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Two dimensional nonassociative Euclidean nearrings and the ring of hyperbolic numbers

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Abstract. The additive group of the ring $\boldsymbol{H}$ of hyperbolic numbers is the two dimensional Euclidean group and the product $v \circ w$ of two elements v and w is defined by $v \circ w = (v_1 w_1 + v_2 w_2, v_1 w_2 + v_2 w_1)$. The ring $\boldsymbol{H}$ has a nontrivial central idempotent (in fact, it has exactly two) and it has exactly four central involutions. We show that each of these properties characterizes $\boldsymbol{H}$, up to isomorphism, within the class of all those nonassociative topological nearrings with a left identity whose additive group is the two dimensional Euclidean group.

1. Introduction and statement of Main Theorem

A *nonassociative Euclidean nearring* $\boldsymbol{N}$ is a triple $(R^n, +, *)$ where $(R^n, +)$ is the n-dimensional Euclidean group, $*$ is a continuous binary operation on R^n, and the following right distributive law holds:

(RD) $\qquad (a + b) * c = a * c + b * c \quad \text{for all} \quad a, b, c \in R^n$

If we wish to emphasize the dimension, we will refer to $\boldsymbol{N}$ as an n-dimensional nonassociative Euclidean nearring. The definition of nonassociative Euclidean nearring doesn't require the multiplication $*$ to be associative, but it doesn't rule it out either. In case the multiplication $*$ is associative, we will refer to $\boldsymbol{N}$ as a Euclidean nearring. Throughout this paper, we will be concerned entirely with two dimensional nonassociative Euclidean nearrings and $(R^2, +)$ will denote the two dimensional Euclidean topological group.

Now let us turn to the ring of *hyperbolic numbers*. We will denote this ring by $\boldsymbol{H}$. Its additive group is simply the two dimensional Euclidean group. However the product $v \circ w$ of two elements is given by $v \circ w =$

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$(v_1w_1 + v_2w_2, v_1w_2 + v_2w_1)$. The hyperbolic numbers have some quite interesting properties and applications. For a nice exposition of some of these, one is invited to read the article [3] by G. Sobczyk along with the references he lists.

An element of a nonassociative nearring $\boldsymbol{N}$ is *central* if it commutes, with respect to multiplication, with all elements of $\boldsymbol{N}$. An idempotent of $\boldsymbol{N}$ will be referred to as *nontrivial* if it is neither the additive identity nor a multiplicative identity of $\boldsymbol{N}$. If $\boldsymbol{N}$ has a multiplicative identity e, then an *involution* is any element $a \in \boldsymbol{N}$ such that $a^2 = e$. It is an easy matter to verify that the ring $\boldsymbol{H}$ of hyperbolic numbers has exactly exactly two nontrivial central idempotents and exactly four central involutions. The two nontrivial central idempotents are $(\frac{1}{2}, -\frac{1}{2})$ and $(\frac{1}{2}, \frac{1}{2})$ and the four involutions are $(1,0)$, $(-1,0)$, $(0,1)$, and $(0,-1)$. What we find remarkable is the fact that the property of having a nontrivial central idempotent as well as the property of having four central involutions characterizes, up to isomorphism, the ring of hyperbolic numbers within the class of all two dimensional nonassociative Euclidean nearrings which have left identities. Specifically, our purpose here is to verify the following

Main Theorem. *Let $(R^2, +, *)$ be a nonassociative two dimensional Euclidean nearring with a left identity. Then the following statements are equivalent:*

(MT1) *$(R^2, +, *)$ contains a nontrivial central idempotent.*

(MT2) *$(R^2, +, *)$ contains an identity e and exactly two nontrivial central idempotents. Moreover, if c is one nontrivial central idempotent, then the other is $e - c$.*

(MT3) *$(R^2, +, *)$ contains an identity and at least three central involutions.*

(MT4) *$(R^2, +, *)$ contains an identity and exactly four central involutions.*

(MT5) *$(R^2, +, *)$ is isomorphic to $\boldsymbol{H}$, the ring of hyperbolic numbers.*

2. Verification of the Main Theorem

We find it convenient to proceed via several lemmas. For any vector $v \in R^2$, we let $v = (v_1, v_2)$ and we will sometimes represent it as a column vector as well.

Lemma 2.1. *Let $(R^2, +, *)$ be a two dimensional nonassociative Euclidean nearring with left identity $(1,0)$. Then there exist two continuous* *functions f and g from R^2 to the space R of real numbers such that*

$$v * w = (v_1 w_1 + v_2 f(w), v_1 w_2 + v_2 g(w))\,. \tag{2.1.1}$$

Proof. According to Theorem 2.10 of [1], there exist four continuous functions f, g, h and k from R^2 to R such that

$$\begin{aligned} v * w &= \begin{bmatrix} h(w) & f(w) \\ k(w) & g(w) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \\ &= \big(v_1 h(w) + v_2 f(w), v_1 k(w) + v_2 g(w)\big). \end{aligned} \tag{2.1.2}$$

One readily verifies that since $(1,0)$ is a left identity for $(R^2, +, *)$, the functions h and k are given by $h(w) = w_1$ and $k(w) = w_2$ for all $w \in R^2$ so that the multiplication $*$ is indeed given as stated in (2.1.1).

Lemma 2.2. *Let $(R^2, +, *)$ be a two dimensional nonassociative nearring with left identity $(1,0)$ and a nontrivial central idempotent (x,y). Then $y \neq 0$ and the multiplication $*$ is given by*

$$v * w = (v_1 w_1 + a v_2 w_2, v_1 w_2 + v_2 w_1 + b v_2 w_2) \tag{2.2.1}$$

where

$$a = \frac{x - x^2}{y^2} \quad \text{and} \quad b = \frac{1 - 2x}{y}. \tag{2.2.2}$$

Furthermore, $(R^2, +, *)$ is isomorphic to the topological ring $\boldsymbol{H}$ of hyperbolic numbers.

Proof. Since (x,y) is idempotent and multiplication is given by (2.1.1), we have

$$(x,y) = (x,y) * (x,y) = \big(x^2 + y f(x,y), xy + y g(x,y)\big). \tag{2.2.3}$$

If $y = 0$, we must have $x^2 = x$ which means that either $x = 0$ or $x = 1$. In either case we have a contradiction since (x,y) is nontrivial. Consequently, $y \neq 0$ and we conclude from (2.2.3) that

$$f(x,y) = \frac{x - x^2}{y} \quad \text{and} \quad g(x,y) = 1 - x. \tag{2.2.4}$$

Let $c = (x - x^2)/y$ and $d = 1 - x$. Then for all $v \in R^2$, we have $v * (x,y) = (v_1 x + c v_2, v_1 y + d v_2)$ and $(x,y) * v = (x v_1 + y f(v), x v_2 + y g(v))$. Since (x,y) commutes with v, it readily follows that

$$f(v) = \frac{c v_2}{y} \quad \text{and} \quad g(v) = v_1 + \frac{d - x}{y} v_2 \tag{2.2.5}$$

for all $v \in R^2$. Now let

$$a = \frac{c}{y} = \frac{x - x^2}{y^2} \quad \text{and} \quad b = \frac{d-x}{y} = \frac{1-2x}{y} \tag{2.2.6}$$

and (2.2.5) becomes $f(v) = av_2$ and $g(v) = v_1 + bv_2$ for all $v \in R^2$. It follows from this and (2.1.2) that the multiplication $*$ is given as in (2.2.1).

It remains for us to show that $(R^2, +, *)$ is isomorphic to $\boldsymbol{H}$. With a little calculation, one can verify that the mapping φ which is defined by $\varphi(v) = (v_1 + rv_2, sv_2)$ where

$$r = \frac{1-2x}{2y} \quad \text{and} \quad s = \frac{1}{2y}$$

is an isomorphism from $(R^2, +, *)$ onto $\boldsymbol{H}$.

Lemma 2.3. *Let $(R^2, +, *)$ be a two dimensional nonassociative near-ring with identity $(1, 0)$ and at least three central involutions. Then one of these involutions is of the form (x, y) where $y \neq 0$ and the multiplication $*$ is given by*

$$v * w = (v_1w_1 + av_2w_2, v_1w_2 + v_2w_1 - bv_2w_2) \tag{2.3.1}$$

where

$$a = \frac{1-x^2}{y^2} \quad \text{and} \quad b = \frac{2x}{y}. \tag{2.3.2}$$

Furthermore $(R^2, +, *)$ is isomorphic to the topological ring $\boldsymbol{H}$ of hyperbolic numbers.

Proof. The proof of this lemma is similar to that of the previous lemma. According to Lemma (2.1), there exist two continuous maps f and g from R^2 to R such that the multiplication $*$ is given (2.1.1). Let (x, y) be a central involution of $(R^2, +, *)$. We then have

$$(1, 0) = (x, y) * (x, y) = \big(x^2 + yf(x, y), xy + yg(x, y)\big). \tag{2.3.3}$$

It follows from (2.3.3) that

$$x^2 + yf(x, y) = 1 \quad \text{and} \quad xy + yg(x, y) = 0. \tag{2.3.4}$$

If $y = 0$, it readily follows from (2.3.4) that $x = \pm 1$ which means (x, y) is either $(1, 0)$ or $(-1, 0)$. Now $(R^2, +, *)$ has a third central involution by hypothesis so we choose (x, y) to be this involution which means $y \neq 0$. It follows from this and (2.3.4) that

$$f(x, y) = \frac{1-x^2}{y} = c \quad \text{and} \quad g(x, y) = -x. \tag{2.3.5}$$

It then follows from (2.3.5) that $v * (x, y) = (v_1x + cv_2, v_1y - v_2x)$ and $(x, y) * v = \big(xv_1 + yf(v), xv_2 + yg(v)\big)$. Since (x, y) commutes with v, it follows from (2.3.5) that $f(v) = av_2$ and $g(v) = v_1 - bv_2$ where

$$a = \frac{1 - x^2}{y^2} \quad \text{and} \quad b = \frac{2x}{y}. \tag{2.3.6}$$

It now follows from (2.1.1) and our previous observations that the multiplication $*$ is, in this case, given by (2.3.1).

Finally define a map φ from $(R^2, +, *)$ to $\boldsymbol{H}$ by $\varphi(v) = (v_1 + rv_2, sv_2)$ where $r = -x/y$ and $s = 1/y$. A few calculations serve to verify the fact that φ is an isomorphism from $(R^2, +, *)$ onto $\boldsymbol{H}$.

Proof of Main Theorem. Let $(R^2, +, \diamond)$ be any nonassociative two dimensional Euclidean nearring with a left identity l and a nontrivial central idempotent. Then $l \neq 0$ and there exists a linear automorphism φ of R^2 such that $\varphi(1, 0) = l$. Define a multiplication $*$ on R^2 by $v * w = \varphi^{-1}\big(\varphi(v) \diamond \varphi(w)\big)$. It follows readily that φ is an isomorphism from $(R^2, +, *)$ onto $(R^2, +, \diamond)$. Moreover, $(1, 0)$ is a left identity for $(R^2, +, *)$ and it contains a nontrivial central idempotent. Thus, $(R^2, +, *)$ is isomorphic to $\boldsymbol{H}$ by Lemma (2.2) which means $(R^2, +, \diamond)$ is isomorphic to $\boldsymbol{H}$. We have now shown that (MT1) implies (MT5). Now suppose (MT5) holds and let φ be an isomorphism from $\boldsymbol{H}$ onto $(R^2, +, *)$. As we mentioned earlier, it is a simple matter to verify that $\boldsymbol{H}$ has exactly two nontrivial central idempotents and that they are $(\frac{1}{2}, -\frac{1}{2})$ and $(\frac{1}{2}, \frac{1}{2})$. Thus, $e = \varphi(1, 0)$ is the identity of $(R^2, +, *)$ and $c = \varphi(\frac{1}{2}, \frac{1}{2})$ is a nontrivial central idempotent of $(R^2, +, *)$. Moreover, we have $e - c = \varphi\big((1, 0) - (\frac{1}{2}, \frac{1}{2})\big) = \varphi(\frac{1}{2}, -\frac{1}{2})$ and we see that $e - c$ is the remaining nontrivial central idempotent of $(R^2, +, *)$. Consequently, (MT5) implies (MT2) and since it is immediate that (MT2) implies (MT1), we have shown that (MT1), (MT2), and (MT5) are equivalent. Now suppose (MT3) holds. The same technique used in verifying that (MT1) implies (MT5) because of Lemma (2.2) also works in verifying that (MT3) implies (MT5) in view of Lemma (2.3). As we mentioned before $(1, 0)$ is the identity of $\boldsymbol{H}$ and one easily verifies that it has exactly four central involutions which are $(1, 0)$, $(-1, 0)$, $(0, 1)$, and $(0, -1)$. Thus, (MT5) implies (MT4) which, in turn, immediately implies (MT3) and the proof of the main theorem is complete.

Example 2.4. The requirement in the Main Theorem that the idempotents and the involutions be central is absolutely essential in the sense that there are many two dimensional (even associative) Euclidean nearrings, certainly different from the ring of hyperbolic numbers which contain many idempotents and many involutions. For example, let $(R^2, +, *)$ be the nearring where the multiplication $*$ is defined by $v * w = (v_1w_1, v_1w_2 + v_2)$. We showed in [2] that this is the unique, up to isomorphism, two dimensional Euclidean nearring which has an identity and is not zero symmetric. That is, there are elements $v \in (R^2, +, *)$ for which $v * 0 \neq 0$. One can verify that, in addition to the identity $(1, 0)$, the idempotents are precisely those elements v where $v_1 = 0$ and, in addition to $(1, 0)$, the involutions are precisely those elements v such that $v_1 = -1$.

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