On the error terms of $\sum_{n \leqq x} \varphi_{f,t}^{(k)}(n)$ and $\sum_{n \leqq x} \psi_{f,t}^{u,v}(n)$

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ABSTRACT

Let f(n) be a nonconstant polynomial with integer coefficients and k, t, u and v positive integers. The first author introduced earlier the generalized Euler totient function $\varphi_{f,t}^{(k)}(n)$ and the generalized Dedekind ψ-functions $\Psi_{f,t}^{(k)}(n)$, $\psi_{f,t}^{(k)}(n)$, and $\psi_{f,t}^{u,v}(n)$ and obtained asymptotic formulae for their summatory functions subject to the condition $N_f(n)=O(n^\varepsilon)$ for some ε, $0<\varepsilon<1$, $N_f(n)$ being the number of incongruent solutions mod n of $f(x)\equiv 0 \pmod n$. In this paper we obtain these asymptotic formulae without any such condition and with better estimates for the error functions.
Key words and phrases: Möbius function, Euler's totient function, Dedekind's ψ-function.

1. Introduction. Let $f=f(x)$ be a nonconstant polynomial with integer coefficients and let k and t be positive integers. In [1] the first author studied the generalized Euler totient function $\varphi_{f,t}^{(k)}(n)$ defined to be the number of incongruent t-tuples of integers $(a_1, a_2, \dots, a_t) \bmod n$ such that $((f(a_1), f(a_2), \dots, f(a_t)), n)_k=1$, it being understood that the t-tuples $(a_1, a_2, \dots, a_t)$ and $(b_1, b_2, \dots, b_t)$ are congruent $\bmod n$ iff $a_i\equiv b_i \pmod n$ for $1\leqq i\leqq t$ and the symbol $(a, b, c, \dots, e)_k$ stands for the largest kth power common divisor of $a, b, c, \dots, e$ and $(a, b, c, \dots, e)_1=(a, b, c, \dots, e)$ with the convention $(0, 0, \dots, 0)_k=0$.

This function $\varphi_{f,t}^{(k)}(n)$ reduces when $f(x)=x$, $t=1$ and $k=1$ to Euler's totient function $\varphi(n)$ and for special choices of t and k, the various extensions of $\varphi(n)$ studied by E. COHEN, E. K. HAVILAND, JORDAN, V. L. KLEE, P. J. MCCARTHY, and SCHEMMEL. $\varphi_{f,t}(n)$ has been studied by P. K. MENON. (For details we refer to [1])

Among other things, in [1] it has been proved that as $x\to\infty$

$$\sum_{n\leqq x} \varphi_{f,t}^{(k)}(n) = \alpha x^{t+1} + O\left(x^{t\varepsilon+\frac{1}{k}}\right), \tag{1.1}$$

where

$$\alpha = \frac{1}{t+1} \sum_{n=1}^{\infty} \frac{\mu(n) N_f^t(n^k)}{n^{k(t+1)}} = \frac{1}{t+1} \Pi_p \left\{1 - \frac{N_f^t(p^k)}{p^{t(k+1)}}\right\} \tag{1.2}$$

under the condition

$$N_f(n) - O(n^\varepsilon) \quad \text{for some} \quad 0<\varepsilon<1; \tag{1.3}$$

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here $N_f(n)$ denotes the number of incongruent solutions mod n of $f(x)\equiv 0 \pmod n$), $N_f^t(n)=(N_f(n))^t$, and $\mu(n)$ the Möbius function.

In the case of $\varphi(n)$ we have the well known result due to F. MERTENS, namely

$$\sum_{n\leqq x}\varphi(n)=\frac{3}{\pi^2}x^2+O(x\log x); \tag{1.4}$$

thus in this special case the main terms of (1.1) and (1.4) agree but the error term in (1.1) is not as good as that in (1.4).

Similarly, in [2] and [3], the first author introduced and studied the generalized Dedekind ψ-functions $\Psi_{f,t}^{(k)}(n)$, $\psi_{f,t}^{(k)}(n)$ and $\psi_{f,t}^{u,v}(n)$. He proved that under the condition (1.3)

$$\sum_{n\leqq x}\Psi_{f,t}^{(k)}(n)=\frac{x^{t+1}}{t+1}\sum_{n=1}^{\infty}\frac{\mu^2(n)N_f^t(n^k)}{n^{k(t+1)}}+O\left(x^{\delta t+\frac{1}{k}}\right), \tag{1.5}$$

$$\sum_{n\leqq x}\psi_{f,t}^{(k)}(n)=\frac{x^{kt+1}}{kt+1}\sum_{n=1}^{\infty}\frac{\mu^2(n)N_f^t(n^k)}{n^{kt+1}}+O(x^{\delta kt+1}) \tag{1.6}$$

for every δ such that

$$1>\delta>\max\left(\varepsilon, 1-\frac{1}{kt}\right); \tag{1.7}$$

and with the condition that ε of (1.3) is $<\frac{1}{u}$,

$$\sum_{n\leqq x}\psi_{f,t}^{u,v}(n)=\frac{x^{vt+1}}{vt+1}\sum_{n=1}^{\infty}\frac{\varrho_{f,t}^{u,v}(n)}{n^{vt+1}}+E(x) \tag{1.8}$$

where

$$E(x)=\begin{cases}O(x^{vt}) & \text{if } \ vt(1-u\varepsilon)>1\\ O(x^{1+u\theta+uvt\varepsilon}) & \text{for every } \ \theta<\dfrac{vt(1-u\varepsilon)}{u}\end{cases} \tag{1.9}$$

if $vt(1-u\varepsilon)\leqq 1$.

Here $\varrho_{f,t}^{u,v}(n)$ is the multiplicative arithmetical function defined by

$$\varrho_{f,t}^{u,v}(n)=\prod_{p^\alpha\parallel n}\binom{u}{\alpha}N_f^{\alpha t}(p^v), \tag{1.10}$$

where $p^\alpha\parallel n$ means that $p^\alpha|n$ and $p^{\alpha+1}\nmid n$ and

$$\binom{u}{\alpha}=\frac{u(u-1)\dots(u-\alpha+1)}{1\cdot 2\dots\alpha},\quad \binom{u}{0}=1 \quad \text{for nonnegative integers}$$

u and α.

In the case when $f(x)=x$, the functions $\Psi_{f,1}^{(k)}(n)$, $\psi_{f,1}^{(k)}(n)$, and $\psi_{f,1}^{k,1}(n)$ reduce respectively to $\Psi_k(n)$, $\psi_k(n)$, and $\psi_{(k)}(n)$ which were studied earlier by D. SURYANARAYANA [6] as extensions of the Dedekind's ψ-function $\psi(n)$ which has the arithmetical form

$$\psi(n)=n\prod_{p|n}\left(1+\frac{1}{p}\right),$$

the product being extended over all prime divisors of n. In fact, $\Psi_1(n)=\psi_1(n)=\psi_{(1)}(n)=\psi(n)$. The estimates for error terms in his asymptotic formulae for $\sum_{n\leq x}\Psi_k(n)$, $\sum_{n\leq x}\psi_k(n)$, and $\sum_{n\leq x}\psi_{(k)}(n)$ turned out to be a little better than those of (1.5), (1.6) and (1.8) in these special cases.

Also, it is known now that (1.3) always holds, ε of course depending upon the polynomial f. In fact, the second author and P. V. Krishnaiah [5] recently proved that

$$N_f(n) \leqq A^{\omega(n)} n^{1-\frac{1}{h}},$$

h being the degrre of f and A an absolute positive constant and $\omega(n)$ the number of distinct prime factors of n. Since $\omega(n)=O\left(\frac{\log n}{\log\log n}\right)$, (1.3) follows and thus the asymptotic formulae given in (1.1), (1.5), (1.6), and (1.8) hold for every nonconstant polynomial f.

In this paper, using entirely different arguments, we give proofs of the above formulae with better estimates for the error terms.

2. Preliminaries. As before, f is a nonconstant polynomial with integer coefficients and h the degree of f. Let D be the g.c.d. of the coefficients of f and $\gamma(n)$ the largest square free divisor of n. It is well known that $N_f(n)$ is a multiplicative function of n, i.e., $N_f(mn)=N_f(m)N_f(n)$ if $(m,n)=1$. Let

$$c = \max\{h, u\}, \tag{2.1}$$

u being the largest prime divisor of D.

We shall also need the following:

$$\varphi_{f,t}^{(k)}(n) = \sum_{d^k\delta=n} \mu(d) N_f^t(d^k)\delta^t, \tag{2.2}$$

$$\Psi_{f,t}^{(k)}(n) = \sum_{d^k\delta=n} \mu^2(d) N_f^t(d^k)\delta^t, \tag{2.3}$$

$$\psi_{f,t}^{(k)}(n) = \sum_{d\delta=n} \mu^2(d) N_f^t(d^k)\delta^{kt}, \tag{2.4}$$

$$\psi_{f,t}^{u,v}(n) = \sum_{d\delta=n} \varrho_{f,t}^{u,v}(d)\delta^{vt}, \tag{2.5}$$

$$\varrho_{f,t}^{1,v}(n) = \mu^2(n) N_f^t(n^v) \tag{2.6}$$

and

$$\varrho_{f,t}^{u,v}(n) = \sum_{d\delta=n} \varrho_{f,t}^{1,v}(d)\varrho_{f,t}^{u-1,v}(\delta). \tag{2.7}$$

Of these (2.2) is Corollary 1 of Theorem 2 in [1], (2.3) is an easy consequence of 2.15 of [2], (2.4) is (3) of 2.18 of [2], (2.6) is (2.9) of [3] and (2.5) and (2.7) are respectively Theorems 2.2 and Lemma 2.1 of [3].

Lemma 2.1. *For all* n,

$$N_f(n) \leqq C^{\omega(n)} \frac{n}{\gamma(n)}.$$

PROOF. By (2.1) we have $N_f(p) \leqq c$ for all primes p. Also it is known that
$$N_f(p^\alpha) \leqq p^{\alpha-1} N_f(p) \quad \text{for all primes} \quad p \quad \text{and}$$
all positive integers α. The multiplicativity of $N_f(n)$ gives the lemma.

Lemma 2.2. For positive integral k,
$$\sum_{n \leqq x} k^{\omega(n)} = O\big(x(\log 2x)^{k-1}\big).$$

PROOF. We use induction on k. First, we note
$$(k+1)^{\omega(n)} \leqq \sum_{d|n} k^{\omega(d)}. \tag{2.8}$$

Clearly the functions appearing on both sides of (2.8) are positive and it is easily seen that they are multiplicative and (2.8) holds when $n=p^\alpha$, p a prime and α a positive integer; thus (2.8) holds for all n.

The lemma is clearly true for $k=1$. Assuming its truth for the positive integer k, we have, by (2.8),
$$\sum_{n \leqq x} (k+1)^{\omega(n)} \leqq \sum_{n \leqq x} \sum_{d\delta = n} k^{\omega(d)} = \sum_{\delta \leqq x} \sum_{d \leqq x/\delta} k^{\omega(d)} =$$
$$= O\left(\sum_{\delta \leqq x} \left(\frac{x}{\delta}\right)\left(\log \frac{2x}{\delta}\right)^{k-1}\right) = O\left(x(\log 2x)^{k-1} \sum_{\delta \leqq x} \frac{1}{\delta}\right) = O\big(x(\log 2x)^k\big),$$
giving the truth of the lemma for $k+1$ and the proof is complete.

Lemma 2.3. *As* $x \to \infty$
$$\sum_{n \leqq x} \mu^2(n) N_f^t(n^k) = O\big(x^{1+(k-1)t}(\log 2x)^{c^t-1}\big), \tag{2.9}$$
$$\sum_{n \leqq x} \frac{\mu^2(n) N_f^t(n^k)}{n^{kt}} = \begin{cases} O\big((\log 2x)^c\big) & \text{if} \quad t = 1 \\ 0(1), & \text{if} \quad t \geqq 2 \end{cases} \tag{2.10}$$
and
$$\sum_{n > x} \frac{\mu^2(n) N_f^t(n^k)}{n^{k(t+1)}} = O\left(\frac{(\log 2x)^{c^t-1}}{x^{k+t-1}}\right). \tag{2.11}$$

PROOF. By Lemmas 2.1 and 2.2, we have
$$\sum_{n \leqq x} \mu^2(n) N_f^t(n^k) \leqq \sum_{n \leqq x} \mu^2(n) n^{(k-1)t} C^{t\omega(n)} \leqq x^{(k-1)t} \sum_{n \leqq x} C^{t\omega(n)} = O\big(x^{1+(k-1)t}(\log 2x)^{c^t-1}\big)$$
and this proves (2.9). Now (2.10) and (2.11) follow from (2.9) by the partial summation theorem (cf. [4], Theorem 421).

Lemma 2.4. *As* $x \to \infty$
$$\sum_{n \leqq x} \varrho_{f,t}^{u,v}(n) = O\big(x^{1+(v-1)t}(\log 2x)^{uc^t-1}\big), \tag{2.12}$$
$$\sum_{n \leqq x} \frac{\varrho_{f,t}^{u,v}(n)}{n^{vt}} = \begin{cases} O\big((\log 2x)^{uc}\big), & \text{if} \quad t = 1 \\ O(1), & \text{if} \quad t \geqq 2 \end{cases} \tag{2.13}$$
and
$$\sum_{n > x} \frac{\varrho_{f,t}^{u,v}(n)}{n^{vt+1}} = O\left(\frac{(\log 2x)^{uc^t-1}}{x^t}\right). \tag{2.14}$$

PROOF. We use induction on u for proving (2.12). For $u=1$, (2.12) is clear in virtue of (2.6) and (2.9). Now, assume (2.12) for $u-1$ where $u\geqq 2$ is an integer. Then, by (2.7) and the induction hypothesis,

$$\sum_{n\leqq x}\varrho_{f,t}^{u,v}(n)=\sum_{d\delta\leqq x}\varrho_{f,t}^{1,v}(d)\varrho_{f,t}^{u-1,v}(\delta)=\sum_{d\leqq x}\varrho_{f,t}^{1,v}(d)\sum_{\delta\leqq x/d}\varrho_{f,t}^{u-1,v}(\delta)=$$

$$=O\left(\sum_{d\leqq x}\varrho_{f,t}^{1,v}(d)\left(\frac{x}{d}\right)^{1+(v-1)t}\left(\log\frac{x}{d}\right)^{(u-1)c^t-1}\right)= \tag{2.15}$$

$$=O\left(x^{1+(v-1)t}(\log x)^{(u-1)c^t-1}\sum_{d\leqq x}\frac{\varrho_{f,t}^{1,v}(d)}{d^{1+(v-1)t}}\right).$$

Now, by (2.6) and partial summation, we get

$$\sum_{n\leqq x}\frac{\varrho_{f,t}^{1,v}(n)}{n^{1+(v-1)t}}=O\big((\log x)^{c^t}\big)$$

and hence (2.12) follows because of (2.15).

(2.13) and (2.14) follow from (2.12) by partial summation.

3. Main results.

Theorem 1.

$$\sum_{n\leqq x}\varphi_{f,t}^{(k)}(n)=\alpha x^{t+1}+\begin{cases}O\big(x(\log x)^c\big), & \text{for } t=1\\ O(x^t), & \text{for } t\geqq 2\end{cases} \tag{3.1}$$

where

$$\alpha=\frac{1}{t+1}\sum_{n=1}^{\infty}\frac{\mu(n)N_f^t(n^k)}{n^{k(t+1)}}=\frac{1}{t+1}\prod_p\left\{1-\frac{N_f^t(p^k)}{p^{k(t+1)}}\right\} \tag{3.2}$$

and the product on the right is extended over all primes p.

PROOF. The series in the definition of α converges absolutely by (2.11) and the general term of the series is a multiplicative function of n. Hence by the Euler's infinite product factorization theorem (cf. [4], Theorem 285) follows the equality of the sum and product in (3.2).

Now, by (2.2), we obtain

$$\sum_{n\leqq x}\varphi_{f,t}^{(k)}(n)=\sum_{d^k\delta\leqq x}\delta^t\mu(d)N_f^t(d^k)=\sum_{d\leqq x^{1/k}}\mu(d)N_f^t(d^k)\sum_{\delta\leqq x/d^k}\delta^t=$$

$$=\sum_{d\leqq x^{1/k}}\mu(d)N_f^t(d^k)\left\{\left(\frac{x}{d^k}\right)^{t+1}\frac{1}{t+1}+O\left(\frac{x}{d^k}\right)^t\right\}=$$

$$=\frac{x^{t+1}}{t+1}\sum_{d=1}^{\infty}\frac{\mu(d)N_f^t(d^k)}{d^{k(t+1)}}+O\left(x^{t+1}\sum_{d>x^{1/k}}\frac{\mu^2(d)N_f^t(d^k)}{d^{k(t+1)}}\right)+O\left(x^t\sum_{d\leqq x^{1/k}}\frac{\mu^2(d)N_f^t(d^k)}{d^{kt}}\right)$$

and the result follows now by an application of (2.10) and (2.11).

Theorem 2.

$$\sum_{n \leqq x} \Psi_{f,t}^{(k)}(n) = \beta x^{t+1} + \begin{cases} O\left(x(\log x)^c\right), & for \quad t = 1 \\ O(x^t), & for \quad t \geqq 2 \end{cases} \tag{3.3}$$

where

$$\beta := \frac{1}{t+1} \sum_{n=1}^{\infty} \frac{\mu^2(n) N_f^t(n^k)}{n^{k(t+1)}} = \frac{1}{t+1} \prod_p \left\{1 + \frac{N_f^t(p^k)}{p^{k(t+1)}}\right\}. \tag{3.4}$$

PROOF. The proof is the same as that of Theorem 1 except that we use (2.3) in place of (2.2).

Theorem 3.

$$\sum_{n \leqq x} \psi_{f,t}^{(k)}(n) = \gamma x^{kt+1} + \begin{cases} O\left(x^k(\log x)^c\right), & for \quad t = 1 \\ O(x^{tk}), & for \quad t \geqq 2 \end{cases} \tag{3.5}$$

where

$$\gamma := \frac{1}{kt+1} \sum_{n=1}^{\infty} \frac{\mu^2(n) N_f^t(n^k)}{n^{kt+1}} = \frac{1}{kt+1} \prod_p \left\{1 + \frac{N_f^t(p^k)}{p^{kt+1}}\right\}. \tag{3.6}$$

PROOF. We use (2.4) in place of (2.2) and the proof runs similar to that of Theorem 1. We omit the details.

Theorem 4.

$$\sum_{n \leqq x} \psi_{f,t}^{u,v}(n) = \delta x^{vt+1} + \begin{cases} O\left(x^v(\log x)^{uc}\right), & for \quad t = 1 \\ O(x^{vt}), & for \quad t > 1 \end{cases} \tag{3.7}$$

where

$$\delta = \frac{1}{vt+1} \sum_{n=1}^{\infty} \frac{\varrho_{f,t}^{u,v}(n)}{n^{vt+1}}.$$

PROOF. We use (2.5), (2.13) and (2.14) in place of (2.2), (2.10) and (2.11) and proceed as in the proof of Theorem 1. We omit the details.

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