On a class of arithmetical functions

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1. Introduction

Let $\mathscr{H}$ denote the set of all arithmetical functions h such that:

$$h\colon N\times C \to C \tag{1.1}$$

$$\sum_{x\leqq n\leqq y} |h(n, z)| \leqq R(y-x)+O\big(y^{\alpha}(\log x)^{\mathrm{Re}\, z-q-2}\big) \tag{1.2}$$

for $|z|\leqq 1$, $0<\alpha<1$, where $q\geqq 0$ is an arbitrary integer number and $R\geqq 1$ is a fixed real number.

$$\sum_{n=1}^{\infty} \frac{h(n, z)}{n^s} = g(s, z)\zeta^z(s), \quad \text{for} \quad \mathrm{Re}\, s>1, \quad |z|\leqq 1 \tag{1.3}$$

(1.4) $g(s, z)$ is a holomorphic function in the region

$$\left\{\sigma > 1-\frac{c_1}{\log(|t|+2)},\ -\infty<t<+\infty,\ |z|\leqq 1\right\};\quad c_1<\frac{\log 2}{2}$$

$$|g(s, z)|\leqq K, \quad \text{for} \quad \sigma\geqq\sigma_1>\frac{1}{2}, \quad \sigma_1<1, \quad |z|\leqq 1. \tag{1.5}$$

In this paper we shall prove the following theorems:

Theorem 1. *Let $h\in\mathscr{H}$, then there exists a sequence of functions $A_j(z)$ defined and continuous in the circle $|z|\leqq 1$ and holomorphic in the circle $|z|<1$ such that for every integer number $q\geqq 0$ we have*

$$\sum_{n\leqq x} h(n, z) = \sum_{j=0}^{q} xA_j(z)(\log x)^{z-j-1}+O\big(x(\log x)^{\mathrm{Re}\, z-q-2}\big) \tag{1.6}$$

uniformly with respect to $|z|\leqq 1$ as $x\to\infty$, where

(1.7)

$$A_j(z) = \frac{B_j(z)}{\Gamma(z-j)}, \quad B_j(z) = \frac{1}{2\pi i}\int_{|s-1|=\delta<1/2} \frac{H(s, z)}{s(s-1)^j}\, ds \quad \text{for} \quad j=0, 1, 2, \dots, q,$$

$$H(s, z) = g(s, z)\exp\big(z\log((s-1)\zeta(s))\big). \tag{1.8}$$

Theorem 2. *Let*

(i) $|h(n,z)| \leqq R$, *where* $R \geqq 1$ *and* $|z| \leqq 1$,

(ii) $\sum_p \frac{|h(p,z)-z|}{p^\sigma} < \infty$, *for* $\sigma \geqq \sigma_1 > \frac{1}{2}$, $|z| \leqq 1$, $\sigma_1 < 1$,

(iii) $h(k,l,z) = h(k,z)\,h(l,z)$ *for all pairs of natural numbers* k, l *such that* $(k,l)=1$.
If the function h *satisfies the condition* (i)—(iii) *and* h *is holomorphic with respect to* z *in the circle* $|z|<1$, *then* $h \in \mathscr{H}$.

From this theorem we get the following corollary:

Corollary. *Let* $h(n,z) = f(n) z^{F(n)}$, *where* $|z| \leqq 1$ *and*

$$f\colon N \to C, \quad |f(n)| \leqq R; \quad R \geqq 1$$

$$F\colon N \to N \cup \{0\}$$

$f(kl) = f(k) f(l)$, $F(kl) = F(k) + F(l)$ *for all pairs of natural numbers* k, l *such that* $(k,l)=1$ *and if* $f(p) = F(p) = 1$ *for every prime number* p, *then* $h = f(n) z^{F(n)} \in \mathscr{H}$.

We remark that from this Corollary and the Theorem 1 we can get some results which were proved by A. Selberg in the paper [2] ($h = z^{F(n)}$, $F = \Omega$ or ω) and by H. Delange in [1], $(h = f(n) z^{F(n)};\ f\colon N \to \{0, 1\})$.

2. Proof of Theorem 1

In the proof of the Theorem 1 we use the following Lemmas:

Lemma 1. *In the region*

$$\sigma > 1 - \frac{c_1}{\log(|t|+2)}, \quad |t| \geqq 2$$

we have

$$\log \zeta(s) \ll \log\log |t|.$$

Lemma 2. *In the region*

$$\sigma > 1 - \frac{c_1}{\log 4} > \frac{1}{2}, \quad |t| < 2,$$

we have

$$\log(s-1)\zeta(s) = O(1).$$

Applying K. Wiertelak's results ([3]) for the case $K=Q$ and utilizing the standard method for estimating $\log\zeta(s)$ and $\log(s-1)\zeta(s)$ we can get the proofs of Lemma 1 and Lemma 2.

Let $P_z(x) = \sum_{n \leqq x} h(n,z)$, then by Perron's classical formula and (3) we get

$$\int_0^x P_z(t)\,dt = \lim_{T\to\infty} \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{x^{s+1}}{s(s+1)} g(s,z) \zeta^z(s)\,ds,$$

where $c>1$, $x>0$. Putting $\zeta^z(s)g(s,z)=H(s,z)(s-1)^{-z}$ and

$$\Phi(s, z, x) = \frac{x^{s+1}}{s(s+1)} H(s, z)(s-1)^{-z}$$

we obtain

$$\int_0^x P_z(t)\,dt = \lim_{T\to\infty} \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \Phi(s, z, x)\,ds. \tag{2.1}$$

Let $\sigma_1=1-\frac{c_1}{\log 4}=1-\eta>\frac{1}{2}$, then $0<\eta<\frac{1}{2}$ and let $r<\eta$ and $0<\varepsilon<$ $<\operatorname{arc\,tg}\frac{2}{\eta}$. Then by Cauchy's theorem the integral on the right-hand side of (2.1) may by replaced by the integrals $I_1, \dots, I_9$ over the paths $\Gamma_1, \dots, \Gamma_9$ which are defined as follows:

Γ_1, is the segment $\left\langle c-iT,\ 1-\frac{c_1}{\log(|T|+2)}-iT\right\rangle$

Γ_2, is the curve described by $1-\frac{c_1}{\log(|t|+2)}+it$ as t increases from $-T$ to -2.

Γ_3, is the segment $\langle 1-\eta-2i,\ 1-\eta-i\eta\operatorname{tg}\varepsilon\rangle$,

Γ_4, is the segment $\langle 1-\eta-i\eta\operatorname{tg}\varepsilon,\ 1-re^{i\varepsilon}\rangle$

Γ_5, is the arc of the circle $1+re^{i\Theta}$ described as Θ increases from $-\pi+\varepsilon$, to $\pi-\varepsilon$.

Γ_6, is the segment $\langle 1-re^{-i\varepsilon},\ 1-\eta+i\eta\operatorname{tg}\varepsilon\rangle$

Γ_7, is the segment $\langle 1-\eta+i\eta\operatorname{tg}\varepsilon,\ 1-\eta+2i\rangle$

Γ_8, is the curve described by $1-\frac{c_1}{\log(|t|+2)}+it$ as t increases from 2 to T.

Γ_9, is the segment $\left\langle 1-\frac{c_1}{\log(T+2)}+iT,\ c+iT\right\rangle$.

We note that Γ_1, Γ_2, Γ_8 and Γ_9 depend only on T, and do not depend on r or ε. For fixed T and r, and for $\varepsilon\to 0$, Γ_3 and Γ_7 become the segments $\Gamma_3'=\langle 1-\eta-2i, 1-\eta)$, $\Gamma_7'=(1-\eta, 1-\eta+2i\rangle$.

For I_4 and I_6 we have

$$\lim_{\varepsilon\to 0} I_4 = \lim_{\varepsilon\to 0}\int_{\Gamma_4} \Phi(s, z, x)\,ds = \int_{1-\eta}^{1-r} H(\sigma, z)(1-\sigma)^{-z}(e^{-i\pi})^{-z}\frac{x^{\sigma+1}}{\sigma(\sigma+1)}\,d\sigma,$$

$$\lim_{\varepsilon\to 0} I_6 = \lim_{\varepsilon\to 0}\int_{\Gamma_6} \Phi(s, z, x)\,ds = \int_{1-r}^{1-\eta} H(\sigma, z)(1-\sigma)^{-z}(e^{i\pi})^{-z}\frac{x^{\sigma+1}}{\sigma(\sigma+1)}\,d\sigma.$$

If $\gamma_r=\{s: |s-1|=r\}$ excluding the point $1-r$, then

$$\lim_{\varepsilon\to 0} I_4+I_5+I_6 = \int_{\gamma_r}\Phi(s, z, x)\,ds+2i\sin\pi z\int_r^\eta \frac{H(1-u, z)}{(1-u)(2-u)}u^{-z}x^{2-u}\,du$$

which does not depend on the choice of r. If $T\to\infty$ then $I_1\to 0$, and $I_9\to 0$ so that

$$\int_0^x P_z(t)\,dt = S_z(x)+\omega(x, z) \tag{2.2}$$

where

$$S_z(x) = \frac{\sin\pi z}{\pi}\int_r^\eta \frac{H(1-u, z)}{(1-u)(2-u)}x^{2-u}\,du+\frac{1}{2\pi i}\int_{\gamma_r}\Phi(s, z, x)\,ds, \tag{2.3}$$

$$\omega(x, z) = J_2+J_3+J_7+J_8 \tag{2.4}$$

and

$$J_2 = \frac{1}{2\pi i}\int_{-\infty}^{-2}\Phi\left(1-\frac{c_1}{\log(|t|+2)}+it, z, x\right)\left(i+\frac{c_1}{(|t|+2)\log^2(|t|+2)}\right)dt, \tag{2.5}$$

$$J_3 = \frac{1}{2\pi i}\int_{\Gamma_3'}\Phi(s, z, x)\,ds;\quad J_7 = \frac{1}{2\pi i}\int_{\Gamma_7'}\Phi(s, z, x)\,ds, \tag{2.6}$$

$$J_8 = \frac{1}{2\pi i}\int_2^{+\infty}\Phi\left(1-\frac{c_1}{\log(t+2)}+it, z, x\right)\left(i+\frac{c_1}{(t+2)\log^2(t+2)}\right)dt. \tag{2.7}$$

Using Lemma 1 we have on Γ_2 and Γ_8

$$\Phi(s, z, x) \ll x^{2-(c_1/\log(|t|+2))}|t|^{-2}\log|t|$$

and we obtain the following estimate:

$$J_2+J_8 \ll x^2\exp\left(-2A\sqrt{\log x}\right)\int_2^{+\infty} t^{-2+\varepsilon}\log t\,dt \tag{2.8}$$

where $A=\sqrt{\varepsilon c_3}$, $c_3>0$.

Since the integral $\int_2^{+\infty} t^{-2+\varepsilon}\log t\,dt$ is bounded, from (2.8) we get

$$J_2+J_8 \ll x^2\exp\left(-2A\sqrt{\log x}\right). \tag{2.9}$$

On Γ_3' and Γ_7' we have

$$\Phi(s, z, x) \ll x^{2-\eta} = x^2\exp(-\eta\log x)$$

and therefore we obtain

$$J_3+J_7 \ll x^2\exp\left(-2A\sqrt{\log x}\right). \tag{2.10}$$

By (2.4), (2.9) and (2.10) it follows that

$$\omega(x, z) \ll x^2\exp\left(-2A\sqrt{\log x}\right). \tag{2.11}$$

It can be noted that for $|z|\leqq 1$ the function $S_z(x)$ is infinitely differentiable. From (2.3) we get

$$(2.12)\quad S_z''(x) = \frac{\sin \pi z}{\pi}\int_r^\eta H(1-u)u^{-z}x^{-u}\,du+\frac{1}{2\pi i}\int_{\gamma_r} H(s, z)(s-1)^{-z}x^{s-1}\,ds.$$

By Lemma 2 and (1.5) it follows that the function $H(s, z)=g(s, z)\exp\big(z\log(s-1)\zeta(s)\big)$ is bounded in the region $|s-1|\leqq\eta$, $|z|\leqq 1$. Hence, we have

$$(2.13)\qquad \int_r^\eta H(1-u, z)u^{-z}x^{-u}\,du \ll \int_{r\log x}^{\eta\log x} v^{-1}e^{-v}\,dv,$$

where

$$u=\frac{v}{\log x}.$$

For $|s-1|=r<\frac{1}{2}$ we obtain

$$(2.14)\qquad \int_{\gamma_r} H(s, z)(s-1)^{-z}x^{s-1}\,ds \ll x^r.$$

Putting $r=\frac{1}{\log x}$ for x sufficiently large from (2.13) and (2.14) we get

$$(2.15)\qquad S_z''(x) = 0(1).$$

Let $0<\xi<\frac{x}{2}$, $t\in\langle x, x+\xi\rangle$, then

$$(2.16)\qquad \left|\frac{1}{\xi}\int_x^{x+\xi} P_z(t)\,dt-P_z(x)\right| \leqq \frac{1}{\xi}\int_x^{x+\xi} |P_z(t)-P_z(x)|\,dt.$$

From (2.16), (1.2) we get

$$(2.17)\qquad |P_z(t)-P_z(x)| \leqq R\xi+O\big(x^\alpha(\log x)^{\operatorname{Re} z-q-2}\big)$$

where $0\leqq t-x\leqq\xi$ and $0<\xi<x/2$. From (2.16) and (2.17) we get

$$(2.18)\qquad \left|\frac{1}{\xi}\int_x^{x+\xi} P_z(t)\,dt-P_z(x)\right| \leqq R\xi+O\big(x^\alpha(\log x)^{\operatorname{Re} z-q-2}\big).$$

On the other hand we have

$$(2.19)\quad \frac{1}{\xi}\int_x^{x+} P_z(t)\,dt = S_z'(x)+\xi\int_0^1 (1-u)S_z''(x+u\xi)\,du+\frac{\omega(x+\xi, z)-\omega(x, z)}{\xi}.$$

By (2.11) and $0<\xi<\frac{x}{2}$ it follows that

$$(2.20)\qquad \omega(x+\xi, z) \ll x^2\exp\big(-2A\sqrt{\log x}\big)$$

and similarly from (2.15) we get

$$(2.21)\qquad S_z''(x+u\xi) = 0(1).$$

From (2.18), (2.20), (2.21) we obtain

$$(2.22)\quad \left|\frac{1}{\xi}\int_x^{x+\xi} P_z(t)\,dt - S_z'(x)\right| \ll \xi + x(\log x)^{\mathrm{Re}\,z-q-2} + \frac{1}{\xi}x^2\exp\left(-2A\sqrt{\log x}\right).$$

Putting $\xi = x\exp\left(-A\sqrt{\log x}\right)$, from (2.22) we have

$$(2.23)\qquad \left|\frac{1}{\xi}\int_x^{x+\xi} P_z(t)\,dt - S_z'(x)\right| \ll x(\log x)^{\mathrm{Re}\,z-q-2}.$$

Since

$$|P_z(x)-S_z'(x)| \leqq \left|\frac{1}{\xi}\int_x^{x+\xi} P_z(t)\,dt - S_z'(x)\right| + \left||P_z(x) - \frac{1}{\xi}\int_x^{x+\xi} P_z(t)\,dt\right|$$

by (2.23) and (2.18) it follows that

$$(2.24)\qquad |P_z(x)-S_z'(x)| \ll x(\log x)^{\mathrm{Re}\,z-q-2}.$$

To finish the proof there remains to evaluate $S_z'(x)$. For $|s-1|=r<\frac{1}{2}$, $|z|\leqq 1$, we have

$$(2.25)\qquad \frac{H(s,z)}{s} = \sum_{j=0}^{q} B_j(z)(s-1)^j + R_q(s,z)(s-1)^{q+1}$$

for every fixed non-negative integer q, where the $B_j(z)$'s are analytic functions. Cauchy's classical inequality for coefficients of a power series implies that $R_q(s,z)= O(1)$. By (2.25) and (2.3) it follows that

(2.25′)

$$S_z'(x) = \sum_{j=0}^{q} xB_j(z)\left[\frac{\sin\pi(z-j)}{\pi}\int_r^{+\infty} u^{j-z}x^{-u}\,du + \frac{1}{2\pi i}\int_{\gamma_r}(s-1)^{j-z}x^{s-1}\,ds\right] + W(x,z)$$

where

$$(2.26)\qquad W(x,z) = -\sum_{j=0}^{q} xB_j(z)\frac{\sin\pi(z-j)}{\pi}\int_\eta^{+\infty} u^{j-z}x^{-u}\,du +$$

$$+x\frac{\sin\pi(z-q-1)}{\pi}\int_r^{\eta} R_q(1-u,z)u^{q+1-z}x^{-u}\,du +$$

$$+\frac{x}{2\pi i}\int_{\gamma_r} R_q(s,z)(s-1)^{q+1-z}x^{s-1}\,ds.$$

Since, for $0\leqq j\leqq q$ we have

$$\frac{\sin\pi(z-j)}{\pi}\int_r^{+\infty} u^{j-z}x^{-u}\,du + \frac{1}{2\pi i}\int_{\gamma_r} x^{s-1}(s-1)^{j-z}\,ds = \frac{(\log x)^{z-j-1}}{\Gamma(z-j)},$$

by (2.25′) it follows that

$$(2.27)\qquad S_z'(x) = \sum_{j=0}^{q} xB_j(z)\frac{(\log x)^{z-j-1}}{\Gamma(z-j)} + W(x,z).$$

We remark that

$$\int_{\eta}^{+\infty} u^{j-z}x^{-u}\,du \ll (\log x)^{-q-3}, \quad \text{for} \quad 0 \leqq j \leqq q, \quad |z| \leqq 1, \tag{2.28}$$

$$\int_{r}^{\eta} R_q(1-u, z)u^{q+1-z}x^{-u}\,du \ll (\log x)^{\mathrm{Re}\,z-q-2}, \tag{2.29}$$

$$\int_{\gamma_r} R_q(s, z)(s-1)^{q+1-z}x^{s-1}\,ds \ll (\log x)^{\mathrm{Re}\,z-q-2} \tag{2.30}$$

and therefore by (2.28)—(2.30) and (2.26) it follows that

$$W(x, z) \ll x(\log x)^{\mathrm{Re}\,z-q-2}. \tag{2.31}$$

From (2.31), (2.27) and (2.24) we obtain

$$P_z(x) = \sum_{j=0}^{q} x\frac{B_j(z)}{\Gamma(z-j)}(\log x)^{z-j-1} + O\big(x(\log x)^{\mathrm{Re}\,z-q-2}\big) \tag{2.32}$$

where

$$B_j(z) = \frac{1}{2\pi i}\int_{|s-1|=\delta<1/2} \frac{H(s, z)}{s(s-1)^{j+1}}\,ds, \quad \text{for} \quad j = 0, 1, \dots, q$$

and theorem 1 is proved.

3. Proof of Theorem 2

From (i)—(iii) we have

$$\left|\sum_{n=1}^{\infty}\frac{h(n, z)}{n^s}\right| \leqq R\sum_{n=1}^{\infty}\frac{1}{n^{\sigma}}, \quad \text{for} \quad \sigma > 1 \quad \text{and} \quad |z| \leqq 1$$

and

$$\sum_{n=1}^{\infty}\frac{h(n, z)}{n^s} = \prod_p \left(1+\sum_{k=1}^{\infty}\frac{h(p^k, z)}{p^{ks}}\right), \quad \mathrm{Re}\,s = \sigma > 1, \quad |z| \leqq 1. \tag{3.0}$$

We remark that the product

$$\prod_p \left(1-\frac{1}{p^s}\right)^z \left(1+\sum_{k=1}^{\infty}\frac{h(p^k, z)}{p^{ks}}\right) \tag{3.1}$$

has the following form

$$\prod_p \big(1+u_p(s, z)\big)\exp\big(-v_p(s, z)\big) \tag{3.2}$$

where

$$u_p(s, z) = \sum_{k=1}^{\infty}\frac{h(p^k, z)}{p^{ks}},$$

(3.3)

$$v_p(s, z) = z\sum_{k=1}^{\infty}\frac{1}{kp^{ks}}.$$

Since $|u_p(s,z)| \leqq R \frac{1}{p^\sigma - 1} = U_p$, we get

$$\sum_p U_p^2 < \infty \quad \text{for} \quad \sigma \geqq \sigma_1 > \frac{1}{2}. \tag{3.4}$$

In a similar way we obtain

$$|u_p(s,z) - v_p(s,z)| \leqq \frac{|h(p,z)-z|}{p^\sigma} + \overline{V}_p = V_p, \tag{3.5}$$

where

$$\overline{V}_p = R_1 \frac{1}{p^{2\sigma}}.$$

From (3.5) and (ii) we get

$$\sum_p V_p < \infty, \quad \text{for} \quad \sigma \geqq \sigma_1 > \frac{1}{2}. \tag{3.6}$$

By Delange's Lemma (see [1]) it follows that the product (3.2) is absolutely and uniformly convergent in the region $\sigma \geqq \sigma_1 = 1-\eta > \frac{1}{2}$, $|z| \leqq 1$ and is bounded in this region. Let

$$g(s,z) = \prod_p \left(1+u_p(s,z)\right) \exp\left(-v_p(s,z)\right),$$

then

$$g(s,z) = \prod_p \left(1-\frac{1}{p^s}\right)^z \left(1+\sum_{k=1}^{\infty} \frac{h(p^k,z)}{p^{ks}}\right) \tag{3.7}$$

and by (3.0) and (3.7) it follows that

$$\sum_{n=1}^{\infty} \frac{h(n,z)}{n^s} = g(s,z)\zeta^z(s)$$

so that (1.3) is fulfilled. We have $|g(s,z)| \leqq K$ for $\operatorname{Re} s \geqq \sigma_1 = 1-\eta > \frac{1}{2}$, $|z| \leqq 1$, thus (1.5) is satisfied. Since $g(s,z)$ is uniformly convergent, $g(s,z)$ is analytic in the region

$$\sigma > 1 - \frac{c_1}{\log(|t|+2)} > \frac{1}{2}, \quad -\infty < t < +\infty, \quad |z| \leqq 1;$$

$c_1 < \frac{\log 2}{2}$ and the theorem is proved.

Corollary 1 is a simple consequence of the Theorem 2.

References

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(Received February 27, 1984.)