Asymptotic behavior of the nonoscillatory solutions of differential equations with integrable coefficients

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Introduction

The object of this paper is to obtain new sufficient conditions which ensure that any nonoscillatory solution $x(t)$ of the equation

$$(*) \qquad (a(t)x')' + q(t)f(x) = h(t)$$

satisfies either $\liminf_{t\to\infty} |x(t)| = 0$ or $x(t) \to 0$ as $t \to \infty$. Previous results under more restrictive hypotheses than those needed here were obtained by HAMMETT [6] for (*) and by KARTSATOS [8] for an n-th order version of (*). Similar results have also been obtained for (*) and generalizations of (*) in [2—5, 9].

Using techniques motivated by the work of KAMENEV [7], we are able to prove our results without requiring $\int^{\infty} q(s)\,ds = \infty$ and/or $q(t) > 0$ as most other authors have done. Examples illustrating the differences in the results here and previous results are given in the body of the paper.

In the last section we extend some of our results to functional differential equations.

Main results

Consider the equation

$$(1) \qquad (a(t)x')' + q(t)f(x) = h(t)$$

where $a, h, q\colon [t_0, \infty) \to R$ and $f\colon R \to R$ are continuous with $a(t) > 0$. All solutions $x(t)$ of (1) considered here are assumed to be defined on an interval $[T, \infty)$ and to satisfy $\sup |x(t)| > 0$ on every interval $[t, \infty)$, $t \geqq T$. Such a solution will be called oscillatory if its set of zeros is unbounded above and it will be called nonoscillatory

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otherwise. We will also utilize the following conditions as needed:

(2) $$xf(x)>0 \quad \text{for all} \quad x\neq 0;$$

(3) $$\int^{\infty} |h(s)|\,ds<\infty;$$

(4) $$\int^{\infty} [1/a(s)]\,ds=\infty;$$

and for every positive constant ε_1 there exists a positive constant ε_2 such that

(5) $$f'(x)\geqq \varepsilon_2 \quad \text{for all} \quad |x|\geqq \varepsilon_1.$$

Also, for notational purposes, we let $w(t)=a(t)x'(t)/f(x(t))$ for any nonoscillatory solution $x(t)$ of (1).

Lemma 1. *In addition to* (2)—(5), *suppose that*

(6) $$\int^{\infty} q(s)\,ds \quad \text{converges.}$$

If $x(t)$ *is a solution of* (1) *such that* $\liminf_{t\to\infty}|x(t)|>0$, *then*

(7) $$\int^{\infty} [f'(x(s))w^2(s)/a(s)]\,ds<\infty,$$

(8) $$w(t)\to 0 \quad as \quad t\to\infty,$$

and

(9) $$w(t)=\int_t^{\infty} [f'(x(s))w^2(s)/a(s)]\,ds+\int_t^{\infty} [q(s)-h(s)/f(x(s))]\,ds$$

for all sufficiently large t.

PROOF. Let $x(t)$ be a solution of (1) satisfying $\liminf_{t\to\infty}|x(t)|>0$. Then there exist positive constants m, B, and $t_1>t_0$ such that $|x(t)|\geqq B$ and $|f(x(t))|\geqq m$ for $t\geqq t_1$. This, together with (3), implies there exists a constant $M_1>0$ satisfying

(10) $$\int_{t_1}^{t} [h(s)/f(x(s))]\,ds\leqq M_1, \quad t\geqq t_1.$$

From the definition of w, we have $w'(t)+w^2(t)f'(x(t))/a(t)=h(t)/f(x(t))-q(t)$, and integrating we obtain

(11) $$w(z)+\int_t^{z} [w^2(s)f'(x(s))/a(s)]\,ds=w(t)+\int_t^{z} [h(s)/f(x(s))]\,ds-\int_t^{z} q(s)\,ds.$$

Using (6), (10), and (11) we see that for $z\geqq t\geqq t_1$

(12) $$\lim_{z\to\infty}\Big[w(z)+\int_t^{z} [w^2(s)f'(x(s))/a(s)]\,ds\Big]=A$$

where A is finite.

Now suppose that (7) does not hold. Then there exists $t_2 > t_1$ such that for all $z \geqq t_2$

$$\left| w(z) \Big/ \int_t^z \left[w^2(s) f'(x(s))/a(s)\right] ds + 1 \right| < 1/2$$

so that

$$w(z) \Big/ \int_t^z \left[w^2(s) f'(x(s))/a(s)\right] ds < -1/2.$$

Also, $|x(t)| \geqq B > 0$ ensures the existence of a constant $b > 0$ such that $f'(x(t)) \geqq b$. Thus for all $z \geqq t_2$ we have

$$\left(w^2(z) f'(x(z))/a(z)\right) \Big/ \left(\int_t^z \left[w^2(s) f'(x(s))/a(s)\right] ds\right)^2 \geqq b/4a(z)$$

and upon integrating we have

$$(b/4) \int_{t_3}^u [1/a(s)]\, ds \leqq -\left(\int_t^z \left[f'(x(s)) w^2(s)/a(s)\right] ds\right)^{-1} \Bigg|_{t_3}^u$$

which contradicts the assumption that (7) does not hold.

To prove (8), first observe that (7) and (12) together imply that $\lim_{z\to\infty} w(z)$ exists. Now if $w(z) \not\to 0$ as $z \to \infty$, then there exist positive constants B and z_1 so that $w^2(z) > B$ for $z \geqq z_1$. But then

$$\int_{z_1}^t \left[f'(x(s)) w^2(s)/a(s)\right] ds \geqq \int_{z_1}^t [Bb/a(s)]\, ds$$

contradicting (7). Finally (9) follows by letting $z \to \infty$ in (11).

For our next result observe that if (3) and (6) are satisfied then

$$h_0(t) = \left(1/a^{1/2}(t)\right) \int_t^\infty \left(q(s) - |h(s)|/k\right) ds$$

is a well defined function on $[t_0, \infty)$ for every positive constant k in the sense that the improper integral converges. As long as the improper integrals involved converge, we can define

$$h_1(t) = \int_t^\infty h_0^2(s)\, ds$$

and

$$h_{n+1}(t) = \int_t^\infty [h_0(s) + K h_n(s)/a^{1/2}(s)]^2\, ds$$

for $n = 1, 2, 3, \ldots,$ where K is any positive constant. In some of the remaining results, we will need the condition that for every constant $K > 0$ there exists a positive integer N such that

(13) $\qquad h_n$ exists for $n = 0, 1, 2, \ldots, N-1,$ and h_N does not exist.

Theorem 2. *Let* (2)—(6) *and* (13) *hold. If, in addition, for every fixed* $k>0$

(14) $$h_0(t) \geqq 0$$

for all large t, *then every solution* $x(t)$ *of* (1) *is either oscillatory or satisfies* $\liminf_{t\to\infty} |x(t)|=0$.

PROOF. Assume that the conclusion of the theorem is false. Then there exists a solution $x(t)$ of (1) such that $\liminf_{t\to\infty} |x(t)|>0$, and there exists a positive constant k such that $|f(x(t))| \geqq k$ for all sufficiently large t. Hence, in view of (9),

(15) $$w(t) \geqq \int_t^\infty (q(s)-|h(s)|/k)\,ds + \int_t^\infty (f'(x(s))w^2(s)/a(s))\,ds,$$

so $w(t)/a^{1/2}(t) \geqq h_0(t) \geqq 0$ from which we have

(16) $$w^2(t)/a(t) \geqq h_0^2(t).$$

Furthermore, (5), (7), and (14) imply that

(17) $$\int^\infty (w^2(s)/a(s))\,ds < \infty$$

and that there exists a constant $K>0$ such that

(18) $$w(t) \geqq a^{1/2}(t)h_0(t) + K\int_t^\infty (w^2(s)/a(s))\,ds.$$

If $N=1$, then (16) and (17) together imply that $h_1(t)=\int_t^\infty h_0^2(s)\,ds<\infty$ which contradicts the nonexistence of $h_1(t)=h_N(t)$. If $N=2$, then (16) and (18) imply that

$$w(t) \geqq a^{1/2}(t)h_0(t) + K\int_t^\infty h_0^2(s)\,ds = a^{1/2}(t)h_0(t)+Kh_1(t).$$

So

$$w^2(t)/a(t) \geqq (h_0(t)+Kh_1(t)/a^{1/2}(t))^2.$$

But then (17) shows that

$$\int_t^\infty (h_0(s)+Kh_1(s)/a^{1/2}(s))^2\,ds < \infty,$$

which contradicts the assumption that $h_N(t)=h_2(t)$ does not exist. A similar argument leads to a contradiction for any integer $N>2$.

An example of an equation satisfying the hypotheses of Theorem 2 is

$$x''+[(2+\sin t-2t\cos t)/2t^{3/2}]x^3 = (2+\sin t-2t\cos t+4t^{3/2})/2t^{9/2}, \quad t \geqq 1,$$

which has the nonoscillatory solution $x(t)=1/t$. Here

$$\int_t^\infty q(s)\,ds = (2+\sin t)/t^{1/2} \geqq 1/t^{1/2}.$$

It is clear that given any positive constant m, $|h(t)| \leqq 9/2t^3 < m/t^{5/2}$ for $t \geqq \max\{1, 81/4m^2\}$, so that

$$\int_t^\infty (|h(s)|/m)\, ds \leqq 2/3t^{3/2}.$$

Thus $h_0(t) \geqq 1/t^{1/2} - 1/t^{3/2} > 0$. Furthermore,

$$\int_{t_0}^\infty h_0^2(s)\, ds \geqq \int_t^\infty (1/s^{1/2} - 1/s^{3/2})^2\, ds = \infty,$$

so $N=1$.

Theorem 2 differs significantly from previous results of this type in that we do not require $q(t) > 0$ or even $\int^\infty q(s)\, ds = \infty$.

In our next results, for notational purposes, we let $F(x) = \int_0^x f(u)\, du$ and for any function H we let $H(u)_+ = \max\{H(u), 0\}$ and $H(u)_- = \max\{-H(u), 0\}$ so that $H(u) = H(u)_+ - H(u)_-$.

Theorem 3. *In addition to* (2), (4)—(6), (13) *and* (14), *let*

$$q(t) > 0, \tag{19}$$

$$\int^\infty [(a(s)q(s))'_+/a(s)q(s)]\, ds < \infty, \tag{20}$$

and

$$\int^\infty [|h(s)|/(a(s)q(s))^{1/2}]\, ds < \infty \tag{21}$$

hold. Then every solution $x(t)$ *of* (1) *is either oscillatory or satisfies* $x(t) \to 0$ *as* $t \to \infty$.

PROOF. Let $x(t)$ be a nonoscillatory solution of (1). First notice that (20) implies that $a(t)q(t)$ is bounded from above which, in view of (21), implies (3), so the hypotheses of Theorem 2 are satisfied. Thus, if $x(t)$ is eventually monotonic, then by Theorem 2 we have $\lim_{t\to\infty} |x(t)| = \liminf_{t\to\infty} |x(t)| = 0$.

Now suppose that $x(t)$ is not eventually monotonic and that $x(t)$ does not tend to zero as $t \to \infty$. Setting

$$V(t) = a(t)[x'(t)]^2/2q(t) + F(x(t))$$

and differentiating we obtain

$$\begin{aligned} V'(t) &= a(t)x'(t)x''(t)/q(t) + [x'(t)]^2[q(t)a'(t) - a(t)q'(t)]/2q^2(t) + x'(t)f(x(t)) = \\ &= x'(t)h(t)/q(t) - [(a(t)q(t))'/a(t)q(t)][a(t)[x'(t)]^2/2q(t)] \geqq \\ &\geqq x'(t)h(t)/q(t) - [(a(t)q(t))'_+/a(t)q(t)]V(t) \geqq \\ &\geqq -[|h(t)|/(a(t)q(t))^{1/2} + (a(t)q(t))'_+/a(t)q(t)]V(t) - |h(t)|/(a(t)q(t))^{1/2}. \end{aligned}$$

Now (20) and (21), together with our assumptions on $x(t)$, imply that there exist positive constants A and $T>t_0$ such that $F(x(T))\geqq 3A$ and

$$\int_T^\infty [|h(s)|/(a(s)q(s))^{1/2}+(a(s)q(s))'_+/a(s)q(s)]\,ds < \max\{1/2, A/2\}.$$

Setting

$$G(t)=\exp\Big[\int_T^t (|h(s)|/(a(s)q(s))^{1/2}+(a(s)q(s))'_+/a(s)q(s))\,ds\Big]$$

we see that

$$[G(t)V(t)]' \geqq -G(t)|h(t)|/(a(t)q(t))^{1/2}.$$

Integrating the last inequality we obtain

$$G(t)V(t)\geqq V(T)-\int_T^t [G(s)|h(s)|/(a(s)q(s))^{1/2}]\,ds$$

and noting that $V(T)\geqq F(x(T))$ and that G is increasing we have

$$V(t)\geqq F(x(T))/G(t)-\int_T^t [|h(s)|/(a(s)q(s))^{1/2}]\,ds \geqq 3A/e^{1/2}-A/2 > A.$$

Thus we have

$$a(t)[x'(t)]^2/2q(t)+F(x(t)) > A$$

for $t\geqq T$ which contradicts Theorem 2.

As an example, we can conclude from Theorem 3 that every solution $x(t)$ of the equation

$$(t^{3/4}x')'+x^3/t\ln^2 t = [2-7\sin(\ln t)-6\cos(\ln t)]/8t^{3/2}+$$
$$+[2+\sin(\ln t)]^3/t^{7/4}\ln^2 t,\quad t\geqq e$$

is either oscillatory or satisfies $x(t)\to 0$ as $t\to\infty$, whereas none of the results in [2—6, 8, 9] give this conclusion. Notice that $x(t)=[2+\sin(\ln t)]/t^{1/4}$ is a nonoscillatory solution of this equation.

Lemma 4. *Suppose that* (2)—(6), (13), (14), *and* (19) *hold. Then every nonoscillatory solution* $x(t)$ *of* (1) *satisfies* $a(t)x'(t)\to 0$ *as* $t\to\infty$.

Proof. Let $x(t)$ be a nonoscillatory solution of (1), say $x(t)$ is eventually negative. For any positive number ε, it follows from (3) that there exists $t_1\geqq t_0$ such that $\int_{t_1}^\infty |h(s)|\,ds<\varepsilon/2$ for $t\geqq t_1$. Now there also exists $T_0>t_1$ so that $a(T_0)x'(T_0)>0$. To see this we observe first that if no such T_0 exists, then $x'(t)$ is eventually nonpositive. But then, since $x(t)$ is eventually negative, $\liminf_{t\to\infty} x(t)<0$ which contradicts Theorem 2. Integrating (1) yields

$$a(t)x'(t)-a(T_0)x'(T_0)+\int_{T_0}^t q(s)f(x(s))\,ds = \int_{T_0}^t h(s)\,ds$$

from which it follows that

$$a(t)x'(t) \geqq -\int_{T_0}^{t} |h(s)|\,ds > -\varepsilon/2 \tag{22}$$

for $t \geqq T_0$.

Notice next that if t is any number in $[T_0, \infty)$, then there exists $T_1 > t$ such that $a(T_1)x'(T_1) < \varepsilon/2$. Otherwise there exists $t_2 \geqq T_0$ such that $a(t)x'(t) \geqq \varepsilon/2$ for $t > t_2$. Then dividing the last inequality by $a(t)$ and integrating gives $x(t) \geqq \geqq x(t_2) + (\varepsilon/2)\int_{t_2}^{t} [1/a(s)]\,ds$ which, in view of (4), contradicts $x(t)$ being eventually negative. Having chosen $T_1 > t$ satisfying $a(T_1)x'(T_1) < \varepsilon/2$, integrate (1) to obtain

$$a(T_1)x'(T_1) - a(t)x'(t) + \int_{t}^{T_1} q(s)f(x(s))\,ds = \int_{t}^{T_1} h(s)\,ds$$

which implies that

$$-a(t)x'(t) > -\varepsilon/2 - \int_{t}^{T_1} |h(s)|\,ds.$$

Thus we have

$$a(t)x'(t) < \varepsilon/2 + \int_{t}^{T_1} |h(s)|\,ds < \varepsilon$$

for $t \geqq T_0$. The last inequality, together with (22), implies that $a(t)x'(t) \to 0$ as $t \to \infty$.

The argument for the case when $x(t)$ is eventually positive is similar and will be omitted.

For our next theorem we will need the condition that if $|x| > \varepsilon_3 > 0$ then there exists positive constants ε_4 and b such that

$$F(x) - \beta|x| > bF(x) \quad \text{for all} \quad \beta \leqq \varepsilon_4. \tag{23}$$

Theorem 5. *In addition to* (2)—(6), (13), (14), (19), *and* (23), *let*

$$\int^{\infty} [(a(s)q(s))'_+/(a(s)q(s))^2]\,ds < \infty. \tag{24}$$

(i) *If*

$$[h(t)/q(t)] \searrow 0 \quad \text{as} \quad t \to \infty, \tag{25}$$

then every nonoscillatory solution $x(t)$ *of* (1) *is eventually positive and satisfies* $x(t) \to 0$ *as* $t \to \infty$.

(ii) *If*

$$[h(t)/q(t)] \nearrow 0 \quad \text{as} \quad t \to \infty, \tag{26}$$

then every nonoscillatory solution $x(t)$ *of* (1) *is eventually negative and satisfies* $x(t) \to 0$ *as* $t \to \infty$.

Proof. Suppose that (25) holds and let $x(t)$ be a nonoscillatory solution of (1). Since $q(t) > 0$ it follows from (25) that $h(t) \geqq 0$. Suppose that $x(t) < 0$ on $[T, \infty)$.

Integrating (1) on $[T_1, t]$, $t \geqq T$, we obtain

$$a(T_1)x'(T_1) - a(t)x'(t) + \int_t^{T_1} q(s)f(x(s))\,ds = \int_t^{T_1} h(s)\,ds.$$

Then by Lemma 4 have

$$a(t)x'(t) = \int_t^{\infty} q(s)f(x(s))\,ds - \int_t^{\infty} h(s)\,ds < 0.$$

The last inequality implies that $x(t)<0$ and decreasing on $[T, \infty)$ which contradicts Theorem 2. Therefore, $x(t)>0$ on $[T, \infty)$.

To complete the proof of (i), notice first that if $x(t)$ is eventually monotonic, then Theorem 2 implies that $x(t)\to 0$ as $t\to\infty$ and (i) follows in this case. Now suppose that $x(t)$ is not eventually monotonic. If $x(t)$ does not tend to zero as $t\to\infty$, then there exist positive constants ε_3 and A such that $\limsup_{t\to\infty} x(t) > 2\varepsilon_3$ and $F(x(t)) > 3A$ whenever $x(t)>\varepsilon_3$. Then (23) and (24) imply that there exist $T_2>T$ and $b>0$ such that

$$x(T_2) > \varepsilon_3, \quad F(x(T_2)) - h(T_2)x(T_2)/q(T_2) > bF(x(T_2)) > 2Ab,$$

$$(a(t)x'(t))^2 < 1 \quad \text{for} \quad t \geqq T_2, \quad \text{and} \quad \int_{T_2}^{\infty} [(a(s)q(s))'_+/2(a(s)q(s))^2]\,ds < Ab.$$

By Theorem 2 there is an increasing sequence $\{t_n\}$ of zeros of $x'(t)$ such that $t_1>T_2$, and $t_n\to\infty$ and $x(t_n)\to 0$ as $n\to\infty$. Multiplying (1) by $x'(t)/q(t)$ and integrating yields

$$\int_{T_2}^{t_n} [(a(s)x'(s))^2(a(s)q(s))'_+/2(a(s)q(s))^2]\,ds + F(x(t_n)) =$$
$$= a(T_2)[x'(T_2)]^2/2q(T_2) + F(x(T_2)) +$$
$$+ \int_{T_2}^{t_n} [(a(s)x'(s))^2(a(s)q(s))'_-/2(a(s)q(s))^2]\,ds + \int_{T_2}^{t_n} [h(s)x'(s)/q(s)]\,ds.$$

By (25) and a mean-value theorem for integrals, we have

$$\int_{T_2}^{t_n} [h(s)x'(s)/q(s)]\,ds = h(t_n)x(t_n)/q(t_n) - h(T_2)x(T_2)/q(T_2) -$$
$$- x(w)[h(t_n)/q(t_n) - h(T_2)/q(T_2)] \geqq -h(T_2)x(T_2)/q(T_2).$$

Therefore, for each $n \geqq 1$

$$Ab + F(x(t_n)) \geqq F(x(T_2)) - h(T_2)x(T_2)/q(T_2) \geqq 2Ab$$

which is clearly impossible in view of the fact that $F(x(t_n))\to 0$ as $n\to\infty$. Thus the proof of (i) is complete. The proof of (ii) is similar and will be omitted.

The equation

$$x'' + x^3/t\ln^2 t = 1/t\ln^5 t + 2/t^2\ln^3 t + 1/t^2\ln^2 t,$$

which has the nonoscillatory solution $x(t)=1/\ln t$, satisfies all the hypotheses of Theorem 5. None of the results in [2—6, 8, 9] can be applied to this example to give the conclusion of Theorem 5.

Notice that Theorem 3 does not apply to the last equation since (21) is not satisfied. Also observe that neither (25) nor (26) is satisfied by the example following the proof of Theorem 3, so Theorem 5 cannot be applied to that example. Thus we see that Theorems 3 and 5 are independent.

Extensions

In this section we extend Lemma 1 and Theorem 2 to the functional equation

$$[a(t)x'(t)]'+Q\big(t, x(t), x(g(t))\big) = h(t) \tag{27}$$

where $g\colon [t_0, \infty)\to R$ and $Q\colon [t_0, \infty)\times R^2\to R$ are continuous, $g(t)\to\infty$ as $t\to\infty$ and $a(t)$ and $h(t)$ are as before. We will also ask that

$$xQ(t, x, y) \geqq 0 \quad \text{if} \quad xy > 0. \tag{28}$$

Lemma 6. *Let $f\colon R\to R$ be a continuous function satisfying* (2) *and* (5) *and, for $x(t)\neq 0$, let $w(t)$ be defined as before. If* (3), (4), *and* (28) *hold and $x(t)$ is a solution of* (27) *such that $\liminf_{t\to\infty}|x(t)|>0$, then* (7) *and* (8) *hold and*

$$w(t) = \int_t^\infty [f'(x(s))w^2(s)/a(s)]\,ds+\int_t^\infty \big[\big(Q\big(s, x(s), x(g(s))\big)-h(s)\big)/f(x(s))\big]\,ds$$

for all sufficiently large t.

Proof. Let $x(t)$ be a solution of (24) satisfying $\liminf_{t\to\infty}|x(t)|>0$. Then there exists $t_1>\max\{t_0, 0\}$ such that both $|x(t)|$ and $|x(g(t))|$ are positive on $[t_1, \infty)$. Then $x(t)$ is a nonoscillatory solution of the unforced equation

$$[a(t)x'(t)]'+\big[\big(Q\big(t, x(t), x(g(t))\big)-h(t)\big)/f(x(t))\big]f(x(t)) = 0.$$

It is well known, see for example [1], that $x(t)$ nonoscillatory implies that $\lim_{t\to\infty}\int_{t_1}^{\infty} \big[\big(Q\big(s, x(s), x(g(s))\big)-h(s)\big)/f(x(s))\big]\,ds\neq\infty$. Also $\liminf_{t\to\infty}|x(t)|>0$ ensures the existence of positive constants m and $t_2>t_1$ such that $|f(x(t))|\geqq m$ for $t\geqq t_2$. This, together with (3), implies that there exists a constant $M_1>0$ so that $\left|\int_{t_2}^{t} [h(s)/f(x(s))]\,ds\right|\leqq \int_{t_2}^{t} |h(s)|\,ds/m\leqq M_1$ for $t\geqq t_2$. Furthermore,

$$\int_{t_2}^{t} \big[\big(Q\big(s, x(s), x(g(s))\big)-h(s)\big)/f(x(s))\big]\,ds \geqq$$

$$\geqq \int_{t_2}^{t} \big[Q\big(s, x(s), x(g(s))\big)/f(x(s))\big]\,ds-\left|\int_{t_2}^{t} [h(s)/f(x(s))]\,ds\right|,$$

so

$$\int_{t_2}^{t} [Q(s, x(s), x(g(s)))/f(x(s))]\, ds \leqq \int_{t_2}^{t} [(Q(s, x(s), x(g(s))) - h(s))/f(x(s))]\, ds + M_1.$$

It then follows that

$$\int_{t_2}^{\infty} [Q(s, x(s), x(g(s)))/f(x(s))]\, ds < \infty$$

since the integrand is nonnegative. Thus $\int_{t}^{z} [Q(s, x(s), x(g(s)))/f(x(s))]\, ds$ converges as $z \to \infty$ for every $t \geqq t_2$.

Differentiating w we obtain

$$w'(t) + f'(x(t))w^2(t)/a(t) = [h(t) - Q(t, x(t), x(g(t)))]/f(x(t)).$$

Therefore for each $t \geqq t_2$ we have

$$w(z) + \int_{t}^{z} [f'(x(s))w^2(s)/a(s)]\, ds = w(t) + \int_{t}^{z} [h(s) - Q(s, x(s), x(g(s)))/f(x(s))]\, ds,$$

and hence

$$\lim_{z\to\infty} \Big[w(z) + \int_{t}^{z} [f'(x(s))w^2(s)/a(s)]\Big]\, ds = M_2$$

for some constant M_2. Thus there exists a positive constant M_3 such that

$$\Big|w(z) + \int_{t}^{z} [f'(x(s))w^2(s)/a(s)]\Big| < M_3.$$

The remainder of the proof is similar to the latter part of the proof of Lemma 1 and hence will be omitted.

In order to extend Theorem 2 to (27) we need $Q(t, x, y)$ to satisfy the condition that there is a continuous function $f\colon R \to R$ such that for any positive constant k there exists a continuous function $q_k\colon [t_0, \infty) \to R$ such that

$$Q(t, x, y)/f(x) \geqq q_k(t) \quad \text{if} \quad xy > 0 \quad \text{and} \quad |x|, |y| \geqq k. \tag{29}$$

Notice that if in addition to (29), (2)—(5) and (28) hold and $x(t)$ is a solution of (27) satisfying $\liminf_{t\to\infty} |x(t)| > 0$, then there exist positive constants t_2, m, m_1, and B such that $|x(t)| \geqq m_1$, $|x(g(t))| \geqq m_1$, $|f(x(t))| \geqq m$, $|f'(x(t))| \geqq B$, and $[Q(t, x(t), x(g(t))) - h(t)]/f(x(t)) \geqq q_{m_1}(t) - |h(t)|/m$ for $t \geqq t_2$. If we define $h_n(t)$, $n = 0, 1, 2, \dots$, as in Theorem 2, we have:

Theorem 7. *Let* (2)—(6), (13), (14), (28), *and* (29) *hold. Then every nonoscillatory solution* $x(t)$ *of* (27) *satisfies* $\liminf_{t\to\infty} |x(t)| = 0$.

The proof of Theorem 7, with simple modifications, is the same as the proof of Theorem 2. As an example, we can conclude from Theorem 7 that every nonoscillatory solution of the equation

$$x''+\sin h\big(x(t)[2+x^2(t^{1/2})]/[1+x^2(t^{1/2})]\big)/t\ln^2 t = 2/t^3, \quad t\geqq 2$$

satisfies $\liminf_{t\to\infty}|x(t)|=0$. Here $h(t)=2/t^3$, $Q(t,x,y)=\sinh(x[2+y^2]/[1+y^2])/t\ln^2 t$, $f(x)=\sinh x$, and $q(t)=1/t\ln^2 t$. Notice that $h_0(t)=\int_t^\infty [q(s)-|h(s)|/m]\,ds=1/\ln t-$ $-1/mt^2$.

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