Convergence rates in the Marcinkiewicz strong law of large numbers for Banach space valued random variables with multidimensional indices

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1. Introduction

Let $(B, |.|)$ be a real separable Banach space and let $\{X_{\mathbf{n}}, \mathbf{n}\in N^d\}$ be independent identically distributed (i.i.d.) B-valued random variables (r.v.'s), where N^d denotes the positive integer d-dimensional lattice points (d is a positive integer). For $\mathbf{n}, \mathbf{m}\in N^d$, $\mathbf{n}\leqq\mathbf{m}$ and $\mathbf{n}<\mathbf{m}$ are defined coordinatewise and $|\mathbf{n}|=\prod_{i=1}^{d} n_i$ if $\mathbf{n}=(n_1, \dots, n_d)$.

Let $S_{\mathbf{n}}=\sum_{\mathbf{k}\leqq\mathbf{n}} X_{\mathbf{k}}$, $\mathbf{n}\in N^d$. In [5] the following strong law of large numbers (SLLN) has been proved: if B is of type p $(1\leqq p<2)$, $E|X_1|^p(\log^+|X_1|)^{d-1}<\infty$ and $EX_1=0$, then $S_{\mathbf{n}}/|\mathbf{n}|^{1/p}\to 0$ almost surely (a.s.) as $\mathbf{n}\to\infty$. This is a common generalization of results due to ACOSTA [1] and GUT [7] who have proved this law for $d=1$ and for $d\geqq 1$, $B=\mathbf{R}$ (the real numbers) respectively.

The aim of this paper is to give convergence rates in the above SLLN. We shall show that under the assumptions given in this SLLN $\sum_{\mathbf{n}} |\mathbf{n}|^{\alpha r-2}P(|S_{\mathbf{n}}|\geqq|\mathbf{n}|^{\alpha}\varepsilon)<\infty$ for every $\varepsilon>0$, where $\alpha r\geqq 1$, $\alpha>1/2$ and $p\alpha>1$. It will also be proved that $P(|S_{\mathbf{n}}|\geqq|\mathbf{n}|^{\alpha}\varepsilon)=o(|\mathbf{n}|^{1-r\alpha})$ as $\mathbf{n}\to\infty$ if $E|X_1|^r<\infty$, $EX_1=0$, $\alpha r\geqq 1$, $r\geqq 1$ and B is of type p for some $2\geqq p>1/\alpha$.

These results are well-known for $d=1$ and $B=\mathbf{R}$ (see e.g. [4]). For $d\geqq 1$ and $B=\mathbf{R}$ the Marcinkiewicz SLLN and the related convergence rates have been treated by GUT [7]. Convergence rates in the SLLN for B-valued r.v.'s have been obtained by LAI [10] and JAIN [9] if B is arbitrary or B is of type 2 (and $d=1$). AZLAROV and VOLODIN [2] and WOYCZYŇSKI [14] deal with the SLLN and the related convergence rates in B-spaces of type p, and in the case when l_p $(1\leqq p<2)$ is not finitely representable in B respectively. In the case $d=1$ BAKŠTYS and NORVAIŠA [3] have proved more general theorems than our results.

In Section 2 we list some known results that will be used in our proofs. Section 3 deals with convergence rates. We follow the method of GUT [7] and use some ideas of WOYCZYŇSKI [14]. In Section 4 a Chung type SLLN is given.

2. Preliminaries

Let $d(k)=\text{Card}\{\mathbf{n}\colon \mathbf{n}\in N^d, |\mathbf{n}|=k\}$ and $M(x)=\sum_{k\le x} d(k)$. We know that $d(x)=o(x^\sigma))$ as $x\to\infty$ for every positive σ and $M(x)\sim\text{const.}\, x(\log^+ x)^{d-1}$ as $x\to\infty$.

We shall denote by X a B-valued r.v. with the same distribution as X_1. The following lemma of Gut [7] plays a fundamental role in the proofs of Theorems 3.1 and 3.2.

Lemma 2.1. *For $r>0$ and $m=0, 1, 2, \ldots$ the following statements are equivalent:*

$$E|X|^r(\log^+|X|)^{d-1+m}<\infty;$$

$$\sum_{\mathbf{n}} |\mathbf{n}|^{\alpha r-1}(\log|\mathbf{n}|)^m P(|X|\ge|\mathbf{n}|^\alpha\varepsilon)<\infty,\quad \alpha>0,\quad \varepsilon>0.$$

Lemma 2.2. *Let $0<r<p\le 2$ and define $Y_{\mathbf{n}}=X_{\mathbf{n}}\chi\{|X_{\mathbf{n}}|\le\varepsilon|\mathbf{n}|^{1/r}\}$, where $\chi(A)$ denotes the indicator of the event A. If $E|X|^r(\log^+|X|)^{d-1+m}<\infty$, then*

$$\sum_{\mathbf{n}}(\log|\mathbf{n}|)^m E\big||\mathbf{n}|^{-1/r}Y_{\mathbf{n}}\big|^p<\infty \quad \textit{for}\quad m=0, 1, 2, \ldots.$$

Proof. This lemma has been proved in [7] for $p=2$ and $B=\mathbf{R}$.

$$\sum_{\mathbf{n}}(\log|\mathbf{n}|)^m E\big||\mathbf{n}|^{-1/r}Y_{\mathbf{n}}\big|^p \le$$

$$\le \sum_{j=1}^{\infty}(\log j)^m j^{-p/r}\, d(j)\sum_{i=1}^{j} i^{p/r}P(i-1<|X|^r\le i) =$$

$$= \sum_{i=1}^{\infty}\Big(\sum_{j=i}^{\infty}(\log j)^m j^{-p/r}\, d(j)\Big)i^{p/r}P(i-1<|X|^r\le i)\le$$

$$\le \text{const.}\sum_{i=1}^{\infty}\big((\log i)^m i^{-p/r}M(i)\big)i^{p/r}P(i-1<|X|^r\le i)\le$$

$$\le \text{const.}\sum_{i=1}^{\infty}(\log i)^{m+d-1} i P(i-1<|X|^r\le i)<\infty.$$

In the third step we applied Theorem 1 of [6].

We shall use the Marcinkiewicz—Zygmund inequality which is valid also for B-valued r.v.'s:

Lemma 2.3. (Woyczyński [14]). *Let B be of type p $(1\le p\le 2)$ and $q\ge p$. There exists c such that for any X with $E|X|^q<\infty$, $EX=0$ the inequality*

$$E|S_{\mathbf{n}}|^q\le c|\mathbf{n}|^{q/p}E|X|^q$$

holds.

Lemma 2.4. (Jain [9]). *Let $\{X_{\mathbf{n}}, \mathbf{n}\in N^d\}$ be i.i.d. symmetric r.v.'s, let j be a positive integer and $t\ge 0$. Then*

$$P(|S_{\mathbf{n}}|\ge 3^j t)\le A_j|\mathbf{n}|P(|X|\ge t)+B_j[P(|S_{\mathbf{n}}|\ge t)]^{2^j},$$

where A_j and B_j are nonnegative constants which depend only on j.

We need the following version of Lévy's inequality:

Lemma 2.5. *Let $\{X_{\mathbf{n}},\ \mathbf{n}\in N^d\}$ be independent symmetric B-valued r.v.'s. Then*

$$P\big(\max_{\mathbf{j}\leqq\mathbf{n}}|S_{\mathbf{j}}|\geqq t\big)\leqq 2^d P(|S_{\mathbf{n}}|\geqq t),\quad t\geqq 0. \tag{2.1}$$

Proof. This can be proved by induction on d using Theorem 2.3 of [8] (which states 2.1 for $d=1$).

The following lemma is a version of Lemma 2 of [3].

Lemma 2.6. *Let $\{Z_{\mathbf{n}},\mathbf{n}\in N^d\}$ be B-valued r.v.'s and let $Z_{\mathbf{n}}^s=Z_{\mathbf{n}}-Z_{\mathbf{n}}'$ be a symmetrization of this sequence. Let ε and δ be positive numbers and suppose that*

$$P(|Z_{\mathbf{n}}|<\varepsilon/2|\mathbf{n}|^{\alpha})>\delta \tag{2.2}$$

for $\mathbf{n}\nless\mathbf{n}_{\varepsilon,\delta}$. Then there exists $\mathbf{n}_0=\mathbf{n}_0(\varepsilon,\delta)$ such that

$$P\big(\max_{\mathbf{j}\leqq\mathbf{n}}|Z_{\mathbf{j}}^s|\geqq|\mathbf{n}|^{\alpha}\varepsilon/2\big)\geqq\delta P\big(\max_{\mathbf{j}\leqq\mathbf{n}}|Z_{\mathbf{j}}|\geqq\varepsilon|\mathbf{n}|^{\alpha}\big)\quad \text{for}\quad \mathbf{n}\nless\mathbf{n}_0. \tag{2.3}$$

If (2.2) holds for $|\mathbf{n}|\geqq k_0$, then

$$P\left(\sup_{|\mathbf{k}|\geqq j}\frac{|Z_{\mathbf{k}}^s|}{|\mathbf{k}|^{\alpha}}\geqq\frac{\varepsilon}{2}\right)\geqq\delta P\left(\sup_{|\mathbf{k}|\geqq j}\frac{|Z_{\mathbf{k}}|}{|\mathbf{k}|^{\alpha}}\geqq\varepsilon\right)\quad\text{for}\quad j\geqq k_0. \tag{2.4}$$

Proof. Let $\mathbf{n}_0$ be so large that

$$\inf_{\mathbf{j}<\mathbf{n}}P(|Z_{\mathbf{j}}|<\varepsilon/2|\mathbf{n}|^{\alpha})>\delta\quad\text{for}\quad\mathbf{n}\nless\mathbf{n}_0.$$

Since

$$\{|Z_{\mathbf{j}}^s|\geqq|\mathbf{n}|^{\alpha}\varepsilon/2\}\supset\{|Z_{\mathbf{j}}|\geqq|\mathbf{n}|^{\alpha}\varepsilon\}\cap\{|Z_{\mathbf{j}}'|\leqq|\mathbf{n}|^{\alpha}\varepsilon/2\},$$

an application of the "lemma for events" (see [11], p. 246) gives (2.3). The proof of (2.4) is similar.

Remark 2.7. Under the assumptions of the preceding lemma

$$P(|Z_{\mathbf{j}}^s|\geqq|\mathbf{n}|^{\alpha}\varepsilon/2)\geqq\delta P(|Z_{\mathbf{j}}|\geqq|\mathbf{n}|^{\alpha}\varepsilon)\quad\text{for}\quad\mathbf{j}\leqq\mathbf{n}\nless\mathbf{n}_{\varepsilon,\delta}.$$

3. Convergence rates

The following results are Banach space analogues of Gut's theorems [7]. The proofs are similar to the proofs given in [7] and will be not given in full detail.

Theorem 3.1. *Let $\{X_{\mathbf{n}},\ \mathbf{n}\in N^d\}$ be i.i.d. B-valued r.v.'s, let $\alpha r\geqq 1$ and $\alpha>1/2$. Suppose that B is of type p for some $2\geqq p>1/\alpha$. Let us consider the following statements:*

(3.1) $E|X|^r(\log^+|X|)^{d-1}<\infty$ *and if* $r\geqq 1$, *then* $EX=0$;

(3.2) $\sum_{\mathbf{n}}|\mathbf{n}|^{\alpha r-2}P(|S_{\mathbf{n}}|\geqq|\mathbf{n}|^{\alpha}\varepsilon)<\infty$ *for every* $\varepsilon>0$;

(3.3) $\sum_{\mathbf{n}} |\mathbf{n}|^{\alpha r-2} P(\max_{\mathbf{k}\leqq\mathbf{n}} |S_{\mathbf{k}}| \geqq |\mathbf{n}|^{\alpha}\varepsilon) < \infty$ *for every* $\varepsilon>0$;

(3.4) $\sum_{j=1}^{\infty} j^{\alpha r-2} P(\sup_{j\leqq|\mathbf{k}|} |S_{\mathbf{k}}|/|\mathbf{k}|^{\alpha} \geqq \varepsilon) < \infty$ *for every* $\varepsilon>0$.

Then (3.1), (3.2) *and* (3.3) *are equivalent. If* $\alpha r>1$, *then all of these statements are equivalent.*

PROOF. (a) (3.1)⇒(3.2). For $r<1$ this follows from Theorem 4.1 of [7]. If $\alpha r=1$, then by $p\alpha>1$, $p>r$. Using Lemma 2.3 (with $q=p$) we obtain

$$\sum_{\mathbf{n}} |\mathbf{n}|^{\alpha r-2} P(|S_{\mathbf{n}}| \geqq 2\varepsilon|\mathbf{n}|^{\alpha}) \leqq c\varepsilon^{-p} \sum_{\mathbf{n}} E\left||\mathbf{n}|^{-1/r} Y_{\mathbf{n}}\right|^{p} + \sum_{\mathbf{n}} P(|X| \geqq |\mathbf{n}|^{1/r}\varepsilon) < \infty$$

by Lemmas 2.2 and 2.1.

In the case $\alpha r>1$ we can suppose that B is of type p for $1\leqq p\leqq r$. First we assume that X has symmetric distribution. Lemma 2.4 and Lemma 2.3 (with $q=r$) give

$$\sum_{\mathbf{n}} |\mathbf{n}|^{\alpha r-2} P(|S_{\mathbf{n}}| \geqq 3^{j}|\mathbf{n}|^{\alpha}\varepsilon) \leqq$$

$$\leqq A_j \sum_{\mathbf{n}} |\mathbf{n}|^{\alpha r-1} P(|X| \geqq |\mathbf{n}|^{\alpha}\varepsilon) + B_j \left(\frac{c}{\varepsilon} E|X|^{r}\right)^{2^j} \sum_{\mathbf{n}} |\mathbf{n}|^{-\beta},$$

where $\beta=2-\alpha r+r\left(\alpha-\frac{1}{p}\right)2^{j}>1$ for an appropriate j. Thus Lemma 2.1 implies that the above expression is finite.

If X is not symmetric, then we consider a symmetrization $X^{S}=X-X'$ of X. According to Theorem 3.2 of [5] $P\left(|S'_{\mathbf{n}}|<\frac{\varepsilon}{2}|\mathbf{n}|^{\alpha}\right)\geqq\delta$ for $\mathbf{n}\nless\mathbf{n}_{\varepsilon,\delta}$ (because $\alpha>1/p$). Since 3.2 holds for the partial sums $S^{S}_{\mathbf{n}}=S_{\mathbf{n}}-S'_{\mathbf{n}}$ of the symmetrized r.v.'s, by Remark 2.7 (3.2) holds for $S_{\mathbf{n}}$.

(b) (3.2)⇒(3.4) if $\alpha r>1$. First we suppose that X is symmetric. In this case we can follow the method used in Proposition 2.1 of [14]. Put $\beta=\alpha r-2>-1$.

$$\sum_{j=1}^{\infty} j^{\beta} P(\sup_{j\leqq|\mathbf{k}|} |S_{\mathbf{k}}|/|\mathbf{k}|^{\alpha} \geqq \varepsilon 2^{\alpha d}) \leqq \text{const.} \sum_{i=0}^{\infty}\sum_{j=i}^{\infty} 2^{id\beta} 2^{id} P(\max_{2^{jd}<|\mathbf{k}|\leqq 2^{(j+1)d}} |S_{\mathbf{k}}|/|\mathbf{k}|^{\alpha} \geqq \varepsilon 2^{\alpha d}) \leqq$$

$$\leqq \text{(by (5.2) of [7])},$$

$$\text{const.} \sum_{i=0}^{\infty}\sum_{j=i}^{\infty} 2^{id\beta} 2^{id} \sum_{|\mathbf{m}|=2^{(j+2)d}} P(|S_{\mathbf{m}}|/|\mathbf{m}|^{\alpha} \geqq \varepsilon/2^{\alpha d}) \leqq$$

$$\leqq \text{const.} \sum_{j=0}^{\infty} 2^{d\beta j} 2^{dj} \sum_{|\mathbf{m}|=2^{(j+2)d}} P(|S_{\mathbf{m}}|/|\mathbf{m}|^{\alpha} \geqq \varepsilon/2^{\alpha d}).$$

Grouping terms in blocks $\{\mathbf{k}\in N^{d}\colon (2^{l_1},\dots,2^{l_d})\leqq\mathbf{k}<(2^{l_1+1},\dots,2^{l_d+1})\}$ we obtain that the last expression is not greater than

$$\text{const.} \sum_{\mathbf{m}\in N^{d}} |\mathbf{m}|^{\beta} P(|S_{\mathbf{m}}| \geqq \varepsilon/2^{2\alpha d}|\mathbf{m}|^{\alpha}) < \infty.$$

(c) (3.4)$\Rightarrow$(3.1). For $d=1$ this implication follows from Theorem 3.3 of Jain [9]. For $d>1$ it can be proved by the method used in [7].

(d) (3.2)$\Rightarrow$(3.1). First we prove for symmetric r.v.'s. If $\alpha r>1$, then (3.2)$\Rightarrow$(3.4)$\Rightarrow$(3.1). In the case $\alpha r=1$ a similar computation as in [2] (p. 585) shows that

$$P(|S_{\mathbf{n}}| \geqq \varepsilon |\mathbf{n}|^{1/r}) \geqq 2^{-d} C \sum_{\mathbf{k} \leqq \mathbf{n}} P(|X_{\mathbf{k}}| \geqq 2\varepsilon |\mathbf{n}|^{1/r})$$

except for finitely many values of $\mathbf{n}$. Thus $\sum_{\mathbf{n}} P(|X| > \varepsilon |\mathbf{n}|^{1/r}) < \infty$, and Lemma 2.1 shows that (3.1) holds.

If X is not symmetric, then let X^S be a symmetrization of X. By the symmetrization inequality the sequence $S_{\mathbf{n}}^S$ satisfies (3.2). Thus, by what has already been proved, $E|X^S|^r(\log^+|X^S|)^{d-1} < \infty$. According to Lemma 2.6 of [9] this expectation is finite for X, too.

(e) We have proved that (3.2) (or (3.3) and (3.4) if $\alpha r>1$) implies (3.1) without assuming that B is of type p. We also know that (3.2)$\Rightarrow$(3.4) in the symmetric case. Now we remove the symmetry assumption. From (3.2) there follows (3.1) and by Theorem 3.2 of [5] $P(|S_{\mathbf{n}}| > |\mathbf{n}|^{\alpha}\varepsilon) > \delta$ finitely often (B is of type p). An application of Lemma 2.6 gives the result, because (3.4) holds for $S_{\mathbf{n}}^S$.

(f) (3.2)$\Rightarrow$(3.3). In the symmetric case this is a consequence of Lévy's inequality (Lemma 2.5). To remove the symmetry assumption one can argue as in (e).

In the case $\alpha r=1$ the following result is valid.

Theorem 3.2. *Let $\{X_{\mathbf{n}}, \mathbf{n} \in N^d\}$ be B-valued i.i.d. r.v.'s. Suppose that B is of type p for some $2 \geqq p > r > 0$. Then the following statements are equivalent:*

$$E|X|^r(\log^+|X|)^d < \infty \quad \text{and} \quad EX = 0 \quad \text{for} \quad r \geqq 1; \tag{3.5}$$

$$\sum_{\mathbf{n}} |\mathbf{n}|^{-1} \log |\mathbf{n}| P(|S_{\mathbf{n}}| \geqq |\mathbf{n}|^{1/r}\varepsilon) < \infty \quad \text{for every} \quad \varepsilon > 0; \tag{3.6}$$

$$\sum_{\mathbf{n}} |\mathbf{n}|^{-1} \log |\mathbf{n}| P\big(\max_{\mathbf{k} \leqq \mathbf{n}} |S_{\mathbf{k}}| \geqq |\mathbf{n}|^{1/r}\varepsilon\big) < \infty \quad \text{for every} \quad \varepsilon > 0; \tag{3.7}$$

$$\sum_{j=1}^{\infty} j^{-1} P\big(\sup_{j \leqq |\mathbf{k}|} |S_{\mathbf{k}}|/|\mathbf{k}|^{1/r} \geqq \varepsilon\big) < \infty \quad \text{for every} \quad \varepsilon > 0. \tag{3.8}$$

Proof. (3.5)$\Rightarrow$(3.6) and in the symmetric case (3.6)$\Rightarrow$(3.8) can be proved as in Theorem 3.1.

One can prove (3.8)$\Rightarrow$(3.5) without assuming symmetry and that B is of type p. First we note that Theorem 3.3 in [9] implies that $E|X|^r < \infty$ and $EX=0$ if $r \geqq 1$. Using this fact, for $d=1$ the proof is the same as in the real-valued case (Theorem 2 of [4]). For $d>1$ the proof proceeds by induction on d as in [7].

(3.6)$\Rightarrow$(3.8) in the non-symmetric case and (3.6)$\Rightarrow$(3.7) follows as in (e) and (f) resp. of the preceding proof.

Theorem 3.3. *Let $\{X_{\mathbf{n}}, \mathbf{n} \in N^d\}$ be i.i.d. B-valued r.v.'s, $E|X|^r < \infty$ ($\alpha r \geqq 1$) and $EX=0$ if $r \geqq 1$. If B is of type p for some $2 \geqq p > \dfrac{1}{\alpha}$, then for every positive ε:*

$$|\mathbf{n}|^{r\alpha-1} P(|S_{\mathbf{n}}| \geqq |\mathbf{n}|^{\alpha}\varepsilon) \to 0 \quad \text{as} \quad \mathbf{n} \to \infty; \tag{3.9}$$

$$|\mathbf{n}|^{r\alpha-1} P\big(\max_{\mathbf{k} \leqq \mathbf{n}} |S_{\mathbf{k}}| \geqq |\mathbf{n}|^{\alpha}\varepsilon\big) \to 0 \quad \text{as} \quad \mathbf{n} \to \infty. \tag{3.10}$$

If $\alpha r>1$, *then*

$$|\mathbf{n}|^{r\alpha-1}P\left(\sup_{\mathbf{n}\leqq\mathbf{k}}\frac{|S_{\mathbf{k}}|}{|\mathbf{k}|^{\alpha}}\geqq\varepsilon\right)\to 0 \quad as \quad \mathbf{n}\to\infty. \tag{3.11}$$

PROOF. (a) If $r\alpha=1$, then (3.9) is the weak law of large numbers. For $r\alpha>1$ an application of Lemmas 2.4 and 2.3 proves (3.9) in the symmetric case. If X is not symmetric, then one can use symmetrization and Remark 2.7.

(b) We prove the equivalence of (3.9), (3.10) and (3.11) without the assumption that $E|X|^r<\infty$ and that B is of type p. (3.9)$\Rightarrow$(3.10) follows from Lévy's inequality by symmetrization.

To prove (3.9)$\Rightarrow$(3.11) we can assume symmetry. Let $2^{\mathbf{m}-1}<\mathbf{n}\leqq 2^{\mathbf{m}}$, where $2^{\mathbf{m}}=(2^{m_1},\ldots,2^{m_d})\in N^d$ and $\mathbf{1}=(1,\ldots,1)\in N^d$. Then by Lévy's inequality

$$|\mathbf{n}|^{r\alpha-1}P\big(\sup_{\mathbf{k}\geqq\mathbf{n}}|S_{\mathbf{k}}|/|\mathbf{k}|^{\alpha}\geqq\varepsilon\big)\leqq|2^{\mathbf{m}}|^{r\alpha-1}2^d\sum_{\mathbf{j}\geqq\mathbf{m}}P(|S_{2^{\mathbf{j}}}|/|2^{\mathbf{j}}|^{\alpha}\geqq\varepsilon/2^{\alpha d})=$$

$$=2^d\sum_{\mathbf{j}\geqq\mathbf{m}}|2^{\mathbf{m}-\mathbf{j}}|^{r\alpha-1}\{|2^{\mathbf{j}}|^{r\alpha-1}P(|S_{2^{\mathbf{j}}}|/|2^{\mathbf{j}}|^{\alpha}\geqq\varepsilon/2^{\alpha d})\}.$$

(3.9) implies that $|\mathbf{n}|^{r\alpha-1}P(|S_{\mathbf{n}}|\geqq|\mathbf{n}|^{\alpha}\varepsilon)>\delta$ occurs finitely often, thus the above expression converges to 0 as $\mathbf{n}\to\infty$.

4. A Chung type SLLN

In this section we deal with a Chung type SLLN for B-valued r.v.'s with multidimensional indices. In the case $d=1$ WOYCZYŇSKI [14] proved an SLLN more general than our theorem. In [13] Smythe presented a Chung type SLLN for real random variables with multidimensional indices.

In the proof we shall use the following version of Kolmogorov's inequality and the Three Series Theorem respectively.

Lemma 4.1. *Let B be of type p $(1<p\leqq 2)$. Let $\{Y_{\mathbf{n}},\ \mathbf{n}\in N^d\}$ be independent B-valued r.v.'s with $E|Y_{\mathbf{n}}|^p<\infty$ and $EY_{\mathbf{n}}=0$ $(\mathbf{n}\in N^d)$. Then there exists a $B_{p,d}$ such that*

$$P\big(\max_{k\leqq n}|Z_{\mathbf{k}}|>\varepsilon\big)\leqq\frac{B_{p,d}}{\varepsilon^p}\sum_{\mathbf{k}\leqq\mathbf{n}}E|X_{\mathbf{k}}|^p$$

for every $\varepsilon>0$, where $Z_{\mathbf{k}}=\sum_{\mathbf{l}\leqq\mathbf{k}}X_{\mathbf{l}}$.

PROOF. One can prove this lemma with the help of the Doob—Cairoli inequality (see e.g. [7]).

Lemma 4.2. (In the case $d=1$ see [12]). *Besides the assumption of Lemma* 4.1 *let us suppose that $\sum_{\mathbf{n}\in N^d}E|Y_{\mathbf{n}}|^p<\infty$. Let $Z_{\mathbf{m}}^{\mathbf{n}}$ denote the sum $\sum_{\mathbf{m}\leqq\mathbf{k}\leqq\mathbf{n}}Y_{\mathbf{k}}$. Then there exists an event M of zero probability with the following properties:*

(a) *For every $\varepsilon>0$ and $\omega\notin M$ one can obtain a $t_0=t_0(\varepsilon,\omega)$ such that $|Z_{\mathbf{n}}^{\mathbf{m}}(\omega)|<\varepsilon$ if at least one coordinate of $\mathbf{n}$ is greater than t_0.*

(b) *For every $\omega\notin M$ there exists $K=K(\omega)$ such that $|Z_{\mathbf{n}}^{\mathbf{m}}(\omega)|\leqq K$ for* $\mathbf{m}>\mathbf{n}$.

PROOF. An application of Lemma 4.1.

Theorem 4.3. *Let B be of type p $(1<p\leqq 2)$. Let $\{X_{\mathbf{n}}, \mathbf{n}\in N^d\}$ be independent B-valued r.v.'s, $EX_{\mathbf{n}}=0$. If $\sum_{\mathbf{n}} E|X_{\mathbf{n}}|^p/|\mathbf{n}|^p<\infty$, then $S_{\mathbf{n}}/|\mathbf{n}|\to 0$ a.s. as $|\mathbf{n}|\to\infty$.*

PROOF. In Lemma 4.2 let $Y_{\mathbf{n}}=X_{\mathbf{n}}/|\mathbf{n}|$ $(\mathbf{n}\in N^d)$. Then

$$\frac{|S_{\mathbf{m}}(\omega)|}{|\mathbf{m}|}=\frac{1}{|\mathbf{m}|}\Big|\sum_{\mathbf{i}\leqq\mathbf{m}} Z_{\mathbf{i}}^{\mathbf{m}}(\omega)\Big|\leqq\frac{1}{|\mathbf{m}|}\sum_{|\mathbf{i}|\leqq t_0^d}|Z_{\mathbf{i}}^{\mathbf{m}}(\omega)|+\frac{1}{|\mathbf{m}|}\sum_{\substack{|\mathbf{i}|>t_0^d\\ \mathbf{i}\leqq\mathbf{m}}}|Z_{\mathbf{i}}^{\mathbf{m}}(\omega)|\leqq$$

$$\leqq\frac{t_0^d K(\omega)}{|\mathbf{m}|}+\varepsilon<2\varepsilon \quad \text{if} \quad |\mathbf{m}|>\frac{t_0^d K(\omega)}{\varepsilon} \quad (\text{for } \omega\notin M).$$

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