta}A_2\Big(\frac{\eta}{1-\xi}\Big)=b_2(\eta)-(1-\eta)^{\beta}B_3\Big(\frac{\xi}{1-\eta}\Big),\tag{3.29}$$

$$a_3(\xi)+(1-\xi)^{\beta}A_3\Big(\frac{\eta}{1-\xi}\Big)=b_3(\eta)+(1-\eta)^{\beta}B_3\Big(\frac{\xi}{1-\eta}\Big).\tag{3.30}$$

In (3.27) we used the transformation φ (see (2.14)) whereas in (3.25) and (3.26) we used the bijective transformation $\gamma: D_{n-1}^{\circ}\to(0,1)^{2(n-1)}$ given by

$$\begin{gathered}\gamma(u,v)=\Big(\frac{u}{1-v},1-v\Big)=(p,q),\\ \gamma^{-1}(p,q)=(pq,1-q)=(u,v),\\ u,v\in D_{n-1}^{\circ},\quad p,q\in(0,1)^{n-1}.\end{gathered}\tag{3.31}$$

From (3.25) to (3.27) we get for some real constants a,b,c:

$$\begin{gathered}a_1(\xi)=c\xi^{\beta}=A_1(\xi),\quad a_1(\xi)=a(1-\xi)^{\beta},\\ A_1(\xi)=a,\quad A_1(\xi)=b(1-\xi)^{\beta};\end{gathered}\tag{3.32}$$

which implies

$$c=a=b\neq 0\quad\text{if}\quad\beta=0\tag{3.33}$$

and

$$c=a=b=0\quad\text{if}\quad\beta\neq 0.\tag{3.34}$$

Thus in both cases (3.33) and (3.34) the functions a_1 and A_1 are determined and $a_1=A_1$. We now determine a_i,b_i,A_i,B_i $(i=2,3)$ in (3.17), (3.18), (3.22) and (3.23), using the equations (3.28)–(3.30), which are $(n-1)$-dimensional functional equations of type (3.1) for which we can use the induction hypothesis.

Case (d1). In (3.28)–(3.30) we first consider the case $\beta = 0$ (and $\alpha_1 = 1$). So that $(\alpha_1, \beta) = (1, 0, \dots, 0) \in \mathcal{U}_n$. Thus we get by induction hypothesis from (3.28) (which is a special case of (3.1) with $h = 0$, that is, $A_1 = 0$, $A_2 = -B_1$, $a_3 = -a_4$)

$$a_2(\xi) B_1' \odot (\log \xi - \log(1-\xi)) + b_3' + a_4' \tag{3.35}$$

$$-A_3(\xi) = -B_1' \odot \log(1-\xi) - a_4' \tag{3.36}$$

$$B_2(\xi) = B_1' \odot (\log \xi - \log(1-\xi)) + b_3' \tag{3.37}$$

for some constants $b_3', a_4' \in \mathbb{R}$ and $B_1' \in \mathbb{R}^{n-1}$. In the same manner we obtain from (3.29) (which is a special case of (3.1) with $f = 0$, that is, $B_1 = 0$, $A_2 = -A_1$, $b_3 = -a_4$)

$$A_2(\xi) = A_1' \odot (\log \xi - \log(1-\xi)) + a_3' \tag{3.38}$$

$$b_2(\xi) = A_1' \odot (\log \xi - \log(1-\xi)) + a_3' + a_4'' \tag{3.39}$$

$$-B_3(\xi) = -A_1' \odot \log(1-\xi) - a_4'' \tag{3.40}$$

where $a_3', a_4'' \in \mathbb{R}$ and $A_1' \in \mathbb{R}^{n-1}$. Finally we consider (3.30) which is of the form (3.1) with solution (3.2) (by induction hypothesis), but A_3 and B_3 have the special forms (3.36) and (3.40), respectively. Taking this into account we get by comparison with (3.2)

$$A_1' = B_1' \tag{3.41}$$

and

$$a_3(\xi) = A_1' \odot \log(1-\xi) + a_4'' + a_5' \tag{3.42}$$

$$b_3(\xi) = A_1' \odot \log(1-\xi) + a_4' + a_5' \tag{3.43}$$

for some constant $a_5' \in \mathbb{R}$. Substituting (3.35)–(3.43) together with (3.32) and (3.33) into (3.17), (3.18), (3.22) and (3.23) we get solutions of the form (3.3) and (3.4) (where $S_{\alpha,A}(x) = S_{(1,0,\dots,0),(a,A_1')}(x_1, \xi)$ and, with the notations of (3.3) and (3.4), $b_1 = b_3'$, $a_3 = a_4'$, $a_4 = a_4' + a_4'' + a_5'$, $a_1 = a_3'$, $a_2 = 0$, $b_2 = a_4''$).

Case (d2). Now we handle the case (3.34), that is, $\alpha = (1, \beta)$ where $\beta \neq 0$ and $\beta \in \mathcal{U}_{n-1}$. Without loss of generality we may assume that $\xi^\beta = x_2$. Like in case (d1) we get by induction hypothesis from (3.28) (which is a special case of (3.3) and (3.4) with $h = 0$, that is, $A = 0$, $a_1 + b_2 = 0$, $a_4 - b_2 = 0$)

$$a_2(\xi) = (b_1' + a_3')x_2 + a_4' - a_3' = (c - c_3)x_2 + c_1 + c_3 \tag{3.45}$$

$$-A_3(\xi) = (-a_4' - a_2')x_2 + a_2' + a_3' = -c_1x_2 - c_3 \tag{3.46}$$

$$B_2(\xi) = (b_1' - a_2')x_2 - a_2' + a_4' = cx_2 + c_1 \tag{3.47}$$

for some constants $a_2', a_3', a_4', b_1' \in \mathbb{R}$. The second equalities in (3.45)–(3.47) arise from the first equalities in (3.45)–(3.47) by putting $c_1 = a_2' + a_4'$, $c_3 = -a_2' - a_3'$, $c = b_1' - a_2'$. In the same manner we obtain from (3.29) (which is a special case of (3.3) and (3.4) with $f = 0$, that is, $A = 0$, $b_1 + a_3 = 0$, $a_4 - a_3 = 0$)

$$A_2(\xi) = (a_1'' - a_2'')x_2 + a_2'' + a_4'' = dx_2 + d_1 \tag{3.48}$$

$$b_2(\xi) = (a_1'' + b_2'')x_2 + a_4'' - b_2'' = (d - d_3)x_2 + d_1 + d_3 \tag{3.49}$$

$$-B_3(\xi) = (-a_4'' - a_2'')x_2 + a_2'' + b_2'' = -d_1x_2 - d_3 \tag{3.50}$$

for some constant $a_1'', a_2'', a_4'', b_2'' \in \mathbb{R}$ and the constants defined by $d_1 = a_2'' + a_4''$, $d_3 = -a_2'' - b_2''$, $d = a_1'' - a_2''$. In (3.45)–(3.50) we may suppose that

$$d_1 = -c_1 \tag{3.51}$$

(Replacing in (3.45)–(3.50) c_1 by $c_1' := c_1 - d_1$ and d_1 by $d_1' := d_1 - c_1$ an immediate calculation shows that we get again solutions of (3.28) and (3.29). But this time we have $c_1' = -d_1'$.) Substituting the forms of A_3 and B_3 (with $d_1 = -c_1$) into (3.30) and separating the variables we arrive at

$$a_3(\xi) - (c_3 - c_1)x_2 + c_3 = b_3(\eta) - (d_3 + c_1)y_2 + d_3 = e \tag{3.52}$$

for some constant e. Putting (3.45)–(3.52) (together with $a_1(\xi) = A_1(\xi) = 0$) into (3.17), (3.18), (3.22), (3.23) we obtain the solution (3.3) and (3.5):

$$\begin{aligned}
f(x_1, \xi) &= cx_1x_2 - c_3(1 - x_1)(1 - x_2) + e + c_1(x_1 - x_2),\\
g(x_1, \xi) &= dx_1x_2 + c_3 - c_1(x_1 - x_2),\\
h(x_1, \xi) &= dx_1x_2 - d_3(1 - x_1)(1 - x_2) + e - c_1(x_1 - x_2),\\
k(x_1, \xi) &= cx_1x_2 + d_3 - d_1(x_1 - x_2).
\end{aligned}$$

Case (d3). Now we treat the case $\alpha = (1, \beta)$, $\beta \neq 0$, $\beta \notin \mathcal{U}_{n-1}$. Like in case (d1) and (d2) by induction hypothesis the solutions of (3.28) and

(3.29) yield (see (3.3) and (3.6) with $h = 0$ (that is, $a_1 = b_2 = a_4 = 0$) and $f = 0$ (that is, $b_1 = a_3 = a_4 = 0$), respectively):

$$a_2(\xi) = b_1'\xi^\beta - a_3'(1-\xi)^\beta, \qquad A_2(\xi) = a_1''\xi^\beta + a_2''(1-\xi)^\beta \tag{3.53}$$

$$-A_3(\xi) = a_2'(1-\xi)^\beta + a_3', \qquad b_2(\xi) = a_1''\xi^\beta - b_2''(1-\xi)^\beta \tag{3.54}$$

$$B_2(\xi) = b_1'\xi^\beta + a_2'(1-\xi)^\beta, \quad -B_3(\xi) = a_2''\xi^\beta + b_2'' \tag{3.55}$$

Here $a_2', a_3', a_1', a_2'', b_1', b_2''$ are constants and again without loss of generality we may assume

$$a_2' = a_2'' \tag{3.56}$$

(otherwise replace a_2' and a_2'' in (3.53)–(3.55) by $a_2' + a_2''$). Using (3.30) and the forms of A_3 and B_3 (with $a_2' = a_2''$) and separating the variables we arrive at

$$a_3(\xi) = a_3'(1-\xi)^\beta + k \tag{3.57}$$

$$b_3(\xi) = b_2''(1-\xi)^\beta + k \tag{3.58}$$

for some constant $k \in \mathbb{R}$. Substituting (3.53)–(3.57) (and $a_1(\xi) = A_1(\xi) = 0$) into (3.17), (3.18), (3.22), (3.23) we arrive at the solutions (3.3) and (3.6) (where $a_1 = a_1'$, $a_2 = -a_2'$, $a_3 = -a_3'$, $a_4 = k$, $b_1 = b_1'$, $b_2 = -b_2''$, $\varphi_1 = \varphi_2 = 0$).

(e) Now let $\alpha_1 = 0$ in (3.16). Like in (d) we get from (3.14)–(3.16) using the 1-dimensional solutions of (3.16) for $d_1 = 1$ the following representations:

$$\begin{aligned} f(x_1,\xi) = {} & b_1(\xi)\log x_1 + (a_1+a_2)(\xi)\log(1-x_1) \\ & + (b_3+a_4)(\xi) \end{aligned} \tag{3.59}$$

$$g(x_1,\xi) = A_1(\xi)\log x_1 + A_2(\xi)\log(1-x_1) + A_3(\xi) \tag{3.60}$$

$$\begin{aligned} h(x_1,\xi) = {} & a_1(\xi)\log x_1 + (a_2+b_1)(\xi)\log(1-x_1) \\ & + (a_3+a_4)(\xi) \end{aligned} \tag{3.61}$$

$$k(x_1,\xi) = B_1(\xi)\log x_1 + A_2(\xi)\log(1-x_1) + B_3(\xi), \tag{3.62}$$

$x_1 \in (0,1)$, $\xi \in (0,1)^{n-1}$. Substitution of these expressions into (3.13) and comparison of the coefficients of $\log(1-x_1)$, $\log x_1$, $\log(1-x_1-y_1)$, $\log y_1$,

$\log(1-y_1)$, 1 yields (in this order) for $(\xi,\eta)\in D^\circ_{n-1}$

$$(3.63)\qquad (a_1+a_2)(\xi)=(1-\xi)^\beta(A_1+A_2)\Big(\frac{\eta}{1-\xi}\Big)$$
$$\text{or}\quad q^{-\beta}(a_1+a_2)(1-q)=(A_1+A_2)(q)=a'$$

$$(3.64)\qquad b_1(\xi)=(1-\eta)^\beta B_1\Big(\frac{\xi}{1-\eta}\Big)\quad\text{or}\quad b_1(pq)=p^\beta B_1(q)$$

$$(3.65)\qquad (1-\xi)^\beta A_2\Big(\frac{\eta}{1-\xi}\Big)=(1-\eta)^\beta A_2\Big(\frac{\xi}{1-\eta}\Big)$$
$$\text{or}\quad \frac{A_2(t)}{(1-t)^\beta}=\frac{A_2(s)}{(1-s)^\beta}=a_2'$$

$$(3.66)\qquad (a_2+b_1)(\eta)=(1-\eta)^\beta(A_2+B_1)\Big(\frac{\xi}{1-\eta}\Big)$$
$$\text{or}\quad q^{-\beta}(a_2+b_1)(1-q)=(A_1+A_2)(q)=b'$$

$$(3.67)\qquad a_1(\eta)=(1-\xi)^\beta A_1\Big(\frac{\eta}{1-\xi}\Big)\quad\text{or}\quad a_1(pq)=p^\beta A_1(q)$$

$$(3.68)\qquad (b_3+a_4)(\xi)=(1-\xi)^\beta A_3\Big(\frac{\eta}{1-\xi}\Big)$$
$$=(a_3+a_4)(\eta)+(1-\eta)^\beta B_3\Big(\frac{\xi}{1-\eta}\Big)$$

Here again the transformations φ and γ (see (2.14) and (3.31)) were used ($p,q,s,t\in(0,1)^{n-1}$), and a',b',a_2' are real constants. From (3.63)–(3.67) we obtain (for some constants a_1',b_1')

$$b_1(\xi)=B_1(\xi)=b_1'\xi^\beta,\qquad a_1(\xi)=A_1(\xi)=a_1'\xi^\beta$$

and

$$a_1'\xi^\beta=A_1(\xi)=((A_1+A_2)-A_2)(\xi)=a'-a_2'(1-\xi)^\beta,$$
$$b_1'\xi^\beta=B_1(\xi)=((A_2+B_1)-A_2)(\xi)=b'-a_2'(1-\xi)^\beta.$$

This implies

$$(3.69)\qquad a'=a_1'+a_2'\quad\text{and } b'=b_1'+a_2'\qquad\text{if}\quad \beta=0,$$
$$(3.70)\qquad a'=a_1'=a_2'=b'=b_1'\qquad\text{if}\quad \beta\in\mathcal{U}_{n-1},$$
$$(3.71)\qquad a'=a_1'=a_2'=b'=b_1'=0\qquad\text{if}\quad \beta\neq0,\ \beta\notin\mathcal{U}_{n-1},$$

If $\beta = 0$ then (3.69) implies (see (3.59)–(3.62))

$$\begin{aligned} f(x_1,\xi) &= b_1' \log x_1 + (a_1' + a_2') \log(1 - x_1) + (b_3 + a_4)(\xi) \\ &=: f'(x_1) + (b_3 + a_4)(\xi) \\ g(x_1,\xi) &= a_1' \log x_1 + a_2' \log(1 - x_1) + A_3(\xi) =: g'(x_1) + A_3(\xi) \\ h(x_1,\xi) &= a_1' \log x_1 + (a_2' + b_1') \log(1 - x_1) + (a_3 + a_4)(\xi) \\ &=: h'(x_1) + (a_3 + a_4)(\xi) \\ k(x_1,\xi) &= b_1' \log x_1 + a_2' \log(1 - x_1) + B_3(\xi) =: k'(x_1) + B_3(\xi). \end{aligned}$$

The still undetermined functions in these expressions satisfy equation (3.68) so that by induction hypothesis it is immediate to see that f, g, h, k have the form (3.2) using the bilinearity of the inner product. (Note that in this case f', g', h', k' and $b_3 + a_4$, A_3, $a_3 + a_4$, B_3 satisfy equation (3.1) for $\alpha_1 = 0$, $(x, y) \in D_1^\circ$ and $\beta = 0$, $(\xi, \eta) \in D_{n-1}^\circ$, respectively.)

If $\beta \in \mathcal{U}_{n-1}$ we may suppose without loss of generality that $\xi^\beta = x_2$, that is $\alpha = (0, 1, 0, \dots, 0) \in \mathcal{U}_n$. But then we get using (3.63)–(3.67) and (3.70)

$$\begin{aligned} f(x_1,\xi) &= a' x_2 \log x_1 + a'(1 - x_2) \log(1 - x_1) + (b_3 + a_4)(\xi) \\ g(x_1,\xi) &= a' x_2 \log x_1 + a'(1 - x_2) \log(1 - x_1) + A_3(\xi) \\ h(x_1,\xi) &= a' x_2 \log x_1 + a'(1 - x_2) \log(1 - x_1) + (a_3 + a_4)(\xi) \\ k(x_1,\xi) &= a' x_2 \log x_1 + a'(1 - x_2) \log(1 - x_1) + B_3(\xi). \end{aligned}$$

Again using induction hypothesis for the solution of equation (3.68) an easy calculation shows that we arrive now at solutions of the form (3.3) where $x^\alpha = \xi^\beta = x_2$.

In case $\beta \neq 0$, $\beta \notin \mathcal{U}_{n-1}$ we see from (3.68), (3.71) and (3.59)–(3.62) that f, g, h, k satisfy (3.68) which by induction hypothesis yields solution (3.3) and (3.5) (or (3.6)). (Note that $x^\alpha = \xi^\beta$.)

(f) Finally we let $\alpha_1 \notin \{0, 1\}$ in (3.16). Like in part (d) we arrive at the following form for the solutions of equation (3.1):

$$f(x_1,\xi) = d_1'(\xi)^{\alpha_1} - a_3'(\xi)(1 - x_1)^{\alpha_1} + a_4'(\xi) \tag{3.72}$$

$$g(x_1,\xi) = c_1'(\xi)^{\alpha_1} + a_2'(\xi)(1 - x_1)^{\alpha_1} + c_3'(\xi) \tag{3.73}$$

$$h(x_1,\xi) = a_1'(\xi)^{\alpha_1} - b_2'(\xi)(1 - x_1)^{\alpha_1} + a_4'(\xi) \tag{3.74}$$

$$k(x_1,\xi) = b_1'(\xi)^{\alpha_1} + a_2'(\xi)(1 - x_1)^{\alpha_1} + c_2'(\xi) \tag{3.75}$$

($x_1 \in (0, 1)$, $\xi \in (0, 1)^{n-1}$). Again substituting these expressions into (3.13) we get by comparison of the coefficients of $x_1^{\alpha_1}$, $(1 - x_1)^{\alpha_1}$, 1,

$(1-y_1)^{\alpha_1}$, $y_1^{\alpha_1}$, $(1-x_1-y_1)^{\alpha_1}$ (in this order) for $(\xi,\eta)\in D^\circ_{n-1}$ and $p,q,s,t\in(0,1)^{n-1}$:

$$d_1'(\xi)=(1-\eta)^\beta b_1'\Big(\frac{\xi}{1-\eta}\Big)\quad\text{or}\quad d_1'(pq)=q^\beta b_1'(p) \tag{3.76}$$

$$a_3'(\xi)=(1-\xi)^\beta a_3'\Big(\frac{\eta}{1-\xi}\Big)\text{ or }(1-\xi)^{-\beta}a_3'(\xi)=c_3'(p)=a_3'' \tag{3.77}$$

$$a_4'(\xi)=a_4'(\eta)=a_4''\quad\text{(say)} \tag{3.78}$$

$$b_2'(\eta)=(1-\eta)^\beta c_2'\Big(\frac{\xi}{1-\eta}\Big)\text{ or }(1-\eta)^{-\beta}b_2'(\eta)=c_2'(p)=b_2'' \tag{3.79}$$

$$a_1'(\eta)=(1-\xi)^\beta c_1'\Big(\frac{\eta}{1-\xi}\Big)\quad\text{or}\quad a_1'(pq)=q^\beta c_1'(p) \tag{3.80}$$

$$(1-\xi)^\beta a_2'\Big(\frac{\eta}{1-\xi}\Big)=(1-\eta)^\beta a_2'\Big(\frac{\xi}{1-\eta}\Big)\quad\text{or}\quad \frac{a_2'(t)}{(1-t)^\beta}=\frac{a_2'(s)}{(1-s)^\beta}=a_2'' \tag{3.81}$$

From (3.76)–(3.81) we get (for some constants a_1'', b_1'')

$$d_1'(\xi)=b_1''\xi^\beta=b_1'(\xi),\quad a_1'(\xi)=a_1''\xi^\beta=c_1'(\xi),\quad a_4'(\xi)=a_4'',\quad c_3'(\xi)=a_3''$$

$$a_3'(\xi)=a_3''(1-\xi)^\beta,\quad c_2'(\xi)=b_2'',\quad b_2'(\xi)=b_2''(1-\xi)^\beta,\quad a_2'(\xi)=a_2''(1-\xi)^\beta.$$

Substituting these expressions into (3.72)–(3.75) we get exactly solutions (3.3) with (3.6) (where $b_1=b_1''$, $a_2=a_2''$, $a_3=a_3''$, $a_4=a_4''$, $b_2=b_2''$).

Remark. The idea of the proof of Theorem 3.1 in the n-dimensional case is very simple: Having the forms of f, g, h and k (see for example (3.17), (3.18), (3.22) and (3.23), where (3.14), (3.15) and (3.21) were used) which are dependent upon unknown functions, we substitute these expressions into (3.1) and by comparison of linearly independent terms we get systems of functional equations for the unknown functions for which we can use the induction hypothesis. The most complicated cases are (in our notation) $\alpha_1=1$, $\beta=0$ and $\alpha_1=1$, $\beta\in\mathcal{U}_{n-1}$, because in the first of these two cases we are lead to the case $\alpha\in\mathcal{U}_n$ and in the second case where $x^\alpha=x_1x_i$ for some $2\le i\le n$ we get not only the expected solution but also the additional solution (3.5). The other cases are rather obvious. For a completely different way to prove Theorem 3.1 in the case $\alpha_1=1$, $\beta=0$ and $f=g=h=k$ we refer the reader to [7] and [8].

References

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