Lattice theoretical characterizations of quantum probability space I

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1. The notion of quantum probability space was introduced by SUPPES [6], and later it was investigated by many authors, for example by GUDDER [2].

Let Ω be a nonempty set and let S_σ be a collection of its subsets which satisfy

(i) $\Omega \in S_\sigma$

(ii) if $A \in S_\sigma$, then $A^c = \Omega \setminus A \in S_\sigma$

(iii) if $A_i \in S_\sigma$ $(i = 1, 2, \dots)$, $A_i \cap A_j = \emptyset$ $(i \neq j)$, then $\bigcup_i A_i \in S_\sigma$

Then we call S_σ a *σ-class*, and (Ω, S_σ, p) is called a *quantum probability space,* if p is a nonnegative set function on S_σ such that $p(\Omega)=1$, and $p(\bigcup_i A_i) = \sum_i p(A_i)$ if $A_i \cap A_j = \emptyset$ $(i \neq j)$.

It is easy to see that quantum probability space is more general than classical probability space, but less general than the usual quantum logic. So the first question that arises: which orthomodular σ-poset (partially ordered set) will be isomorphic to a σ-class?

A simple theorem of Gudder gives a characterization of this type of orthomodular σ-poset with the help of two-valued measures on it. See, for example [3], Theorem 3.28. In this paper we shall give lattice theoretical characterizations of those orthomodular lattices that are isomorphic to a class S of a nonempty set Ω with the following properties:

(i) $\Omega \in S$

(ii) if $A \in S$, then $A^c = \Omega \setminus A \in S$

(iii) if $A_1, A_2 \in S$, $A_1 \cap A_2 = \emptyset$, then $A_1 \cup A_2 \in S$

Such an S will be called an *n-class.**)

First we recall the basic notions that we shall use.

2. Let $\mathscr{L}(\vee, \wedge, \perp, 0, 1)$ be a complemented lattice with least and greatest elements 0 an 1, respectively. If the complementation $\perp$ satisfies also (i) $(a^\perp)^\perp = a \forall a \in \mathscr{L}$, and (ii) $b^\perp \leqq a^\perp$ if $a \leqq b$, $a, b \in \mathscr{L}$, then we call $\perp$ *orthocomplementa-*

*) The isomorphism to a σ-class will be examined in Part II.

tion and $\mathscr{L}$ *orthocomplemented lattice*. We remark that the De Morgan laws are valid in every orthocomplemented lattice.

A complemented lattice is *weakly modular* if

$$a\vee b = a\vee\big(a^{\perp}\wedge(a\vee b)\big) \quad \text{for all} \quad a,\, b\in\mathscr{L}.$$

An *orthomodular lattice* is an orthocomplemented and weakly modular lattice. The relation of *orthogonality* ($\perp$) for elements a, b of an orthocomplemented lattice is defined by $a\perp b$ if $a\leqq b^{\perp}$. For elements a, b of an orthocomplemented lattice $\mathscr{L}$ we say that they are *compatible*, in symbols $a\leftrightarrow b$, if there exists a Boolean subalgebra in $\mathscr{L}$ containing a and b.

In the following lemma we collected the most important properties of compatibility.

Lemma 1. *In every orthomodular lattice $\mathscr{L}$ the following implications are true:*

(a) *if one of the three elements a, b, c of $\mathscr{L}$ is compatible with each of the two others, then*

$$(a\vee b)\wedge c = (a\wedge c)\vee(b\wedge c)$$

$$(a\wedge b)\vee c = (a\vee c)\wedge(b\vee c)$$

(b) *if $a\leqq b$ or $a\perp b$, then $a\leftrightarrow b$*

(c) *if $a\leftrightarrow b$, then $a\leftrightarrow b^{\perp}$*

(d) $a\leftrightarrow b\Leftrightarrow(a\vee b^{\perp})\wedge b=a\wedge b$

(e) *$\mathscr{L}$ is a Boolean algebra $\Leftrightarrow$ $a\leftrightarrow b$ for all $a, b\in\mathscr{L}$*

(f) $a\leftrightarrow b_i\Rightarrow a\leftrightarrow\bigvee_i b_i,\ a\leftrightarrow\bigwedge_i b_i$ *if $\bigvee_i b_i$ and $\bigwedge_i b_i$ exist*

Let $\mathscr{L}$ be a complemented lattice. Let us introduce the *star-product* by

$$a*b = (a\vee b^{\perp})\wedge b; \quad a, b\in\mathscr{L}.$$

By an elementary computation one can prove the following

Lemma 2. (*See also Proposition* 3 *in* [4]). *If $\mathscr{L}$ is an orthomodular lattice and $a, b, c\in\mathscr{L}$, then*

(a) $a*a = a$

(b) $c*a = 0$ *if* $c\perp a$

(c) $(c*a)*b = 0$ *if* $a\perp b$

(d) $(c*a)*a = c*a$

(e) $c*(c*a) = c*a$

(f) $c*a = c\wedge a \Leftrightarrow c\leftrightarrow a$

(g) $(c*a)*b = (c*b)*a = c*a$ *if* $a\leqq b$

(h) $\big(\bigvee_i c_i\big)*a = \bigvee_i (c_i*a)$ *if* $c_i\in\mathscr{L}$ *and* $\bigvee_i c_i,\ \bigvee_i (c_i*a)$ *exist*

(i) $(a*b)*c = a*(b*c)$ *if* $c\leftrightarrow b$

Let $\mathscr{L}$ be an orthocomplemented lattice. A mapping m of $\mathscr{L}$ into the real line $\mathbf{R}$ is a *probability measure* if (i) $0 \leqq m(a) \leqq 1 \quad \forall a \in \mathscr{L}$, (ii) $m(1)=1$, (iii) $m(a \vee b) = m(a)+m(b)$, if $a, b \in \mathscr{L}$, and $a \perp b$.

A probability measure m is a *probability σ-measure* if $m\left(\bigvee_{i=1}^{\infty} a_i\right) = \sum_{i=1}^{\infty} m(a_i)$ for all pairwise orthogonal $a_i \in \mathscr{L}$, provided that $\bigvee_{i=1}^{\infty} a_i$ exists. We call a probability measure (probability σ-measure) *a two-valued measure (two-valued σ-measure)*, if it has only two values 0 and 1.

3. We need some new concepts before going on.

Definition. Let $\mathscr{L}$ be a complemented lattice. A nonempty subset $\mathscr{N}$ of $\mathscr{L}$ will be called a *proper $*$-filter* if

(i) $0 \in \mathscr{N}$

(ii) if $a \in \mathscr{N}$ and $a \leqq b$, then $b \in \mathscr{N}$

(iii) if $a, b \in \mathscr{N}$, then $a * b \in \mathscr{N}$

If moreover $\mathscr{N}$ satisfies also

(iv) $a \in \mathscr{N}$ or $a^\perp \in \mathscr{N}$ for every $a \in \mathscr{L}$,

then $\mathscr{N}$ is called a *$*$-ultrafilter*.

Of course, if $\mathscr{N}$ is a $*$-ultrafilter, then it is not a proper subset of a proper $*$-filter of $\mathscr{L}$, so $\mathscr{N}$ is maximal in this sense.

Lemma 3. *If $\mathscr{L}$ is an orthomodular lattice, then a nonempty subset $\mathscr{N}$ of $\mathscr{L}$ is a proper $*$-filter if and only if*

(i) $0 \in \mathscr{N}$;

(ii) *if* $a \in \mathscr{N}$ *and* $a \leqq b$, *then* $b \in \mathscr{N}$;

(iii)′ *if* $a, b \in \mathscr{N}$ *and* $a \leftrightarrow b$, *then* $a \wedge b \in \mathscr{N}$.

Proof. If $\mathscr{N}$ is a proper $*$-filter, then (i) and (ii) hold by definition. If $a, b \in \mathscr{N}$ and $a \leftrightarrow b$, then $a * b = a \wedge b \in \mathscr{N}$. Conversely, if $\mathscr{N}$ satisfies (i), (ii) and (iii)′, then $a, b \in \mathscr{N}$ implies $a \vee b^\perp \in \mathscr{N}$. However, $a \vee b^\perp \leftrightarrow b$, so $a * b = (a \vee b^\perp) \wedge b \in \mathscr{N}$, i.e. $\mathscr{N}$ is a proper $*$-filter, which was to be proved.

A *block* of a complemented lattice $\mathscr{L}$ is a maximal Boolean subalgebra of.

Definition. Let $\hat{\mathbf{B}}_j$, $j \in \mathscr{J}$ be the class of the blocks of a complemented lattice $\mathscr{L}$. Let $\mathscr{J}' \subseteq \mathscr{J}$ and let us suppose that for every $j \in \mathscr{J}'$ there is a maximal filter $\mathscr{N}_j$ in $\hat{\mathbf{B}}_j$ such that

(i) $\bigcup_{j \in \mathscr{J}'} \mathscr{N}_j$ does not contain orthogonal elements

(ii) $\bigcup_{j \in \mathscr{J}'} \mathscr{N}_j$ is maximal, that is for every $i \in \mathscr{J} \setminus \mathscr{J}'$

and for every maximal filter $\mathcal{N}_i$ of $\hat{\mathbf{B}}_i$, $(\bigcup_{j\in\mathcal{J}'}\mathcal{N}_j)\cup\mathcal{N}_i$ has orthogonal elements. In this case $\mathcal{R}=\bigcup_{j\in\mathcal{J}'}\mathcal{N}_j$ will be called a *realization*.

If in the preceding definition $\mathcal{J}'=\mathcal{J}$, then we say that $\mathcal{R}=\bigcup_{j\in\mathcal{J}'}\mathcal{N}_j$ is a *complete realization.*

Between the notions of $*$-ultrafilter and complete realization there is a close connection.

Lemma 4. If $\mathcal{L}$ is an orthomodular lattice and $\mathcal{R}\subset\mathcal{L}$, then the following two statements are equivalent:

(i) $\mathcal{R}$ is a complete realization
(ii) $\mathcal{R}$ is a $*$-ultrafilter

PROOF. Let $\mathcal{R}=\bigcup_{j\in\mathcal{J}}\mathcal{N}_j$ be a complete realization. Then a) $0\notin\mathcal{R}$, b) If $a\in\mathcal{R}$ and $a\leqq b$, then there exists a block $\hat{\mathbf{B}}_j$, $j\in\mathcal{J}$ in $\mathcal{L}$ such that $a, b\in\hat{\mathbf{B}}_j$. If $a\in\mathcal{N}_j$, then $a^\perp\in\mathcal{N}_j$, which contradicts $a\in\mathcal{R}$, so $a\in\mathcal{N}_j$. Hence $b\in\mathcal{N}_j$. c) If $a, b\in\mathcal{R}$, then by the preceding property $a\vee b^\perp\in\mathcal{R}$, and from $a\vee b^\perp\leftrightarrow b$ there exists a block $\hat{\mathbf{B}}_k$, $k\in\mathcal{J}$ such that $a\vee b^\perp$, $b\in\hat{\mathbf{B}}_k$. Then $a\vee b^\perp$, $b\in\mathcal{N}_k\subset\hat{\mathbf{B}}_k$ and consequently $(a\vee b^\perp)\wedge b=a*b\in\mathcal{N}_k\subset\mathcal{R}$. d) If $c\in\mathcal{L}$, then $c\in\mathcal{R}$ or $c^\perp\in\mathcal{R}$.

a), b), c) and d) mean exactly that $\mathcal{R}$ is a $*$-ultrafilter.

Conversely, let us suppose that $\mathcal{R}$ is a $*$-ultrafilter in $\mathcal{L}$. Let $\hat{\mathbf{B}}_j$, $j\in\mathcal{J}$ be the blocks of $\mathcal{L}$ and let $\mathcal{N}_j=\mathcal{R}\cap\hat{\mathbf{B}}_j$, $j\in\mathcal{J}$. Then $\mathcal{N}_j$ is a maximal filter in $\hat{\mathbf{B}}_j$ and $\mathcal{R}=\bigcup_{j\in\mathcal{J}}\mathcal{N}_j$ is a complete realization in $\mathcal{L}$.

Lemma 5. *Let $\mathcal{L}$ be an orthocomplemented lattice. Then*

a) *if m is a two-valued measure on $\mathcal{L}$, then $\mathcal{R}=\{a\in\mathcal{L}|m(a)=1\}$ is a $*$-ultrafilter*

b) *if $\mathcal{R}$ is a $*$-ultrafilter, then $m:\mathcal{L}\to\{0, 1\}$ defined by*

$$m(a)=\begin{cases}1, & a\in\mathcal{R}\\ 0. & a\in\mathcal{L}\setminus\mathcal{R}\end{cases}$$

is a two-valued measure on $\mathcal{L}$.

PROOF. The proof is somewhat trivial and so it will be not presented here.

Definitions. Let $\mathcal{L}$ be a complemented lattice and denote by G a class of subsets of $\mathcal{L}$. We say that G is *order determining* if $a\in A\Rightarrow b\in A$ for all $A\in G$ implies $a\leqq b$. Similarly, a set $\mathfrak{M}$ of probability measures on an orthocomplemented lattice is *order determining* if $m(a)\leqq m(b)$ for all $m\in\mathfrak{M}$ implies $a\leqq b$.

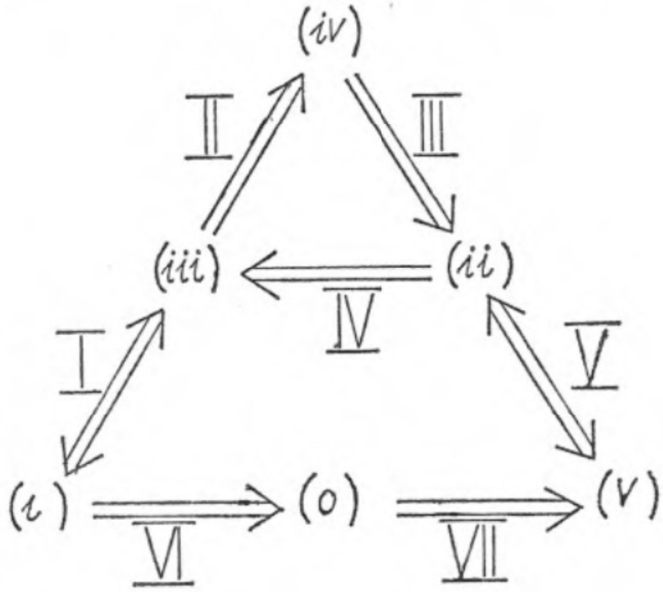
Now we prove our theorem characterizing orthomodular lattices isomorphic to an n-class:

Theorem. *Let $\mathcal{L}$ be a complemented lattice. Then the following six statements are equivalent.*

(0) *$\mathcal{L}$ is orthomodular and isomorphic to an n-class.*
(i) *$\mathcal{L}$ has the following three properties:*

(A) *If $a \in \mathscr{L}$, $a \neq 0$, then there exists a complete realization of $\mathscr{L}$ containing a.*
(B) *The set of the complete realizations of $\mathscr{L}$ is order determining.*
(C) *If $a, b \in \mathscr{L}$ and $a \leqq b$, then there exists a Boolean subalgebra of $\mathscr{L}$ which contains a and b.*
(ii) *$\mathscr{L}$ has the following two properties:*
(A') *If $a \in \mathscr{L}$, $a \neq 0$, then there exists a $*$-ultrafilter which contains a.*
(B') *The set of the $*$-ultrafilters of $\mathscr{L}$ is order determining.*
(iii) *$\mathscr{L}$ is orthomodular and*
(D) *for all $a, b \in \mathscr{L}$, $a \not\perp b$ there exists a complete realization of $\mathscr{L}$ which contains a and b.*
(iv) *$\mathscr{L}$ is orthomodular and*
(D') *for all $a, b \in \mathscr{L}$, $a \not\perp b$ there exists a $*$-ultrafilter of $\mathscr{L}$ which contains a and b.*
(v) *The complementation of $\mathscr{L}$ is an orthocomplementation and the set of two-valued probability measures on $\mathscr{L}$ is order determining.*

PROOF. The sketch of the proof is the following



I. (i)$\Rightarrow$(iii) because (C) implies that $\mathscr{L}$ is orthomodular and if $a, b \in \mathscr{L}$, $a \not\perp b$, then there exists a complete realization of $\mathscr{L}$ containing a. If every such complete realization $\mathscr{R}$ contained also $b^{\perp}$, then from (B) we should get $a \leqq b^{\perp}$. This contradicts $a \not\perp b$, so there exists a complete realization of $\mathscr{L}$ containing a and b.

Conversely, (iii)$\Rightarrow$(i), because the orthomodularity implies (C). Furthermore, if $a \in \mathscr{L}$, $a \neq 0$, then there exists a complete realization containing a and 1, so (A) holds. Finally, if $a \in \mathscr{R} \Rightarrow b \in \mathscr{R}$ for every complete realization $\mathscr{R}$ of $\mathscr{L}$, then there exists no complete realization containing a and b. Therefore $a \perp b^{\perp}$, that is $a \leqq b$. So (B) holds, too.

II. (iii)$\Rightarrow$(iv) follows immediately from Lemma 4.

III. Let us assume that $\mathscr{L}$ is orthomodular and (D') is satisfied in $\mathscr{L}$. Then (A') holds. If $a, b \in \mathscr{L}$ and $a \in \mathscr{R} \Rightarrow b \in \mathscr{R}$ for all $*$-ultrafilters $\mathscr{R}$ containing a, then there exists no $*$-ultrafilter containing a and $b^\perp$. Hence by (D') $a \perp b^\perp$, that is $a \leqq (b^\perp)^\perp = b$, which was to be proved for (B').

IV. Let us assume that (A') and (B') are satisfied in $\mathscr{L}$. If $a, b \in \mathscr{L}$ and $a \leqq b$, then there are two different cases. Firstly, when $b^\perp = 0$, then trivially $b^\perp \leqq a^\perp$. Secondly, when $b^\perp \neq 0$, then there exists a $*$-ultrafilter $\mathscr{R}$ in $\mathscr{L}$ which contains $b^\perp$. But for every such $\mathscr{R}$ we have $a^\perp \in \mathscr{R}$ because $a \in \mathscr{R}$ would contradict $b^\perp \in \mathscr{R}$. Hence by (B') $b^\perp \leqq a^\perp$. On the other hand, there is no $*$-ultrafilter in $\mathscr{L}$ which contains $(0^\perp)^\perp$, so $(0^\perp)^\perp = 0$, and if $a \in \mathscr{L}$, $a \neq 0$, then for every $*$-ultrafilter $\mathscr{R}$ which contains a we have $a^\perp \notin \mathscr{R}$, $(a^\perp)^\perp \in \mathscr{R}$, i.e. $a \leqq (a^\perp)^\perp$. Contrarily, if $(a^\perp)^\perp \in \mathscr{R}'$ for a $*$-ultrafilter $\mathscr{R}'$, then $a^\perp \notin \mathscr{R}'$, $a \in \mathscr{R}'$ which imply $(a^\perp)^\perp = a$. Summarizing our results we can state that $\perp$ is an orthocomplementation.

Now we prove the weakly modularity of $\mathscr{L}$. Let $a, b \in \mathscr{L}$, $a \leqq b$. Then $a, b \wedge a^\perp \leqq b$. Let us assume that $a, b \wedge a^\perp \leqq c$, $c \in \mathscr{L}$. To prove $b = a \vee (b \wedge a^\perp)$ it is sufficient to see $b \leqq c$. Let us assume that $\mathscr{R}$ is a $*$-ultrafilter and $b \in \mathscr{R}$. Then there are two different cases: 1. If $a \in \mathscr{R}$, then $c \in \mathscr{R}$, 2. If $a \notin \mathscr{R}$, then $a^\perp \in \mathscr{R}$, $b * a^\perp = (b \vee a) \wedge a^\perp = b \wedge a^\perp \in \mathscr{R}$. By using $b \wedge a^\perp \leqq c$ we get $c \in \mathscr{R}$, that is $b \in \mathscr{R}$ implies $c \in \mathscr{R}$, which means that $b \leqq c$, which was to be proved. Thus the orthomodularity of $\mathscr{L}$ is proved.

In order to prove (D) it will be sufficient to show that for every $a, b \in \mathscr{L}$, $a \not\leq b$ there is a complete realization $\mathscr{R}$ of $\mathscr{L}$ such that $a, b \in \mathscr{R}$.

Let $a, b \in \mathscr{L}$, $a \not\leq b$. Since $a \neq 0$, we have a $*$-ultrafilter $\mathscr{R}$ in $\mathscr{L}$ containing a. If there is no $*$-ultrafilter containing a and b, then $b^\perp \in \mathscr{R}$. Hence by (B') we have $a \leqq b^\perp$ which is a contradiction. This means that there is a $*$-ultrafilter $\mathscr{R}'$ which contains a and b, and by Lemma 4. $\mathscr{R}'$ is a complete realization.

V. We assume first that $\mathscr{L}$ is an orthocomplemented lattice and the set $\mathfrak{M}_0$ of two-valued probability measures on $\mathscr{L}$ is order determining. Let $a \in \mathscr{L}$, $a \neq 0$, then there exists an element m of $\mathfrak{M}_0$ satisfying $m(a) = 1$, because $m(a) = 0$ for all $m \in \mathfrak{M}_0$ would imply $a = 0$. So $a \in \mathscr{R} = \{x \in \mathscr{L} | m(x) = 1\}$, where $\mathscr{R}$ is a $*$-ultrafilter by Lemma 5, which shows that (A') holds in $\mathscr{L}$.

To prove (B') let us suppose now that $a, b \in \mathscr{L}$, $a \neq 0$ and every $*$-ultrafilter $\mathscr{R}$ containing a contains also b. By Lemma 5. this means that for every $m \in \mathfrak{M}_0$, $m(a) = 1$ implies $m(b) = 1$. Hence $v(a) \leqq v(b)$ for all $v \in \mathfrak{M}_0$. Since $\mathfrak{M}_0$ is order determining we obtain $a \leqq b$. So (B') also holds in $\mathscr{L}$.

Contrarily, let us suppose that (A') and (B') are satisfied in $\mathscr{L}$. Then $\mathscr{L}$ is an orthomodular and consequently orthocomplemented lattice. Since there exists $*$-ultrafilters of $\mathscr{L}$ so $\mathfrak{M}_0 \neq \emptyset$. Moreover, if $a, b \in \mathscr{L}$, $a \neq 0$ and $m(a) \leqq m(b)$ for all $m \in \mathfrak{M}_0$, then for every $*$-ultrafilter $\mathscr{R}$ satisfying $a \in \mathscr{R}$ we can define a two-valued probability measure m:

$$m(x) = \begin{cases} 1, & x \in \mathscr{R} \\ 0, & x \in \mathscr{L} \setminus \mathscr{R} \end{cases}$$

which satisfies $m(a) = 1$. Therefore $m(b) = 1$ and $b \in \mathscr{R}$. With the help of (B') we obtain $a \leqq b$. So $\mathfrak{M}_0$ is order determining.

VI. Let us suppose that $\mathscr{L}$ is satisfies (A), (B) and (C). Denote by $\Omega(\mathscr{L})$ the set of complete realizations of $\mathscr{L}$, and for all $a \in \mathscr{L}$ let

$$X_a = \begin{cases} & \text{if } \ a = 0 \\ \{\mathscr{R} \in \Omega(\mathscr{L}) | a \in \mathscr{R}\} & \text{if } \ a \neq 0 \end{cases}$$

Let $\mathscr{L}'=\{X_a|a\in\mathscr{L}\}$ and $h(a)=X_a$ for all $a\in\mathscr{L}$. Then $\mathscr{L}'$ is an n-class and $h\colon \mathscr{L}\to\mathscr{L}'$ is an isomorphism to $\mathscr{L}'$, because

1. $\Omega(\mathscr{L})\in\mathscr{L}'$ by $X_1=\Omega(\mathscr{L})$.
2. If $a\in\mathscr{L}$, then $X_{a^\perp}=\Omega(\mathscr{L})\setminus X_a$, so $A\in\mathscr{L}'\Rightarrow A'=\Omega\setminus A\in\mathscr{L}'$.
3. Let $a, b\in\mathscr{L}$. Then trivially $a\leqq b\Leftrightarrow X_a\subseteqq X_b$.

The orthomodularity of $\mathscr{L}$ follows from (C).

VII. (0)⇒(v) is somewhat trivial, because if $\mathscr{L}$ is isomorphic to an n-class $\mathscr{L}'$, then $\mathscr{L}'$ is orthocomplemented and the set of two-valued probability measures on $\mathscr{L}'$ is order determining, which implies the same properties in $\mathscr{L}$.

This completes the proof of the Theorem.

Remark. Consider the above defined isomorphism $h\colon \mathscr{L}\to\mathscr{L}'$. The following three conditions are equivalent:

(i) $a \leftrightarrow b$

(ii) $\sup\big(h(a), h(b)\big) = h(a)\cup h(b)$

(iii) $\inf\big(h(a), h(b)\big) = h(a)\cap h(b))$

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