# Iterative methods of order two and three for computing inverse elements in Banach algebra with identity

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*Abstract.* Using the Banach's theorems and some properties of the Fréchet derivatives of nonlinear operators, proofs are given for theorems about the convergence and error estimates of two iteration methods of order two and three for calculating inverse elements in Banach algebra with identity.

*Key words:* fixed point theorem, iterative method, error estimate, inverse element in Banach algebra, Fréchet derivatives of nonlinear operator.

## 1. Extension of the quadratic method of Hotelling and Schultz

Let us consider the iterative method due to H. HOTELLING [1] and SCHULTZ [2] for the inversion of a quadratic and nonsingular matrix A:

$$(1.1)\qquad \begin{cases} C_0 = I-AB_0, \\ B_n = B_{n-1}(I+C_{n-1}), \quad C_n = I-AB_n; \quad n = 1, 2, \dots \end{cases}$$

i.e.

$$(1.2)\qquad \begin{cases} B_{n+1} = F(B_n); \quad n = 0, 1, \dots \quad \text{where} \\ F(B) \doteq (2I-BA)B. \end{cases}$$

It is known that $\lim_{n\to\infty} B_n = A^{-1}$ if $\|C_0\|<1$ and $\exists A^{-1}$.

We try to extend this method in complex (or real) Banach algebra with unit element and thereafter in chapter 2 to increase the rapidity of the method obtained. For this reason let $X$ be a real or complex Banach algebra with identity $e$ (i.e. let

1) $X$ be a Banach space over $\mathbf{R}$ (or $\mathbf{C}$);
2) $X$ be a linear algebra with identity $e$ over $\mathbf{R}$ (or $\mathbf{C}$);
3) $\|xy\| \leqq \|x\| \cdot \|y\|, \ x, y \in X$;
4) $\|e\|=1$).

(1.3) Let a be an element of $X$ having an inverse element $a^{-1}$. Let us take an arbitrary element $x_0 \in X$ and denote $c_0 \doteq e - ax_0$.
Let us consider the iteration

$$x_n = x_{n-1}(e+c_{n-1}), \quad c_n = e-ax_n; \quad n = 1, 2, \dots$$

i.e.

$$x_{n+1} = F(x_n); \quad n = 0, 1, 2, \dots$$

where

$$F(x) \doteq x(2e-ax) = 2x - xax.$$

Obviously, the element $a^{-1}$ is a fixed point of the iteration function $F$:

$$F(a^{-1}) = a^{-1}.$$

**Lemma 1.1.** *The mapping* $F: X \to X$ *defined by* (1.3) *on the whole Banach algebra* $X$ *with identity* $e$ *is twice Fréchet differentiable in* $X$ *and the Fréchet differentials of* $F$ *take the forms*

$$\begin{cases} F'(x)h = 2h - xah - hax, \\ F''(x)(h_1, h_2) = -h_2ah_1 - h_1ah_2, \\ x, h, h_1, h_2 \in X, \end{cases} \tag{1.4}$$

further

$$\|F'(x)\| \leqq \|e - xa\| + \|e - ax\|,$$

$$\|F''(x)\| \leqq 2\|a\|, \quad x \in X.$$

PROOF. It follows from

$$F(x+h) - F(x) = 2(x+h) - (x+h)a(x+h) - 2x + xax =$$

$$= 2h - xah - hax - hah; \quad x, h \in X$$

that

$$\|F(x+h) - F(x) - F'(x)h\| = \|hah\| \leqq \|a\| \cdot \|h\|^2.$$

Consequently, $\lim_{\|h\| \to 0} \varepsilon(\|h\|) = 0$ if one chooses

$$\varepsilon(\|h\|) \doteq \|a\| \cdot \|h\|,$$

therefore, according to [2] (pp. 15—18), the map $F$ is Fréchet differentiable in $x (\in X)$ and the value of its Fréchet differential $F'(x)h$ is really equal to (1.4). Let us show that the operator $F'(x)$ is linear and bounded.

The operator $F'(x)$ is additive and homogeneous because

$$F'(x)(h_1 + h_2) = 2(h_1 + h_2) - xa(h_1 + h_2) - (h_1 + h_2)ax =$$

$$= (2h_1 - xah_1 - h_1ax) + (2h_2 - xah_2 - h_2ax) = F'(x)h_1 + F'(x)h_2; \quad x, h_1, h_2 \in X$$

and

$$F'(x)(\lambda h) = 2(\lambda h) - xa(\lambda h) - \lambda hax = \lambda(2h - xah - hax) = \lambda F'(x)h;$$

$$x, h \in X; \quad \lambda \text{ is a scalar.}$$

$F'(x)$ is bounded: $\|F'(x)\| \leqq M \doteq \|e - xa\| + \|e - ax\|$ for

$$\|F'(x)h\| = \|(h - xah) + (h - hax)\| = \|(e - xa)h + h(e - ax)\| \leqq M\|h\|; \quad x, h \in X.$$

In order to get the second Fréchet derivatives let's consider the expression

$$F'(x + \Delta x) - F'(x) = 2. - (x + \Delta x)a. - .a(x + \Delta x) - 2. + xa. + .ax = -\Delta xa. - .a\Delta x$$

We define the linear operation $B(\Delta x, .)$ in the following way:

$$B(\Delta x, .) = -\Delta xa. - .a\Delta x$$

i.e. the value of the differential can be given as

$$B(h_1, h_2) = -h_1ah_2 - h_2ah_1; \quad \Delta x, a, h_1, h_2 \in X.$$

Then the operator $B$ is linear in $h_1$ and in $h_2$ too, further it is bounded:

$$\|B\| \leqq 2\|a\|$$

because

$$\|B(h_1, h_2)\| \leqq \|h_1ah_2\| + \|h_2ah_1\| \leqq 2\|a\| \cdot \|h_1\| \cdot \|h_2\|.$$

Thus we have

$$\|F'(x+\Delta x) - F'(x) - B(\Delta x, .)\| = 0.$$

In virtue of the point 2.3 of § 2, XVII in [3] and the uniqueness theorem of the Fréchet derivatives ([2], pp. 15—16) we get $F''(x) = B$. Qu.e.d.

**Theorem 1.1.** *Let $X$ be a real or complex Banach algebra with identity $e$ and let the element $a \in X$ have an inverse element $a^{-1}$. Let $q \in [0, 1)$ be an arbitrary but fixed real number further let us take the set*

$$G \doteq \{x \in X \mid \|e-ax\| + \|e-xa\| \leqq q\} \subset X$$

*and define the map $F: G \to X$ in such a manner:*

$$F(x) := x(2e-ax), \quad x \in G.$$

*Then the following assertions hold:*

1° *$F$ has exactly one fixed point in $G$:*

$$F(a^{-1}) = a^{-1}.$$

2° *The sequence $\{x_n\}$ generated by the iteration formula*

$$x_{n+1} = F(x_n); \quad n = 0, 1, 2, \ldots$$

*tends to $a^{-1}$ for an arbitrary $x_0 \in G$.*

3° $$\|x_n - a^{-1}\| \leqq \frac{\|x_0\|}{1-q} q^{2^n}; \quad n = 0, 1, 2, \ldots$$

*("a priori" error estimate)*

4° *The real sequence $\{\|x_n - a^{-1}\|\}_{n=0}^{\infty}$ is monotonously decreasing.*

5° $$\|x_n - a^{-1}\| \leqq \|x_n - x_{n-1}\|; \quad n = 1, 2, \ldots$$

*if $q \leqq 1/2$ ("a posteriori" error estimate).*

6° *If $x_n \neq a^{-1}$, $n \in \mathbf{N}$ then the method is of order 2, i.e.*

$$0 < \lim_{n\to\infty} \frac{\|x_{n+1} - a^{-1}\|}{\|x_n - a^{-1}\|^2} < \infty.$$

PROOF. We shall apply the Banach's fixed point theorem, therefore, we have to show that the set $G$ is non-empty, closed and convex, further, that $F(G) \subset G$ and $F$ is contraction in $G$. Here

$$G \neq \emptyset \quad \text{for} \quad a^{-1} \in G.$$

It is known that any normed algebra is also a topological algebra ([4], p. 175) so the scalar multiplication, addition and multiplication are continuous operations. The set $G$ is closed due to the continuity of the norm.

We are going to show the convexity of $G$. Let $x'$ and $x''$ be arbitrary elements of $G$, i.e.

$$\|e-ax'\|+\|e-x'a\| \leqq q$$

and

$$\|e-ax''\|+\|e-x''a\| \leqq q.$$

If $t\in(0,1)$ then

$$x \doteq tx''+(1-t)x'\in G$$

since

$$\|e-ax\|+\|e-xa\| =$$

$$= \|te-tax''+(1-t)e-(1-t)ax'\|+\|te-tx''a+(1-t)e-(1-t)x'a\| \leqq$$

$$\leqq t\|e-ax''\|+(1-t)\|e-ax'\|+t\|e-x''a\|+(1-t)\|e-x'a\| =$$

$$= t(\|e-ax''\|+\|e-x''a\|)+(1-t)(\|e-ax'\|+\|e-x'a\|) \leqq$$

$$\leqq tq+(1-t)q = q.$$

(This can be done because $X$ is a real or complex Banach algebra!)

$F(y)\in G$ for any arbitrary $y\in G$ because

$$\|e-aF(y)\|+\|e-F(y)a\| = \|(e-ay)+ayay-ay\|+\|(e-ya)+yaya-ya\| =$$
$$= \|(e-ay)^2\|+\|(e-ya)^2\| \leqq (\|e-ay\|+\|e-ya\|)^2 \leqq q^2 \leqq q.$$

We prove that $F$ is a contraction in $G$. By virtue of lemma 1.1 there exists the Fréchet derivative $F'(x)$, $x\in X$ and the inequality

$$\|F'(x)\| \leqq \|e-xa\|+\|e-ax\|$$

holds. If $x\in G$ then we can write

$$\|F'(x)\| = \sup_{\|h\|=1} \|F'(x)h\| \leqq \sup_{\|h\|=1} [(\|e-xa\|+\|e-ax\|)\|h\|] =$$
$$= \|e-xa\|+\|e-ax\| \leqq q.$$

Using the generalization of the mean value theorem of Lagrange in normed vector-space ([2], pp. 28—29) and the convexity of the set $G$, we obtain that the mapping $F$ is a contraction in $G$:

$$(1.5)\qquad \begin{cases} \|F(x)-F(y)\| \leqq \big\|F'\big(x+\vartheta(y-x)\big)\big\|\,\|x-y\| \leqq q\|x-y\|; \\ x,\ y\in G;\quad \vartheta\in(0,1). \end{cases}$$

Now we apply the fixed point theorem of Banach and Caccioppoli ([2], pp. 42—43; [3], pp. 510—511) thus the following assertions hold

1) $$\exists!\, p\in G\colon\ p = F(p);$$

$\big(p=a^{-1}$ because $a^{-1}\in G$ and $a^{-1}=F(a^{-1})\big)$;

2) the iteration sequence

$$x_{n+1} = F(x_n); \quad n = 0, 1, 2, \dots$$

tends to $a^{-1}$ for an arbitrary initial point $x_0 \in G$;

3) the estimate

$$\|x_n - a^{-1}\| \leqq \frac{q^n}{1-q} \|F(x_0) - x_0\|; \quad n = 0, 1, 2, \dots$$

holds.

Since

$$\|F(x_0) - x_0\| = \|(e - x_0 a) x_0\| \leqq \|e - x_0 a\| \cdot \|x_0\|$$

so

$$\|x_n - a^{-1}\| \leqq \frac{q^n}{1-q} \|x_0\| \cdot \|e - x_0 a\|; \quad n = 0, 1, \dots .$$

But one can prove a more efficient estimate, namely 3°. For this reason one takes an arbitrary element $x_0 \in G$ and thereafter the sequences $\{c_n\}$ and $\{x_n\}$ generated by (1.3). In virtue of (1.3) one writes

$$c_n = e - ax_n = e - ax_{n-1}(e + c_{n-1}) = e - (e - c_{n-1})(e + c_{n-1}) = c_{n-1}^2; \quad n = 1, 2, \dots$$

thus

$$c_n = c_{n-1}^2 = c_{n-2}^4 = \dots = c_0^{2^n}; \quad n = 0, 1, 2, \dots .$$

For

$$\|e - ax_n\| = \|c_n\| \leqq \|c_0\|^{2^n} \leqq q^{2^n} \to 0 \quad \text{if} \quad n \to \infty$$

so

$$\lim_{n \to \infty} ax_n = e.$$

One is going to show $\lim_{n \to \infty} x_n = a^{-1}$, i.e. $\lim_{n \to \infty} \|x_n - a^{-1}\| = 0$. Since $\|c_0\| \leqq q < 1$, in virtue of a generalization of another theorem of Banach ([3], pp. 139—140) there exists $(e - c_0)^{-1}$ and the inequalities

$$\|(e - c_0)^{-1}\| \leqq \frac{1}{1 - \|c_0\|} \leqq \frac{1}{1-q}$$

hold ([4], p. 177).

By help of (1.3) one can write

$$x_0 = a^{-1}(ax_0) = a^{-1}(e - c_0),$$

i.e.

$$x_0(e - c_0)^{-1} = a^{-1},$$

i.e.

$$x_0(e - c_0)^{-1} a = e.$$

Then, for the element

$$u := x_0(e - c_0)^{-1}$$

the relation $ua = e$ is valid.

On the other hand, by virtue of (1.3) one can write

$$au = ax_0(e - c_0)^{-1} = ax_0(ax_0)^{-1} = e$$

thus

$$u = a^{-1} = x_0(e - c_0)^{-1}.$$

Consequently,

$$\|a^{-1}\| \leqq \|x_0\| \cdot \|(e-c_0)^{-1}\| \leqq \frac{\|x_0\|}{1-q}.$$

Using (1.3), it can be written

$$c_n = e - ax_n$$

and

$$x_n - a^{-1} = -a^{-1}c_n,$$

therefore one gets the error estimate 3°:

$$\|x_n - a^{-1}\| \leqq \|a^{-1}\| \cdot \|c_n\| \leqq \frac{\|x_0\|}{1-q} q^{2^n}; \quad n = 0, 1, 2, \ldots.$$

By virtue of (1.5), we can write

$$\|x_{n+1} - a^{-1}\| = \|F(x_n) - F(a^{-1})\| \leqq q\|x_n - a^{-1}\|$$

which proves the assertion 4°.

In order to show 5° we take the inequalities

$$\|x_n - a^{-1}\| \leqq \|x_n - x_{n+1}\| + \|x_{n+1} - a^{-1}\| \leqq q\|x_{n-1} - x_n\| + q\|x_n - a^{-1}\|; \quad n \in \mathbf{N}.$$

So we can write that

$$(1-q)\|x_n - a^{-1}\| \leqq q\|x_n - x_{n-1}\|,$$

from which the assertion 5° follows:

$$\|x_n - a^{-1}\| \leqq \frac{q}{1-q}\|x_n - x_{n-1}\| \leqq \frac{1/2}{1-\frac{1}{2}}\|x_n - x_{n-1}\|.$$

This can be considered as an "a posteriori" error estimate.

At the point $a^{-1}$ the method is of order two if and only if $F(a^{-1}) = a^{-1}$, $F'(a^{-1}) = 0$, $F''(a^{-1}) \neq 0$ [5]. Clearly, $a^{-1}$ is a fixed point of $F$. The linear operator $F'(a^{-1})$ is the 0-operator because

$$\|F'(a^{-1})\| \leqq [\|e - xa\| + \|e - ax\|]_{x=a^{-1}} = 0.$$

Using (1.4) from lemma 1.1, we get

$$F''(a^{-1})(h_1, h_2) = -h_1 a h_2 - h_2 a h_1$$

which does not vanish in general. Consequently, the method is quadratic.

Qu.e.d.

## 2. An iterative procedure of order three for computing inverse element

Let us return to the real case: let $X = \mathbf{R}$. The method described in the previous chapter, is similarly quadratic as the Newton iteration. The question arises, whether or not this procedure can be taken into consideration as a Newton method for zero determination of some real function $f$. Of course, the answer is yes if one choses $f(x) = a - x^{-1}$, $x \neq 0$ where $a \neq 0$ is an arbitrary but fixed real number.

[The form of our iteration function is $F(x)=2x-ax^2$ and the Newton iteration function has the form $N(x)=x-\frac{f(x)}{f'(x)}$. Thus the function $f$ in question satisfies the differential equation

$$x-ax^2 = -\frac{f(x)}{f'(x)}$$

i.e.

$$\ln f(x) = \int \frac{dx}{ax^2-x}\left(=\int\frac{adx}{ax-1}-\int\frac{dx}{x}\right).$$

Consequently

$$f(x) = \text{const.}\,\frac{ax-1}{x}.]$$

This train of thoughts suggests the idea of increasing the speed of the method in such a way: let us apply the Chebyshev iteration method of order three [5]

$$x_{n+1} = \varphi(x_n); \quad n = 0, 1, \dots$$

$$\varphi(x) \doteq x-\frac{f(x)}{f'(x)}-\frac{f^2(x)f''(x)}{2f'^3(x)}$$

to solve the equation $f(x)=0$. The iteration function $\varphi$ has the form

$$\varphi(x) = 3x-3ax^2+a^2x^3$$

because $f(x)=a-x^{-1}$.

From now let $X$ be again an arbitrary real (or complex) Banach algebra with identity. We are going to investigate the operator form of $\varphi$.

**Lemma 2.1.** *Let $X$ be a Banach algebra with identity $e$ and let $a$ be an arbitrary element of $X$. Then the operator $\varphi: X\to X$,*

$$\varphi(x) \doteq x(3e-3ax+axax)$$

*is three times Fréchet differentiable in $X$ and the Fréchet differentials of $F$ have the following forms:*

$$\varphi'(x)h = 3h-3xah-3hax+xaxah+xahax+haxax,$$

$$\varphi''(x)(h_1, h_2) = -3h_1ah_2-3h_2ah_1+h_1axah_2+h_2axah_1+$$
$$+xah_1ah_2+xah_2ah_1+h_1ah_2ax+h_2ah_1ax,$$

$$\varphi'''(x)(h_1, h_2, h_3) = h_1ah_2ah_3+h_1ah_3ah_2+$$
$$+h_2ah_1ah_3+h_2ah_3ah_1+h_3ah_1ah_2+h_3ah_2ah_1; \quad x, h, h_1, h_2, h_3\in X$$

*further*

$$\|\varphi'(x)\| \leqq (\|e-xa\|+\|e-ax\|)^2,$$

$$\|\varphi''(x)\| \leqq 2\|a\|(2\|e-xa\|+\|e-ax\|),$$

$$\|\varphi'''(x)\| \leqq 6\|a\|^2; \quad x\in X.$$

PROOF. For calculating $\varphi'(x)$ we form the difference

$$\varphi(x+h)-\varphi(x) = 3h-3hah-3xah-3hax+hahah+hahax+haxah+xahah+ +xaxah+xahax+haxax.$$

We define the additive and homogeneous (i.e. linear) operator $l$ in the following way:

$$lh \doteq 3h-3xah-3hax+xaxah+xahax+haxax.$$

The linear operator $l$ is bounded for

$$\|l\| = \sup_{\|h\|=1} \|lh\| =$$

$$= \sup_{\|h\|=1} \|(h-2xah+xaxah)+(h-hax-xah+xahax)+(h-2hax+haxax)\| =$$

$$= \sup_{\|h\|=1} \|(e-xa)^2h+(e-xa)h(e-ax)+h(e-ax)^2\| \leqq$$

$$\leqq \sup_{\|h\|=1} [(\|e-xa\|^2+\|e-xa\|\,\|e-ax\|+\|e-ax\|^2)\|h\|] \leqq (\|e-xa\|+\|e-ax\|)^2.$$

On the basis of the definition of the Fréchet differentiability ([2], [3]), the Fréchet derivative of $\varphi$ in $x$ is $l$

$$\big(\varphi'(x) = l = 3e. -3xa. -3.ax+xaxa.+xa.ax+.axax\big)$$

because

$$\|\varphi(x+h)-\varphi(x)-lh\| = \|hahah+hah(ax-e)+ha(xa-e)h+(xa-e)hah\| \leqq$$

$$\leqq \|h\|(\|h\|\,\|a\|[\|ah\|+\|ax-e\|+2\|xa-e\|]) \doteq \|h\|\varepsilon(\|h\|)$$

and $\lim_{\|h\|\to 0} \varepsilon(\|h\|)=0.$

In order to determine the bilinear operation $\varphi''(x)$ we build the difference

$$\varphi'(x+\Delta x)-\varphi'(x) = -3(x+\Delta x)a.-3.a(x+\Delta x)+(x+\Delta x)a(x+\Delta x)a.+$$

$$+(x+\Delta x)a.a(x+\Delta x)+.a(x+\Delta x)a(x+\Delta x)+3xa.+3.ax-xaxa.-xa.ax-$$

$$-.axax = -3\Delta xa.-3.a\Delta x+\Delta xaxa.+xa\Delta xa.+xa.a\Delta x+$$

$$+\Delta xa.ax+.axa\Delta x+.a\Delta xax+\Delta xa\Delta xa.+\Delta xa.a\Delta x+.a\Delta xa\Delta x$$

Let us take the linear operator $b(\Delta x,.)=-3\Delta xa.-3.a\Delta x+\Delta xaxa.+xa\Delta xa.+ +xa.a\Delta x+\Delta xa.ax+.axa\Delta x+.a\Delta xax$. It is clear that $b$ is linear in $\Delta x$ too, therefore, $b$ is a bilinear operator in virtue of the point 2.3 of § 2, XVII, [3].

The second derivative of the mapping $\varphi$ in $x$ is $b$ because

$$\|\varphi'(x+\Delta x)-\varphi'(x)-b\Delta x\| = \|(\Delta xa)^2.+\Delta xa.a\Delta x+.(a\Delta x)^2\| \leqq$$

$$\leqq \|\Delta x\|\,\|a\|\,\|\Delta x\|\,\|a.\|+\|\Delta x\|\,\|a.\|\,\|a\|\,\|\Delta x\|+\|.a\|\,\|\Delta x\|\,\|a\|\,\|\Delta x\| =$$

$$= \|\Delta x\|[\|\Delta x\|\,\|a\|(2\|a.\|+\|.a\|)] \doteq \|\Delta x\|\varepsilon(\|\Delta x\|)$$

where $\lim_{\|\Delta x\|\to 0} \varepsilon(\|\Delta x\|) = 0.$

Thus the second Fréchet differential of $\varphi$ has the form

$$\varphi''(x)(h_1, h_2) = -3h_1ah_2 - 3h_2ah_1 + xah_1ah_2 + xah_2ah_1 + \\ + h_1axah_2 + h_2axah_1 + h_1ah_2ax + h_2ah_1ax.$$

The boundedness is evident for

$$\|\varphi''(x)\| = \sup_{\|h_1\|=\|h_2\|=1} \|\varphi''(x)(h_1, h_2)\| = \sup_{\|h_1\|=\|h_2\|=1} \|(xa-e)h_1ah_2 + (xa-e)h_2ah_1 + \\ + h_1a(xa-e)h_2 + h_2a(xa-e)h_1 + h_1ah_2(ax-e) + h_2ah_1(ax-e)\| \leqq \\ \leqq 2\|a\|(\|e-ax\| + 2\|e-xa\|).$$

If we define a 3 — linear operation $t$ as

$$t(\Delta x, h_1, h_2) \doteq h_1a\Delta xah_2 + h_1ah_2a\Delta x + h_2ah_1a\Delta x + \\ + h_2a\Delta xah_1 + \Delta xah_1ah_2 + \Delta xah_2ah_1$$

then

$$[\varphi''(x+\Delta x) - \varphi''(x) - t\Delta x](h_1, h_2) = \\ = \varphi''(x+\Delta x)(h_1, h_2) - \varphi''(x)(h_1, h_2) - t(\Delta x, h_1, h_2) = 0; \quad h_1, h_2 \in X.$$

Thus the bilinear operator

$$\varphi''(x+\Delta x) - \varphi''(x) - t\Delta x$$

s the bilinear 0-operator, therefore, by taking $\varepsilon(\|\Delta x\|) \equiv 0$ the relations

$$(0 =)\|\varphi''(x+\Delta x) - \varphi''(x) - t\Delta x\| \leqq \|\Delta x\|\,\varepsilon(\|\Delta x\|),$$
$$\lim_{\|\Delta x\| \to 0} \varepsilon(\|\Delta x\|) = 0$$

are evidently satisfied. The 3-linear operator $t$ is bounded because

$$\|t\| = \sup_{\|h_1\|=\|h_2\|=\|h_3\|=1} \|h_1ah_2ah_3 + h_1ah_3ah_2 + \\ + h_2ah_1ah_3 + h_2ah_3ah_1 + h_3ah_1ah_2 + h_3ah_2ah_1\| \leqq 6\|a\|^2.$$

Consequently, $\varphi'''(x) = t$ and the third Fréchet-differential of $\varphi$ has the form

$$\varphi'''(x)(h_1, h_2, h_3) = h_1ah_2ah_3 + h_1ah_3ah_2 + \\ + h_2ah_1ah_3 + h_2ah_3ah_1 + h_3ah_1ah_2 + h_3ah_2ah_1; \quad x, h_1, h_2, h_3 \in X$$

which was to be proved.

**Theorem 2.1.** *Let $X, e, a, q$ and $G$ denote the same as in the theorem* 1.1. *Let us define the operator $\varphi: G \to X$ in such a manner:*

$$\varphi(x) \doteq x(3e - 3ax + axax); \quad x \in G.$$

*Then the following statements hold:*

1° *the mapping $\varphi$ has the only fixed point $a^{-1}$ in $G$:*

$$\varphi(a^{-1}) = a^{-1};$$

2° *the sequence* $\{x_n\}$ *generated by the iteration formula*

$$x_{n+1} = \varphi(x_n); \quad n = 0, 1, \ldots$$

*converges to* $a^{-1}$ *for arbitrary* $x_0$ *from* $G$;

3° $$\|x_n - a^{-1}\| \leqq \frac{\|x_0\|}{1-q} q^{3^n}; \quad n = 0, 1, \ldots$$

(*"a priori" estimate*);

4° *the sequence*

$$\{\|x_n - a^{-1}\|\}_{n=0}^{\infty}$$

*is monotonously decreasing;*

5° $$\|x_n - a^{-1}\| \leqq \|x_n - x_{n-1}\|; \quad n = 1, 2, \ldots \quad \text{if} \quad q \leqq \frac{1}{2}$$

(*"a posteriori" estimate*);

6° *the order of the convergence of the method is three:*

$$0 < \lim_{n\to\infty} \frac{\|x_{n+1} - a^{-1}\|}{\|x_n - a^{-1}\|^3} < \infty$$

*if* $x_n \neq a^{-1}$, $n \in \mathbf{N}$.

PROOF. We have seen in the proof of the theorem 1.1 that the set $G$ is closed, convex and not empty. If $x \in G$ then

$$\|e - a\varphi(x)\| + \|e - \varphi(x)a\| = \|(e-ax)^3\| + \|(e-xa)^3\| \leqq$$
$$\leqq (\|e-ax\| + \|e-xa\|)^3 \leqq q^3 \leqq q,$$

consequently $\varphi(G) \subset G$.

In order to apply the fixed point theorem, we have only to show that the operator $\varphi$ is contraction in $G$. According to the lemma 2.1,

$$\exists \varphi'(x), \quad x \in X$$

and

$$\|\varphi'(x)\| \leqq (\|e - xa\| + \|e - ax\|)^2.$$

Therefore

$$\|\varphi'(x)\| \leqq q^2 \leqq q$$

if $x \in G$. We use the convexity of $G$ and apply again the extension of the Lagrange mean value theorem ([2], pp. 28—29) so we get

$$\text{(2.1)} \qquad \begin{cases} \|\varphi(x) - \varphi(y)\| \leqq \|\varphi'(x + \Theta(y-x))\| \|x-y\| \leqq q\|x-y\|, \\ x, y \in G; \quad \Theta \in (0, 1) \end{cases}$$

i.e. $\varphi$ is a contraction.

Using the fixed point theorem of Banach and Caccioppoli ([2], pp. 42—43 and [3], pp. 510—511) it follows that

1° $$\exists!\, \alpha \in G \colon \alpha = \varphi(\alpha).$$

($\alpha = a^{-1}$ because $a^{-1} = \varphi(a^{-1}) \in G$.)

2° For an arbitrary $x_0 \in G$, the sequence $\{x_n\}$ generated by the iteration formula $x_{n+1}=\varphi(x_n)$; $n=0, 1, \dots$ tends to $a^{-1}$.

Let us prove the à priori error estimate 3°. The iterative procedure generated by $\varphi$ can be written in the following way:

$$\begin{cases} c_0 = e-ax_0; \\ x_{n+1} = x_n(e+c_n+c_n^2); \quad c_{n+1} = e-ax_{n+1} \\ n = 0, 1, 2, \dots . \end{cases} \tag{2.2}$$

Using induction we shall show that

$$c_n = c_0^{3^n}; \quad n = 0, 1, 2, \dots . \tag{2.3}$$

For $n=0$ the statement is evident. Supposing (2.3) to be valid for $k \in [0, n] \cap \mathbf{Z}$, we have to prove it for $k=n+1$.

In virtue of (2.2) one can write

$$c_{n+1} = e-ax_{n+1} = e-ax_n(e+c_n+c_n^2) = e-(e-c_n)(e+c_n+c_n^2) = c_n^3$$

which is equal to

$$(c_0^{3^n})^3 = c_0^{3^{n+1}}$$

if one uses the inductive condition.

Let us follow the train of thoughts in the proof of the theorem 1.1. One can see immediately that

$$\lim_{n\to\infty} ax_n = e, \quad \lim_{n\to\infty} x_n = a^{-1},$$

$$\|a^{-1}\| \leqq \frac{\|x_0\|}{1-q}$$

and

$$\|x_n-a^{-1}\| \leqq \frac{\|x_0\|}{1-q} q^{3^n}; \quad n = 0, 1, 2, \dots .$$

In virtue of (2.1) we have

$$\|x_{n+1}-a^{-1}\| = \|\varphi(x_n)-\varphi(a^{-1})\| \leqq q\|x_n-a^{-1}\|; \quad n = 0, 1, 2, \dots$$

therefore 4° holds.

The estimate 5° can be proved as in the theorem 1.1.

At the point $a^{-1}$ our procedure is of order three if and only if the following relations hold:

$$\varphi(a^{-1}) = a^{-1}, \quad \varphi'(a^{-1}) = 0, \quad \varphi''(a^{-1}) = 0, \quad \varphi'''(a^{-1}) \not\equiv 0 \quad \text{(s. [5])}.$$

The first equality is evident.

It arises from the lemma 2.1 that

$$\varphi'(a^{-1})h = 0; \quad h \in X$$

and

$$\varphi''(a^{-1})(h_1, h_2) = 0; \quad h_1, h_2 \in X.$$

But the value of

$$\varphi'''(a^{-1})(h_1, h_2, h_3)$$

does not vanish in general, therefore, the 3-linear operator $\varphi'''(a^{-1})$ is not equal to the 0-operator.

Qu.e.d.

The following applications can be mentioned.

I. Determination of inverse matrices in the noncommutative Banach algebra of $k \times k$ matrices with real or complex entries.

II. Building the inverse operator of a linear and bounded operator defined in a (real or complex) Banach space.

III. Calculation of inverse operator of a linear integral operator in the normed algebra of continuous functionals defined on compact topological spaces.

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*(Received August 13, 1983.)*