On the Lebesgue function on infinite interval, II.

By S. A. N. ENEDUANYA (Minna)

1. In this paper we shall continue our previous investigations [1] and obtain a bound for the Lebesgue function on the interval $(-\infty, +\infty)$. Moreover, we shall establish a convergence theorem for the Lagrange interpolation on this interval.

Let a weight function $p(x)$ be given such that

$$(1.1) \qquad \int_{-\infty}^{\infty} x^k p(x)\,dx < \infty \quad (k = 0, 1, 2, \ldots),$$

then there exists one and only one system of orthonormal polynomials $\{\omega_n(x)\}_{n=0}^{\infty}$ such that $\omega_n(x)$ has exactly n simple real roots and

$$(1.2) \qquad -\infty < x_n < x_{n-1} < \ldots < x_{\nu+1} < x_\nu < \ldots < x_2 < x_1 < +\infty.$$

For the orthonormal Hermite polynomials defining the n-th polynomial as

$$(1.3) \qquad \omega_n^*(x) = \bar{H}_n(x) = \frac{(-1)^n}{\sqrt{2^n n! \sqrt{\pi}}} e^{x^2}\{e^{-x^2}\}^{(n)}$$

the weight function is $p(x) = e^{-x^2}$.

We define the Lebesgue function using the roots (1.2)

$$(1.4) \qquad \lambda_n(x) \overset{\text{def}}{=} \sum_{\nu=1}^{n} |l_\nu(x)|$$

where

$$(1.5) \qquad l_\nu(x) = \frac{\omega_n(x)}{\omega_n'(x_\nu)(x - x_\nu)}.$$

Grünwald and Turán [2] proved that given a system of orthonormal polynomials $\{\bar{\omega}_n(x)\}_{n=0}^{\infty}$ with weight functions $p(x) \geqq m > 0$ on the interval $[-1, 1]$, the Lebesgue function using the roots of $\bar{\omega}_n(x)$ can be estimated thus:

$$\lambda_n(x) \leqq c_1 \sqrt{n}, \quad x \in (-1, 1)$$

and

$$\lambda_n(x) \leqq c_2 n \qquad x \in [-1, 1]$$

where c_1 and c_2 are constants independent of n.

In the first of this series [1] we extend our investigation to the interval $[0, \infty)$ and the bound

$$\lambda_n(x) = \begin{cases} c_3 x^{(\alpha/2)-(1/4)} n^{1/4}, & \alpha \geqq -\frac{1}{2}, \quad \textit{for fixed} \quad x \in (0, \infty), \\ c_4 n^{1/4}, & -1 < \alpha \leqq -\frac{1}{2}, \quad \textit{for fixed} \quad x \in [0, \infty) \end{cases}$$

if the weight function satisfy the condition $p(x)x^{-\alpha}e^{x} \geqq m > 0$. The c_3 and c_4 constants are independent of x and n.

We now state the following

Theorem 1. *Let* $p(x)e^{x^2} \geqq m > 0$ *and*

$$\int_{-\infty}^{\infty} x^k p(x)\, dx < \infty \quad (k = 0, 1, 2, \ldots),$$

then the Lebesgue function, using the roots of $\omega_n(x)$

$$\lambda_n(x) = O(1) n^{1/4}, \quad x \in [-A, A]$$

where A *is an arbitrary fixed real number.*

Remark. *If* $p(x) = e^{-x^2}$, *then* $m = 1$.

2. The following results, the proofs of which may be found similar to those in [1], are true.

(a) The Cotes numbers

$$\int_{-\infty}^{\infty} l_\nu(x) p(x)\, dx = \int_{-\infty}^{\infty} l_\nu(x)^2 p(x)\, dx = \mu_\nu \tag{2.1}$$

$$(\nu = 1, 2, \ldots, n;\ n = 1, 2, 3, \ldots).$$

(b) From the orthogonality of $\{\omega_n(x)\}_{n=0}^{\infty}$

$$\int_{-\infty}^{\infty} l_i(x) l_j(x) p(x)\, dx = \begin{cases} 0, & i \neq j \\ \mu_i, & i = j \end{cases} \tag{2.2}$$

$$(i, j = 1, 2, \ldots, n;\ n = 1, 2, 3, \ldots).$$

Using (2.1), (2.2) and the fact that

$$\sum_{\nu=1}^{n} l_\nu(x) \equiv 1 \quad (n = 1, 2, 3, \ldots),$$

we have

$$\int_{-\infty}^{\infty} \Big\{\sum_{\nu=1}^{n} l_\nu(x)\Big\}^2 p(x)\, dx = \int_{-\infty}^{\infty} \Big\{\sum_{\nu=1}^{n} l_\nu^2(x)\Big\} p(x)\, dx = \tag{2.3}$$

$$= \int_{-\infty}^{\infty} \Big\{\sum_{\nu=1}^{n} l_\nu(x)\Big\} p(x)\, dx = \int_{-\infty}^{\infty} p(x)\, dx < \infty.$$

Let us define $\varepsilon_\nu \overset{\text{def}}{=} \operatorname{sign}\{l_\nu(x_0)\}$, where $x_0 \geqq 0$ is an arbitrary fixed value and let

$$\Psi_n(x) = \sum_{\nu=1}^{n} \varepsilon_\nu l_\nu(x). \tag{2.4}$$

The polynomial $\Psi_n(x)$ is of degree $n-1$ and can be expanded by the Fourier series of orthonormal Hermite polynomials $\{\omega_n^*(x)\}$. Hence from the inequality Cauchy's we get

$$[\Psi_n(x)]^2 = \Big[\sum_{i=0}^{n-1} c_i \omega_i^*(x)\Big]^2 \leqq \sum_{i=0}^{n-1} c_i^2 \sum_{i=0}^{n-1} [\omega_i^*(x)]^2. \tag{2.5}$$

Thus, and from (2.4), (2.3) we have

$$\begin{aligned}\int_{-\infty}^{\infty} [\Psi_n(x)]^2 e^{-x^2}\, dx &= \sum_{i=0}^{n-1} c_i^2 = \int_{-\infty}^{\infty} \Big\{\sum_{\nu=1}^{n} \varepsilon_\nu l_\nu(x)\Big\}^2 e^{-x^2}\, dx \leqq \\ &\leqq \frac{1}{m} \int_{-\infty}^{\infty} \Big\{\sum_{\nu=1}^{\nu} \varepsilon_\nu l_\nu(x)\Big\}^2 p(x)\, dx = \frac{1}{m} \int_{-\infty}^{\infty} p(x)\, dx < \infty.\end{aligned} \tag{2.6}$$

This value which is also independent of n.

From (2.5) and (2.6) we get

$$[\Psi_n(x)]^2 = O(1) \sum_{i=0}^{n-1} [\omega_i^*(x)]^2. \tag{2.7}$$

Considering [3] (formulas (5.5.1), (5.6.1), (7.6.9) and (7.6.10)) it is easy to verify that

$$|\omega_i^*(x)| = O(1)(i+1)^{-1/4}, \quad -A \leqq x \leqq A \tag{2.8}$$

where A is an arbitrary real number.

From (2.7) and (2.8) the following inequality

$$[\Psi_n(x)]^2 \leqq O(1) n^{1/2}, \quad -A \leqq x \leqq A \tag{2.9}$$

holds.

Hence, from (2.4) and (2.9) we have

$$|\Psi_n(x_0)| = \lambda_n(x_0) = O(1) n^{1/4}, \quad -A \leqq x_0 \leqq A.$$

Thus, the theorem is proved, because x_0 is an arbitrary value.

3. We shall in this section consider the Lagrange interpolation polynomial of a continuous function $f(x)$ on the interval $(-\infty, \infty)$.

It is interesting at this point to consider a function

$$f(x) = e^{ax^2} \varphi(x) \quad (a > 0, -\infty < x < \infty) \tag{3.1}$$

where $\varphi(x) \in \mathrm{Lip}_M \gamma$, $\frac{1}{2} < \gamma \leqq 1$.

Let us define polynomials $L_n(x, f)$ of degree $n-1$, satisfying the following equalities on the roots of Hermite polynomials

$$(3.2)\qquad \begin{gathered} L_n(x_\nu;\, f) = e^{-ax_\nu^2} f(x_\nu) = \varphi(x_\nu) = y_\nu \\ (\nu = 1, 2, 3, \ldots, n,\ n = 1, 2, \ldots). \end{gathered}$$

The polynomials $L_n(x;\, f)$ have the explicit forms

$$(3.3)\qquad L_n(x;\, f) = \sum_{\nu=1}^{n} y_\nu l_\nu(x)$$

where

$$(3.4)\qquad l_\nu(x) = \frac{\omega_n^*(x)}{\omega_n^{*\prime}(x_\nu)(x-x_\nu)}.$$

From [3] (formula (5.6.1)) we have, if n is even

$$(3.5)\qquad l_\nu(x) = \frac{\omega_n^*(x)}{\omega_n^{*\prime}(x_\nu)(x-x_\nu)} = \frac{L_{n/2}^{(-1/2)}(x^2)}{2x_\nu L_{n/2}^{(-1/2)\prime}(x_\nu^2)(x-x_\nu)} \quad (\nu = 1, 2, \ldots, n)$$

and if n is odd

$$(3.6)\qquad l_\nu(x) = \frac{\omega_n^*(x)}{\omega_n^{*\prime}(x_\nu)(x-x_\nu)} = \frac{x L_{(n-1)/2}^{(1/2)}(x^2)}{2x_\nu^2 L_{(n-1)/2}^{(1/2)\prime}(x_\nu^2)(x-x_\nu)} \quad (\nu = 1, 2, \ldots, n)$$

where $L_k^{(\alpha)}(x)$ is the polynomial Laguerre's.

We shall now prove the following statement.

Theorem 2. *If* $f(x)$ *satisfies the condition* (3.1), *then the inequalities*

$$|f(x) - e^{ax^2} L_n(x;\, f)| \leqq O(1)\, n^{(1/4)-(\gamma/2)}, \quad x \in [-A, A]$$

hold where A *is an arbitrary fixed real number.*

PROOF. Let x_ν denote the roots of polynomials $\omega_n^*(x)$, then the following inequalities are true (SZEGŐ [3])

$$(3.7)\qquad c_1 \frac{\nu}{\sqrt{n}} < x_\nu < c_2 \frac{\nu}{\sqrt{n}} \quad \left(0 \leqq \nu \leqq \left[\frac{n}{2}\right]\right)$$

and $x_\nu = -x_{n-\nu+1}$, if $\left[\frac{n}{2}\right] + 1 \leqq \nu \leqq n$, where c_1 and c_2 constants are independent of ν and n.

Let $\varphi(x) \in \mathrm{Lip}_M\, \gamma$, $\frac{1}{2} < \gamma \leqq 1$, and $x \in [-x_{[n/2]}, x_{[n/2]}]$, then by using [1] it is easy to verify that there exists a polynomial $Q_n(x;\, \varphi)$ of degree $n-1$, for which the relations

$$|Q_n(x;\, \varphi) - \varphi(x)| = O(1)\, n^{-\gamma/2}$$

hold.

It is well known that

$$Q_n(x;\ \varphi) \equiv \sum_{\nu=1}^{n} Q_n(x_\nu;\ \varphi)\, l_\nu(x) \tag{3.9}$$

is true.

From (3.9), (3.8) and (3.5) we have, that for $x \in [-A, A]$, n even

$$\begin{gathered} |\varphi(x)-L_n(x;\ \varphi)| \leqq |\varphi(x)-Q_n(x;\ \varphi)| + \sum_{\nu=1}^{n} |Q_n(x_\nu;\ \varphi)-\varphi(x_\nu)|\,|l_\nu(x)| \leqq \\ \leqq O(1)n^{-\gamma/2} + O(1)n^{-\gamma/2} \sum_{0 \leqq x_\nu \leqq 2A} |l_\nu(x)| + O(1)n^{-\gamma/2} \sum_{x_\nu > 2A} |L_{n/2}^{(-1/2)}(x^2)|\,|L_{n/2}^{(-1/2)\prime}(x_\nu^2)|^{-1}. \end{gathered} \tag{3.10}$$

The following inequalities are taken from SZEGŐ [3] for $x \in [-A, A]$

$$\sum_{\nu=1}^{n} |L_n^{(\alpha)\prime}(x_\nu^2)|^{-1} \leqq \sqrt{n}\, \Big\{ \sum_{\nu=1}^{n} [L_n^{(\alpha)\prime}(x_\nu^2)]^{-2} \Big\}^{1/2} = O(1)n^{(1/2)-(\alpha/2)} \tag{3.11}$$

and

$$|L_{n/2}^{(-1/2)}(x^2)| = O(1)n^{-1/2}, \quad |xL_{(n-1)/2}^{(1/2)}(x^2)| = O(1). \tag{3.12}$$

Using (3.1), Theorem 1, (3.10), (3.12) and (3.11) we get for $x \in [-A, A]$, if n is even

$$\begin{gathered} |f(x)-e^{ax^2}L_n(x;\ f)| = e^{ax^2}|\varphi(x)-L_n(x;\ \varphi)| = O(1)n^{(1/4)-(\alpha/2)} \\ (n = 1, 2, 3, \ldots). \end{gathered} \tag{3.13}$$

These relations are also true, if n is odd. The proof is similar to that of (3.13).

Thus Theorem 2 is proved.

References

[1] S. A. N. ENEDUANYA, On Lebesgue function on infinite interval, I. *Publ. Math. (Debrecen).* **31.**
[2] G. GRÜNWALD—P. TURÁN, Über Interpolation, *Annali della Scuola Sup. di Pisa* (2), **7** (1938), 137—146.
[3] G. SZEGŐ, Orthogonal Polynomials, *Amer. Math. Soc. Coll.*, XXIII, 1959.

S. A. N. ENEDUANYA
DEPARTMENT OF MATHEMATICS, FEDERAL UNIVERSITY OF TECHNOLOGY, MINNA, NIGERIA

(Received November 11, 1983.)