New classes of almost contact metric structures

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In this paper, we introduce two new classes of almost contact structures, called trans-Sasakian and almost trans-Sasakian structures, which are obtained from certain classes of almost Hermitian manifolds closely related to locally conformal Kähler or almost Kähler manifolds, respectively. In particular, although trans-Sasakian structures are normal almost contact metric structures containing both cosymplectic and Sasakian structures, they are different from quasi-Sasakian structures [1], as it is shown constructing explicit examples, and in fact no inclusion relation between these classes exists.

1. *Almost contact metric structures*

A $(2n+1)$-dimensional real differentiable manifold M of class C^∞ is said to have an *almost contact structure* (φ, ξ, η) if it admits a tensor field φ of type $(1, 1)$, a vector field ξ, and a 1-form η such that

$$\eta(\xi) = 1, \quad \varphi^2 = -I+\eta\otimes\xi,$$

where I denotes the identity transformation. Then $\varphi\xi=0$ and $\eta\varphi=0$; moreover, the endomorphism φ has rank $2n$. If M has an almost contact structure (φ, ξ, η), we can find a Riemannian metric g on M such that

$$g(\varphi X, \varphi Y) = g(X, Y)-\eta(X)\eta(Y),$$

where X and Y are vector fields on M. Then g is called a *compatible metric* and M is said to have a (φ, ξ, η, g)*-structure* or an *almost contact metric structure*. Clearly, η is the covariant form of ξ with respect to g, that is $\eta=g(\xi, \cdot)$. The 2-form Φ on M defined by

$$\Phi(X, Y) = g(X, \varphi Y)$$

is called the *fundamental* 2*-form* of the almost contact metric structure.

Let ∇ denote covariant differentiation with respect to the Riemannian connection of g. Then

$$(\nabla_X\eta)Y = g(\nabla_X\xi, Y), \tag{1.1}$$

$$(\nabla_X\eta)Y = (\nabla_X\Phi)(\xi, \varphi Y). \tag{1.2}$$

The exterior derivatives of η and Φ are given by

$$2d\eta(X, Y) = (\nabla_X \eta)Y - (\nabla_Y \eta)X,$$

$$3d\Phi(X, Y, Z) = (\nabla_X \Phi)(Y, Z) + (\nabla_Y \Phi)(Z, X) + (\nabla_Z \Phi)(X, Y).$$

If $\{X_i, \varphi X_i, \xi;\ i=1, \dots, n\}$ is a frame field on an open subset of M, the coderivatives of η and Φ are computed to be

$$\delta\eta = -\sum_{i=1}^{n} \{(\nabla_{X_i}\eta)X_i + (\nabla_{\varphi X_i}\eta)\varphi X_i\}, \tag{1.3}$$

$$\delta\Phi(X) = -\sum_{i=1}^{n} \{(\nabla_{X_i}\Phi)(X_i, X) + (\nabla_{\varphi X_i}\Phi)(\varphi X_i, X)\} - (\nabla_\xi \Phi)(\xi, X). \tag{1.4}$$

An almost contact structure (φ, ξ, η) is said to be *normal* if the almost complex structure J on $M \times \mathbf{R}$ given by

$$J\left(X, a\frac{d}{dt}\right) = \left(\varphi X - a\xi, \eta(X)\frac{d}{dt}\right), \tag{1.5}$$

where a is a C^∞ function on $M \times \mathbf{R}$, is integrable, which is equivalent to the condition $[\varphi, \varphi] + 2d\eta \otimes \xi = 0$, where $[\varphi, \varphi]$ denotes the Nijenhuis torsion of φ.

An almost contact metric structure (φ, ξ, η, g) is said to be:

(a) *cosymplectic* if $d\Phi=0$, $d\eta=0$ and (φ, ξ, η) is normal,
(b) *almost cosymplectic* if $d\Phi=0$ and $d\eta=0$,
(c) *Sasakian* if $\Phi=d\eta$ and (φ, ξ, η) is normal,
(d) *a contact metric structure* if $\Phi=d\eta$,
(e) *quasi-Sasakian* if $d\Phi=0$ and (φ, ξ, η) is normal.

Now, let g be a Riemannian metric on the manifold M with an almost contact structure (φ, ξ, η), and define Riemannian metrics h, h° on $M \times \mathbf{R}$ by

$$h\left(\left(X, a\frac{d}{dt}\right), \left(Y, b\frac{d}{dt}\right)\right) = g(X, Y) + ab \tag{1.6}$$

and

$$h^\circ = e^{2\sigma}h,$$

where $\sigma\colon M \times \mathbf{R} \to \mathbf{R}$ is defined by $\sigma(x, t) = t$ for all $(x, t) \in M \times \mathbf{R}$. Then, the identity of $M \times \mathbf{R}$ is a conformal diffeomorphism between the Riemannian manifolds $(M \times \mathbf{R}, h)$ and $(M \times \mathbf{R}, h^\circ)$. Now, one easily proves

Proposition 1.1. *The following conditions are equivalent:*

(i) g *is compatible with the* (φ, ξ, η)*-structure on* M,
(ii) h *is a Hermitian metric on* $(M \times \mathbf{R}, J)$,
(iii) h° *is a Hermitian metric on* $(M \times \mathbf{R}, J)$.

Hereafter, we suppose that the equivalent conditions of Proposition 1.1 are satisfied.

The Kähler form F of $(M\times\mathbf{R}, J, h)$ is given by

$$F\left(\left(X, a\frac{d}{dt}\right), \left(Y, b\frac{d}{dt}\right)\right) = h\left(\left(X, a\frac{d}{dt}\right), J\left(Y, b\frac{d}{dt}\right)\right),$$

and hence,

$$F\left(\left(X, a\frac{d}{dt}\right), \left(Y, b\frac{d}{dt}\right)\right) = \Phi(X, Y) - b\eta(X) + a\eta(Y), \tag{1.7}$$

and for the Kähler form F° of $(M\times\mathbf{R}, J, h^\circ)$, we have

$$F^\circ\left(\left(X, a\frac{d}{dt}\right), \left(Y, b\frac{d}{dt}\right)\right) = e^{2\sigma}\{\Phi(X, Y) - b\eta(X) + a\eta(Y)\}.$$

The manifold $(M\times\mathbf{R}, J, h)$ (respectively $(M\times\mathbf{R}, J, h^\circ)$) is almost Kählerian if $dF=0$ (resp. $dF^\circ=0$) and, as it is well known, an almost complex manifold is Kählerian if and only if it is almost Kählerian and its almost complex structure is integrable. Thus, taking into account

$$3dF\left(\left(X, a\frac{d}{dt}\right), \left(Y, b\frac{d}{dt}\right), \left(Z, c\frac{d}{dt}\right)\right) = \tag{1.8}$$

$$= 3d\Phi(X, Y, Z) - 2\{cd\eta(X, Y) + ad\eta(Y, Z) + bd\eta(Z, X)\}$$

and

$$3dF^\circ\left(\left(X, a\frac{d}{dt}\right), \left(Y, b\frac{d}{dt}\right), \left(Z, c\frac{d}{dt}\right)\right) = e^{2\sigma}(3d\Phi(X, Y, Z) +$$

$$+ 2c\{\Phi(X, Y) - d\eta(X, Y)\} + 2b\{\Phi(Z, X) - d\eta(Z, X)\} + 2a\{\Phi(Y, Z) - d\eta(Y, Z)\}),$$

we obtain

Proposition 1.2. (i) (φ, ξ, η, g) *is almost cosymplectic if and only if* $(M\times\mathbf{R}, J, h)$ *is almost Kählerian,*

(ii) (φ, ξ, η, g) *is cosymplectic if and only if* $(M\times\mathbf{R}, J, h)$ *is Kählerian,*

(iii) (φ, ξ, η, g) *is a contact metric structure if and only if* $(M\times\mathbf{R}, J, h^\circ)$ *is almost Kählerian,*

(iv) (φ, ξ, η, g) *is Sasakian if and only if* $(M\times\mathbf{R}, J, h^\circ)$ *is Kählerian.*

2. *Trans-Sasakian and almost trans-Sasakian structures*

In the classification of A. Gray and L. M. Hervella [3] of almost Hermitian manifolds, there appear, in particular, two classes named ω_4 and $\omega_2\oplus\omega_4$. The former is a class of Hermitian manifolds which contains locally conformal Kähler manifolds, and the latter is a class of almost Hermitian manifolds which contains locally conformal almost Kähler manifolds. Moreover, these classes are preserved under conformal diffeomorphisms. Thus, $(M\times\mathbf{R}, J, h^\circ)$ belongs to the class ω_4 if and only if $(M\times\mathbf{R}, J, h)$ does too, and this is equivalent to

$$(D_{(X, ad/dt)}F)\left(\left(Y, b\frac{d}{dt}\right), \left(Z, c\frac{d}{dt}\right)\right) = \tag{2.1}$$

$$= -\frac{1}{2n}\left\{h\left(\left(X, a\frac{d}{dt}\right), \left(Y, b\frac{d}{dt}\right)\right)\bar{\delta}F\left(Z, c\frac{d}{dt}\right) - \right.$$

$$-h\left(\left(X, a\frac{d}{dt}\right), \left(Z, c\frac{d}{dt}\right)\right)\bar{\delta}F\left(Y, b\frac{d}{dt}\right) -$$

$$-h\left(\left(X, a\frac{d}{dt}\right), J\left(Y, b\frac{d}{dt}\right)\right)\bar{\delta}F\left(J\left(Z, c\frac{d}{dt}\right)\right) +$$

$$\left. +h\left(\left(X, a\frac{d}{dt}\right), J\left(Z, c\frac{d}{dt}\right)\right)\bar{\delta}F\left(J\left(Y, b\frac{d}{dt}\right)\right)\right\},$$

where D and $\bar{\delta}$ are the Riemannian connection and the coderivative, respectively, of $(M\times\mathbf{R}, J, h)$. Note that $\dim(M\times\mathbf{R})=2n+2$.

Definition 1. An almost contact metric structure (φ, ξ, η, g) is called a *trans-Sasakian structure* if $(M\times\mathbf{R}, J, h)$ (or, equivalently, $(M\times\mathbf{R}, J, h^\circ)$) belongs to the class ω_4 of almost Hermitian manifolds.

Because of (ii) and (iv) of Proposition 1.2, both cosymplectic and Sasakian structures are trans-Sasakian.

Definition 2. An almost contact metric structure (φ, ξ, η, g) is called an *almost trans-Sasakian structure* if the almost Hermitian manifold $(M\times\mathbf{R}, J, h)$ (or, equivalently, $(M\times\mathbf{R}, J, h^\circ)$) belongs to the class $\omega_2\oplus\omega_4$, that is

$$dF = F\wedge\theta, \tag{2.2}$$

where θ is the Lee form of $(M\times\mathbf{R}, J, h)$, given by

$$\theta\left(X, a\frac{d}{dt}\right) = \frac{1}{n}\bar{\delta}F\left(J\left(X, a\frac{d}{dt}\right)\right)$$

If $\{X_1, \dots, X_n, \varphi X_1, \dots, \varphi X_n, \xi\}$ is an orthonormal frame field on an open subset U of M, then $\left\{(X_1, 0), \dots, (X_n, 0), (\varphi X_1, 0), \dots, (\varphi X_n, 0), (\xi, 0), \left(0, \frac{d}{dt}\right)\right\}$ is an orthonormal frame field with respect to h on the open subset $U\times\mathbf{R}$ of $M\times\mathbf{R}$. Now, a simple verification yields

$$\bar{\delta}F\left(X, a\frac{d}{dt}\right) = \delta\Phi(X) - a\delta\eta, \tag{2.3}$$

and hence

$$\theta\left(X, a\frac{d}{dt}\right) = \frac{1}{n}\{\delta\Phi(\varphi X) - \eta(X)\delta\eta - a\delta\Phi(\xi)\}. \tag{2.4}$$

Theorem 2.1. (φ, ξ, η, g) *is trans-Sasakian if and only if*

$$(2.5)\qquad (\nabla_X \Phi)(Y, Z) = -\frac{1}{2n}\{(g(X, Y)\eta(Z)-g(X, Z)\eta(Y))\delta\Phi(\xi)+g(X, \varphi Y)\eta(Z)\delta\eta-g(X, \varphi Z)\eta(Y)\delta\eta\}$$

for any vector fields X, Y, Z *on* M.

Proof. By using (1.5), (1.6), (2.3) and the formula

$$(D_{(X, ad/dt)}F)\left(\left(Y, b\frac{d}{dt}\right), \left(Z, c\frac{d}{dt}\right)\right) = (\nabla_X \Phi)(Y, Z)-c(\nabla_X\eta)Y+b(\nabla_X\eta)Z,$$

the equation (2.1) is separated into the equations

$$\text{(a)}\quad (\nabla_X \Phi)(Y, Z) = -\frac{1}{2n}\{g(X, Y)\delta\Phi(Z)-g(X, Z)\delta\Phi(Y)-g(X, \varphi Y)\delta\Phi(\varphi Z)+ +g(X, \varphi Y)\eta(Z)\delta\eta+g(X, \varphi Z)\delta\Phi(\varphi Y)-g(X, \varphi Z)\eta(Y)\delta\eta\},$$

$$\text{(b)}\quad (\nabla_X\eta)Y = -\frac{1}{2n}\{g(X, Y)\delta\eta-g(X, \varphi Y)\delta\Phi(\xi)+\eta(X)\delta\Phi(\varphi Y)-\eta(X)\eta(Y)\delta\eta\},$$

$$\text{(c)}\quad -\eta(Y)\delta\Phi(\varphi Z)+\eta(Z)\delta\Phi(\varphi Y) = 0,$$

$$\text{(d)}\quad -\delta\Phi(Y)+\eta(Y)\delta\Phi(\xi) = 0.$$

Setting $Z=\xi$ in (c) we obtain $\delta\Phi(\varphi Y)=0$, and hence $\delta\Phi(Y)=\eta(Y)\delta\Phi(\xi)$; therefore (c) implies (d). Conversely, replacing Y by φY in (d), we obtain $\delta\Phi(\varphi Y)=0$ and (c) becomes an identity. Thus, the conditions (c), (d) and $\varphi^*(\delta\Phi)=0$ are mutually equivalent. Now, it is easy to prove that (a) and (d) imply (2.5). On the other hand, from (2.5) and making use of (1.4) we obtain (c) and hence (a). Finally, setting $Z=\xi$ and replacing Y by φY in (2.5), and using the equation (1.2), we obtain (b). Thus the equation (2.5) is equivalent to the system of equations (a)—(d).

Corollary 2.2. *If* (φ, ξ, η, g) *is trans-Sasakian then* $\varphi^*(\delta\Phi)=0$.

Now, we define a 1-form ω on M by $\omega(X)=\theta(X, 0)$, that is,

$$\omega = \frac{1}{n}(\varphi^*(\delta\Phi)-(\delta\eta)\eta),$$

by virtue of (2.4).

Theorem 2.3. (φ, ξ, η, g) *is almost trans-Sasakian if and only if the following conditions are satisfied*:

$$(2.6)\qquad d\Phi = \Phi\wedge\omega, \quad d\eta = \frac{1}{2n}\{\delta\Phi(\xi)\Phi-2\eta\wedge\varphi^*(\delta\Phi)\}.$$

Proof. By (1.7), (1.8), (2.3) and (2.4), the equation (2.2) is separated into the equations (2.6).

Since $\omega_4 = \omega_2 \oplus \omega_4 \cap \omega_3 \oplus \omega_4$ ([3]), where $\omega_3 \oplus \omega_4$ is the class of Hermitian manifolds, an almost contact metric structure is trans-Sasakian if and only if it is almost trans-Sasakian and normal. Then, by Corollary 2.2 and Theorem 2.3, we obtain

Theorem 2.4. *A normal almost contact metric structure (φ, ξ, η, g) is trans-Sasakian if and only if the following conditions are satisfied:*

$$d\Phi = -\frac{1}{n}\,\delta\eta(\Phi \wedge \eta), \quad d\eta = \frac{1}{2n}\,\delta\Phi(\xi)\Phi, \quad \varphi^*(\delta\Phi) = 0.$$

3. *Relationship between quasi-Sasakian and trans-Sasakian structures*

For a quasi-Sasakian structure (φ, ξ, η, g) we have ([1])

$$(\nabla_X \eta)Y = -(\nabla_Y \eta)X. \tag{3.1}$$

This together with (1.3) gives

Proposition 3.1. *If (φ, ξ, η, g) is a quasi-Sasakian structure then $\delta\eta = 0$.*

Theorem 3.2. *A trans-Sasakian structure (φ, ξ, η, g) is quasi-Sasakian if and only if $\delta\eta = 0$.*

PROOF. This follows from Theorem 2.4 and Proposition 3.1.

Lemma 3.3. *Let $\{X_i, \varphi X_i, \xi;\ i = 1, \dots, n\}$ be an orthonormal frame field on an open subset of the manifold M with an almost contact metric structure (φ, ξ, η, g). Then*

$$\delta\Phi(\varphi X) = -3\sum_{i=1}^{n}\{d\Phi(X_i, \varphi X_i, X)\} + \eta(X)\delta\eta - (\nabla_\xi \eta)X$$

for any vector field X on M.

We omit the proof, which is straightforward.

Proposition 3.4. *If (φ, ξ, η, g) is a quasi-Sasakian structure then $\varphi^*(\delta\Phi) = 0$.*

PROOF. It follows from (1.1) and (3.1) that if (φ, ξ, η, g) is a quasi-Sasakian structure then $\nabla_\xi \eta = 0$. The result is now a consequence of Proposition 3.1 and Lemma 3.3.

Theorem 3.5. *A quasi-Sasakian structure (φ, ξ, η, g) is trans-Sasakian if and only if $d\eta = \frac{1}{2n}\,\delta\Phi(\xi)\,\Phi$.*

PROOF. This follows from Theorem 2.4 and Propositions 3.1 and 3.4.

Finally, we shall prove that the classes of quasi-Sasakian and trans-Sasakian structures are not related by inclusion.

For instance, if (M, J, h) is a Kähler manifold and $(\varphi', \xi', \eta', g')$ is a Sasakian structure on a manifold M', then the almost contact metric structure (φ, ξ, η, g) on $M \times M'$ defined by

$$\varphi(X, X') = (JX, \varphi' X'), \quad \xi = (0, \xi'), \quad \eta(X, X') = \eta'(X'),$$
$$g((X, X'), (Y, Y')) = h(X, Y) + g'(X', Y'),$$

is quasi-Sasakian but not trans-Sasakian.

Conversely, let (φ, ξ, η, g) be the almost contact metric structure on $\mathbf{R}^{2n+1}$ given by

$$\varphi\left(\sum_{i=1}^{2n+1} \lambda^i \frac{\partial}{\partial x^i}\right) = \sum_{i=1}^{n} \left\{\lambda^i \frac{\partial}{\partial x^{n+i}} - \lambda^{n+i} \frac{\partial}{\partial x^i}\right\}, \quad \xi = e^{-x^{2n+1}} \frac{\partial}{\partial x^{2n+1}},$$
$$\eta = e^{x^{2n+1}} dx^{2n+1}, \quad g = e^{2x^{2n+1}} k,$$

where $(x^1, \ldots, x^{2n+1})$ are Cartesian coordinates and k is the Euclidean Riemannian metric on $\mathbf{R}^{2n+1}$. Then (φ, ξ, η, g) is a trans-Sasakian structure on $\mathbf{R}^{2n+1}$ which is not quasi-Sasakian. In particular, this also provides an example of trans-Sasakian structure which is neither cosymplectic nor Sasakian.

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