# Latent additivity and a differential equation

By ERICH NEUWIRTH (Wien)

## 1. Statement of the problem

In psychology one sometimes studies two-parameter families of distribution functions $V(z/x, y)$ where $x$ and $y$ are the parameters. These distribution functions often depend on $x$ and $y$ only through a real-valued function $F$ such that $V(z/x, y)= =V_1(z/F(x, y))$. A very important class of models related to Rasch's theory of specific objectivity (see [2], [3], [4] e.g.) are models where $F$ is of the form $F(x, y)= =f(g(x)+h(y))$. Functions of this form will be called latent additive in the following. (In some references these functions also are called nomographic.) It will be shown that the differential equation

$$F_{xy}(F_{xx}F_y^2-F_{yy}F_x^2) = F_xF_y(F_{xxy}F_y-F_{xyy}F_x)$$

nearly characterizes the class of latent additive functions. (The full result is given in corrollary 1 at the end of the paper.)

## 2. Theorems about latent additive functions

Let $A$ be some subset of $\mathbf{R}^k$. $C^n(A)$ is the set of all functions defined on $A$ such that all partial derivatives up to order $n$ exist and are continuous.

**Theorem 1.** *Let $I_1$ and $I_2$ be (possibly infinite) intervals in $\mathbf{R}_1$ and let $g\in C^3(I_1)$ $h\in C^3(I_2)$ and $f\in C^3(g(I_1)+h(I_2))$. Then $F(x, y)=f(g(x)+h(y))$ fulfills the equation*

$$F_{xy}(F_{xx}F_y^2-F_{yy}F_x^2) = F_xF_y(F_{xxy}F_y-F_{xyy}F_x).$$

PROOF. Trivial.

This result is essentially known (see [1] e.g.) and it is also known that there exist functions fulfilling our differential equation without being latent additive. (An example can be found in [1].)

Therefore we will try to impose further conditions on $F$ such that fulfillment of our differential equation also yields latent additivity.

For technical reasons we first prove

**Lemma 1.** *Let $I_1$ and $I_2$ be intervals (possibly infinite) in $\mathbf{R}_1$ and let $R=I_1\times I_2$. For $n\geqq 1$ let $F\in C^n(R)$ fulfill the partial differential equation $F_x=F_y$. Then there exists $f\in C(I_1+I_2)$ with $F(x, y)=f(x+y)$.*

PROOF. Let $T(x, y)=(x+y, x-y)$. Then $T^{-1}=\frac{1}{2}T$ and the projection of $T^{-1}(R)$ onto the $x$-axis is

$$S_1 = \frac{1}{2}(I_1+I_2).$$

Let $S=T^{-1}(R)$ and $G(x, y)=F(T(x, y))$. ($G$ is defined on $S$). Since $T$ is a linear isomorphism between $S$ and $R$ $G\in C^n(S)$ and we have

$$G_y(x, y) = F_x(x, y)-F_y(x, y) = 0.$$

Therefore there exists $g\in C^n(S_1)$ with $G(x, y)=g(x)$ and therefore

$$F(x, y) = G\left(\frac{1}{2}(x+y, x-y)\right) = g\left(\frac{1}{2}(x+y)\right).$$

Defining $f(x)=g\left(\frac{1}{2}x\right)$ for $x$ $I_1+I_2$ we have

$$F(x, y) = f(x+y) \quad \text{and} \quad f\in C^n(I_1+I_2).$$

With the help of this lemma we can state sufficient conditions for latent additivity.

**Theorem 2.** *Let $I_1$ and $I_2$ be (possibly infinite) intervals in $\mathbf{R}_1$. Let $R=I_1\times I_2$ and $F\in C^3(R)$. Let furthermore $F_x>0$ and $F_y>0$ on $R$ and $F$ fulfill the third-order partial differential equation $F_{xy}(F_{xx}F_y^2-F_{yy}F_x^2)=F_xF_y(F_{xxy}F_y-F_{xyy}F_x)$. Then there exist $f\in C^3(I_1)$, $g\in C^3(I_2)$ and $h\in C^3(f(I_1)+h(I_2))$ with $f'>0$, $g'>0$ and $h'>0$ on their domains such that $F(x, y)=f(g(x)+h(y))$ on $R$.*

PROOF. Define $H(x, y)=(F_yF_{xx}-F_xF_{xy})/F_xF_y$. Then one easily sees $H_y=0$ and $H\in C^1(R)$. Therefore there exists $r\in C^1(I_1)$ with $H(x, y)=r(x)$. Let $s(x)$ be an indefinite integral of $r(x)$ and let $g(x)$ be an indefinite integral of $\exp(s(x))$. Then $g\in C^3(I_1)$ and $g'>0$ on $I_1$. Furthermore we have

$$\frac{g''}{g'} = (\ln g')' = (F_yF_{xx}-F_xF_{xy})/F_xF_y.$$

Therefore defining $T(x, y)=g'(x)\cdot F_y(x, y)/F_x(x, y)$ we have

$$T_x=g'(F_{xy}F_x-F_{xx}F_y)/F_x^2+g''F_yF_x$$

and we easily see $T_x=0$.

We also have $T\in C^2(R)$ and therefore there exists $t\in C^2(I_2)$ with $T(x, y)=t(y)$. Let $h(y)$ be an indefinite integral of $t(y)$. Since $T(x, y)=t(y)>0$ $h$ is strictly increasing. Since $g$ also is strictly increasing the inverse functions $g^{-1}$ and $h^{-1}$ are defined on the intervals $J_1=g(I_1)$ and $J_2=h(I_2)$. We have

$$g'(x)F_y(x, y)/F_x(x, y) = h'(y)$$

or equivalently

$$F_y(x, y)/h'(y) = F_x(x, y)/g'(x) \quad \text{for all} \quad (x, y) \in K.$$

Now we define

$$G(x, y) = F\big(g^{-1}(x), h^{-1}(y)\big).$$

$G$ is defined on $J_1 \times J_2$ and we have

$$G_x(x, y) = F_x\big(g^{-1}(x), h^{-1}(y)\big)/g'\big(g^{-1}(x)\big)$$
$$G_y(x, y) = F_y\big(g^{-1}(x), h^{-1}(y)\big)/h'\big(h^{-1}(y)\big).$$

Therefore we have $G_x = G_y$ on $J_1 \times J_2$. Furthermore we have $1/(g'og^{-1}) \in C^2(J_1)$ and $1/(h'oh^{-1}) \in C^2(J_2)$. Therefore $G \in C^3(J_1 \times J_2)$ and according to Lemma 1 there exists $f \in C^3(J_1 + J_2)$ with $G(x, y) = f(x+y)$. Since $F(x, y) = G\big(g(x), h(y)\big)$ we have $F(x+y) = f\big(g(x)+h(y)\big)$ and our theorem is proved.

Combining both theorems we get the following characterization result for latent additive functions:

**Corollary 1.** *Let $I_1$ and $I_2$ be (possibly infinite) intervals in $\mathbf{R}_1$, let $F \in C^3(I_1 \times I_2)$ and $F_x > 0$ and $F_y > 0$ on $I_1 \times I_2$. Then $F(x, y)$ is of the form $F(x, y) = f\big(g(x)+h(y)\big)$ with $g \in C^3(I_1)$, $h \in C^3(I_2)$ and $f \in C^3\big(g(I_1)+h(I_2)\big)$ if and only if $F_{xy}(F_{xx}F_y^2 - F_{yy}F_x^2) = F_xF_y(F_{xxy}F_y - F_{xyy}F_x)$.*

*with $g \in C^3(I_1)$, $h \in C^3(I_2)$ and $f \in C^3\big(g(I_1)+h(I_2)\big)$ if and only if $F_{xy}(F_{xx}F_y^2 - F_{yy}F_x^2) = F_xF_y(F_{xxy}F_y - F_{xyy}F_x)$.*

[1] R. C. Buck, Approximate Complexity and Functional Representation. *Journal of Mathematical Analysis and Applications* **70** (1979), 280—298.

[2] G. Fischer, Einführung in die Theorie psychologischer Tests. Verlag Hans Huber, Bern 1974, S. 407ff.

[3] G. Rasch, On general laws and the meaning of measurement in psychology. *Berkeley symposium on mathematical statistics and probability. Univ. of California Press* 1961.

[4] G. Rasch, An informal report on a theory of objectivity in comparisons. *Proc of the NUFFIC international summer session, The Hague, July 14—28*, 1966.

INSTITUT FÜR STATISTIK UND INFORMATIK
DER UNIVERSITÄT WIEN

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