# On some conditions for the strong law of large numbers

By A. KUCZMASZEWSKA and D. SZYNAL (Lublin)

**1. Introduction.** Let $\{X_n, n\geqq 1\}$ be a sequence of independent random variables with $EX_n=0$, $\sigma_n^2=\sigma^2 X_n<\infty$, $n\geqq 1$. H. TEICHER [4] has proved that if

(i) $$\sum_k = \sum_{j_k=k}^{\infty} j_k^{-2k}\sigma_{j_k}^2 \sum_{j_{k-1}=k-1}^{j_k-1} \sigma_{j_{k-1}}^2 \cdots \sum_{j_1=1}^{j_2-1} \sigma_{j_1}^2 < \infty, \quad k \geqq 2,$$

(ii) $$n^{-2} \sum_{k=1}^{n} \sigma_k^2 = o(1) \quad \text{as} \quad n \to \infty,$$

(iii) $$\sum_{n=1}^{\infty} P[|X_n| \geqq a_n] < \infty$$

for some positive numerical sequence $\{a_n, n\geqq 1\}$ with $\sum_{n=1}^{\infty} n^{-4}a_n^2\sigma_n^2<\infty$, then $\{X_n, n\geqq 1\}$ satisfies the strong law of large numbers, i.e. $S_n/n\to 0$, a.s., $n\to\infty$, where $S_n=\sum_{j=1}^{n} X_j$.

Note that the Kolmogorov series $\sum_1=\sum_{j=1}^{\infty} j^{-2}\sigma_j^2$ can be considered as the first term of the sequence $\{\sum_k; k\geqq 1\}$ where

$$\sum_1 = \sum_{j=1}^{\infty} j^{-2}\sigma_j^2, \quad \sum_k = \sum_{j_k=k}^{\infty} j_k^{-2k}\sigma_{j_k}^2 \sum_{j_{k-1}=k-1}^{j_k-1} \sigma_{j_{k-1}}^2 \cdots \sum_{j_1=1}^{j_2-1} \sigma_{j_1}^2, \quad k \geqq 2.$$

KAI-LAI CHUNG [2] has proved that if a function $\varphi\colon R\to R^+$ is nonnegative, even, continuous and nondecreasing on $(0,\infty)$ with $\lim_{x\to\infty}\varphi(x)=\infty$ such that (a) $\varphi(x)/x\downarrow$ or (b) $\varphi(x)/x\uparrow$, $\varphi(x)/x^2\downarrow$ as $x\uparrow\infty$, and if $\{X_k, k\geqq 1\}$ is a sequence of independent random variables with $EX_n=0$, $n\geqq 1$, and $\sum_{n=1}^{\infty} E\varphi(X_n)/\varphi(n)<\infty$, then $\{X_n, n\geqq 1\}$ satisfies the strong law of large numbers.

In [5] there have been given Chung—Teicher type conditions for the strong law of large numbers. Namely, it was shown that if $\{X_n, n\geqq 1\}$ is a sequence of independent random variables with $EX_n=0$, $n\geqq 1$, and $\varphi$ is a function satisfying the condition (a) and

$$\text{(I)}\qquad \sum_{j_k=k}^{\infty}\frac{E\varphi^2(X_{j_k})}{j_k^{2(k-1)}\varphi^2(j_k)}\sum_{j_{k-1}=k-1}^{j_k-1} j_{k-1}^2\frac{E\varphi^2(X_{j_{k-1}})}{\varphi^2(j_{k-1})}\cdots$$

$$\cdots\sum_{j_1=1}^{j_2-1} j_1^2\frac{E\varphi^2(X_{j_1})}{\varphi^2(j_1)}<\infty,\quad k\geqq 2,$$

$$\text{(II)}\qquad n^{-2}\sum_{i=1}^{n} i^2E\varphi^2(X_i)/\varphi^2(i)=o(1),$$

$$\text{(III)}\qquad \sum_{n=1}^{\infty}P[|X_n|\geqq a_n]<\infty$$

for some positive numerical sequence $\{a_n, n\geqq 1\}$ with

$$\text{(IV)}\qquad \sum_{n=1}^{\infty}\varphi^2(a_n)E\varphi^2(X_n)/\varphi^4(n)<\infty,$$

or if $\varphi$ is a function satisfying the condition (b) and

$$\text{(I}'\text{)}\qquad \sum_{j_k=k}^{\infty}\frac{E\varphi(X_{j_k})}{j_k^{2(k-1)}\varphi(j_k)}\sum_{j_{k-1}=k-1}^{j_k-1} j_{k-1}^2\frac{E\varphi(X_{j_{k-1}})}{\varphi(j_{k-1})}\cdots$$

$$\cdots\sum_{j_1=1}^{j_2-1} j_1^2\frac{E\varphi(X_{j_1})}{\varphi(j_1)}<\infty,\quad k\geqq 2,$$

$$\text{(II}'\text{)}\qquad n^{-2}\sum_{i=1}^{n} i^2E\varphi(X_i)/\varphi(i)=o(1),$$

and (III) is satisfied for some numerical sequence $\{a_n, n\geqq 1\}$ with

$$\text{(IV}'\text{)}\qquad \sum_{n=1}^{\infty}\varphi(a_n)E\varphi(X_n)/\varphi^2(n)<\infty,$$

then $n^{-1}\sum_{k=1}^{n}(X_k-EX_kI_{[|X_k|<k]})\to 0$ a.s., $n\to\infty$.

Note that for a sequence $\{X_n, n\geqq 1\}$ of independent random variables such that

$$P[X_1=0]=1,\quad P\left[X_j=\pm\frac{j}{(\ln\ln j)^{2/(1+\delta)}}\right]=\frac{(\ln\ln j)^2+1}{j\ln j},\quad 0<\delta<1$$

$$P[X_j=\pm j^3]=\frac{1}{j^{1+\delta}},\quad P[X_j=0]=1-\frac{2}{j^{1+\delta}}-\frac{2(\ln\ln j)^2+2}{j\ln j},$$

$$j\geqq 2\quad\text{and}\quad \varphi(x)=|x|^{(1+\delta)/2}$$

the series

$$\sum_{j=2}^{\infty}j^{-2}\frac{E\varphi^2(X_j)}{\varphi^2(j)}\sum_{i=1}^{j-1}i^2\frac{E\varphi^2(X_i)}{\varphi^2(i)}$$

diverges, so the mentioned results do not allow to verify whether that sequence satisfies or not the strong law of large numbers.

The aim of our considerations is to give some new conditions allowing us to verify the strong law of large numbers for larger classes of random variables than the conditions given above.

## 2. Results

**Theorem 1.** *Let* $\{X_n, n\geqq 1\}$ *be a sequence of independent random variables with* $EX_n=0$, $n\geqq 1$. *Let a function* $\varphi: R\to R^+$ *be nonnegative, even, continuous and nondecreasing on* $(0, \infty)$ *with* $\lim_{x\to\infty}\varphi(x)=\infty$ *such that*

$$\text{(a) } \varphi(x)/x\downarrow \quad\text{or}\quad \text{(b) } \varphi(x)/x\uparrow,\quad \varphi(x)/x^2\downarrow \quad\text{as}\quad x\uparrow\infty.$$

*Suppose that in the case* (a)

$$\text{(A)}\qquad \sum_{j=2}^{\infty} j^{-2}E\frac{\varphi^2(|X_j|)}{\varphi^2(j)+\varphi^2(|X_j|)}\sum_{i=1}^{j-1} i^2E\frac{\varphi^2(|X_i|)}{\varphi^2(i)+\varphi^2(|X_i|)}<\infty,$$

$$\text{(B)}\qquad n^{-2}\sum_{i=1}^{n} i^2E\frac{\varphi^2(|X_i|)}{\varphi^2(i)+\varphi^2(|X_i|)}=o(1),$$

$$\text{(C)}\qquad \sum_{n=1}^{\infty} P[|X_n|\geqq a_n]<\infty$$

*for some positive numerical sequence* $\{a_n, n\geqq 1\}$ *with*

$$\text{(D)}\qquad \sum_{n=1}^{\infty}\varphi^2(a_n)E\frac{\varphi^2(|X_n|)}{\varphi^4(n)+\varphi^4(|X_n|)}<\infty,$$

*or in the case* (b)

$$\text{(A}_1\text{)}\qquad \sum_{j=2}^{\infty} j^{-2}E\frac{\varphi(|X_j|)}{\varphi(j)+\varphi(|X_j|)}\sum_{i=1}^{j-1} i^2E\frac{\varphi(|X_i|)}{\varphi(i)+\varphi(|X_i|)}<\infty,$$

$$\text{(B}_1\text{)}\qquad n^{-2}\sum_{i=1}^{n} i^2E\frac{\varphi(|X_i|)}{\varphi(i)+\varphi(|X_i|)}=o(1),$$

*and* (C) *is satisfied for some numerical sequence* $\{a_n, n\geqq 1\}$ *with*

$$\text{(D}_1\text{)}\qquad \sum_{n=1}^{\infty}\varphi(a_n)E\frac{\varphi(|X_n|)}{\varphi^2(n)+\varphi^2(|X_n|)}<\infty.$$

*Then*

$$(1)\qquad n^{-1}\sum_{k=1}^{n}(X_k-EX_kI_{[|X_k|<k]})\to 0\quad\text{a.s.},\quad n\to\infty.$$

PROOF. Put $X'_n = X_n I_{[|X_n|<n]}$, and let us consider

$$n^{-2}\Big[\sum_{i=1}^{n}(X'_i - EX'_i)\Big]^2 = n^{-2}\sum_{i=1}^{n}(X'_i - EX'_i)^2 + \tag{2}$$

$$+2n^{-2}\sum_{j=2}^{n}(X'_j - EX'_j)\sum_{i=1}^{j-1}(X'_i - EX'_i)$$

Note that in the case (a) $|X'_n|/n \leqq \varphi(|X'_n|)/\varphi(n)$ while in the case (b) $|X'_n|^2/n^2 \leqq \varphi(|X'_n|)/\varphi(n)$. Taking into account these facts and setting $Y_n = (X'_n - EX'_n)^2 I_{[|X_n|<a_n]}$, $Z_n = Y_n - EY_n$ we get in the case (a)

$$\sum_{n=1}^{\infty} E(Z_n/n^2)^2 \leqq \sum_{n=1}^{\infty} E(Y_n^2/n^4) = \sum_{n=1}^{\infty} n^{-4}E(X'_n - EX'_n)^4 I_{[|X_n|<a_n]} \leqq$$

$$\leqq 8\sum_{n=1}^{\infty} n^{-4}E\{(|X'_n|^4 + E^4|X'_n|)I_{[|X_n|<a_n]}\} \leqq 8\sum_{n=1}^{\infty} n^{-4}E|X'_n|^4 I_{[|X_n|<a_n]} +$$

$$+8\sum_{n=1}^{\infty} n^{-4}E^4|X'_n| = 8\sum_{n=1}^{\infty} n^{-4}E|X'_n|^4 I_{[|X_n|<a_n]} +$$

$$+8\sum_{n=1}^{\infty} n^{-4}E(|X'_n|^4 I_{[|X_n|<a_n]}) + 8\sum_{n=1}^{\infty} n^{-4}E(|X'_n|^4 I_{[|X_n|\geqq a_n]}) \leqq$$

$$\leqq 16\sum_{n=1}^{\infty} n^{-4}E|X'_n|^4 I_{[|X_n|<a_n]} + 8\sum_{n=1}^{\infty} P[|X_n| \geqq a_n] \leqq$$

$$\leqq 16\sum_{n=1}^{\infty} E\frac{\varphi^4(|X_n|)I_{[|X_n|<n]}I_{[|X_n|<a_n]}}{\varphi^4(n)} + 8\sum_{n=1}^{\infty} P[|X_n| \geqq a_n] \leqq$$

$$\leqq 32\sum_{n=1}^{\infty} E\frac{\varphi^4(|X_n|)}{\varphi^4(n)+\varphi^4(|X_n|)} I_{[|X_n|<a_n]} + 8\sum_{n=1}^{\infty} P[|X_n| \geqq a_n] \leqq$$

$$\leqq 32\sum_{n=1}^{\infty} \varphi^2(a_n)E\frac{\varphi^2(|X_n|)}{\varphi^4(n)+\varphi^4(|X_n|)} + 8\sum_{n=1}^{\infty} P[|X_n| \geqq a_n] < \infty,$$

and in the case (b)

$$\sum_{n=1}^{\infty} E(Z_n/n^2)^2 \leqq 16\sum_{n=1}^{\infty} n^{-4}E|X'_n|^4 I_{[|X_n|<a_n]} +$$

$$+8\sum_{n=1}^{\infty} P[|X_n| \geqq a_n] \leqq 16\sum_{n=1}^{\infty} E\frac{\varphi^2(|X_n|)}{\varphi^2(n)} I_{[|X_n|<a_n]} +$$

$$+8\sum_{n=1}^{\infty} P[|X_n| \geqq a_n] \leqq 32\sum_{n=1}^{\infty} E\frac{\varphi^2(|X_n|)}{\varphi^2(n)+\varphi^2(|X_n|)} I_{[|X_n|<a_n]} +$$

$$+8\sum_{n=1}^{\infty} P[|X_n| \geqq a_n] \leqq 32\sum_{n=1}^{\infty} \varphi(a_n)E\frac{\varphi(|X_n|)}{\varphi^2(n)+\varphi^2(|X_n|)} + 8\sum_{n=1}^{\infty} P[|X_n| \geqq a_n] < \infty.$$

This proves that

$$\sum_{n=1}^{\infty} (Z_n/n^2) < \infty \quad \text{a.s.},$$

which implies by Kronecker's lemma

$$n^{-2} \sum_{i=1}^{n} (Y_i - EY_i) \to 0 \quad \text{a.s.}, \quad n \to \infty.$$

Moreover, we note that in the case (a) the assumption (B) implies

$$n^{-2} \sum_{i=1}^{n} EY_i = n^{-2} \sum_{i=1}^{n} E(X_i' - EX_i')^2 I_{[|X_i| < a_i]} \leqq$$

$$\leqq 4n^{-2} \sum_{i=1}^{n} E(X_i)^2 I_{[|X_i| < a_i]} \leqq 4n^{-2} \sum_{i=1}^{n} i^2 E(X_i^2/i^2) \leqq$$

$$\leqq 8n^{-2} \sum_{i=1}^{n} i^2 E \frac{\varphi^2(|X_i|)}{\varphi^2(i) + \varphi^2(|X_i|)} = o(1),$$

while in the case (b) the assumption ($B_1$) implies

$$n^{-2} \sum_{i=1}^{n} EY_i \leqq 8n^{-2} \sum_{i=1}^{n} i^2 E \frac{\varphi(|X_i|)}{\varphi(i) + \varphi(|X_i|)} = o(1).$$

Thus in the case (a) or in the case (b)

$$n^{-2} \sum_{i=1}^{n} (X_i' - EX_i')^2 \xrightarrow{\text{a.s.}} 0 \quad \text{as} \quad n \to \infty. \tag{3}$$

Put now

$$U_{2j} = (X_j' - EX_j') \sum_{i=1}^{j-1} (X_i' - EX_i').$$

Then $EU_{2j} = 0$, and by (A) in the case (a)

$$\sum_{j=2}^{\infty} E(U_{2j}/j^2)^2 =$$

$$= \sum_{j=2}^{\infty} Ej^{-4}\big[(X_j' - EX_j') \sum_{i=1}^{j-1} (X_i' - EX_i')\big]^2 \leqq \sum_{j=2}^{\infty} j^{-2} E \frac{(X_j')^2}{j^2} \sum_{i=1}^{j-1} i^2 E \frac{(X_i')^2}{i^2} \leqq$$

$$\leqq \sum_{j=2}^{\infty} j^{-2} E \frac{\varphi^2(|X_j'|)}{\varphi^2(j)} \sum_{i=1}^{j-1} i^2 E \frac{\varphi^2(|X_i'|)}{\varphi^2(i)} \leqq$$

$$\leqq 2^2 \sum_{j=2}^{\infty} j^{-2} E \frac{\varphi^2(|X_j|)}{\varphi^2(j) + \varphi^2(|X_j|)} \sum_{i=1}^{j-1} i^2 E \frac{\varphi^2(|X_i|)}{\varphi^2(i) + \varphi^2(|X_i|)} < \infty$$

while in the case (b), by ($A_1$),

$$\sum_{j=2}^{\infty} E(U_{2j}/j^2)^2 \leqq \sum_{j=2}^{\infty} j^{-2} E\frac{(X_j')^2}{j^2} \sum_{i=1}^{j-1} i^2 E\frac{(X_i')^2}{i^2} \leqq$$

$$\leqq 2^2 \sum_{j=2}^{\infty} j^{-2} E\frac{\varphi(|X_j|)}{\varphi(j)+\varphi(|X_j|)} \sum_{i=1}^{j-1} i^2 E\frac{\varphi(|X_i|)}{\varphi(i)+\varphi(|X_i|)} < \infty.$$

But $\{U_{2j},\ j\geqq 2\}$ is a martingale difference sequence, so that $\sum_{j=2}^{\infty} U_{2j}/j^2 < \infty$ a.s., which implies

$$n^{-2}\sum_{j=2}^{\infty} U_{2j} = n^{-2}\sum_{j=2}^{\infty}(X_j'-EX_j')\sum_{i=1}^{j-1}(X_i'-EX_i') \xrightarrow{\text{a.s.}} 0 \tag{4}$$

as $n\to\infty$. Therefore, by (2)—(4), we obtain

$$n^{-2}\Big[\sum_{i=1}^{n}(X_i'-EX_i')\Big]^2 \xrightarrow{\text{a.s.}} 0 \quad \text{as} \quad n\to\infty.$$

Thus we have obtained

$$n^{-1}\sum_{i=1}^{n}(X_i'-EX_i') \xrightarrow{\text{a.s.}} 0 \quad \text{as} \quad n\to\infty.$$

Now taking into account the properties of $\varphi$ and the condition (C), we have

$$\sum_{n=1}^{\infty} P[X_n' \neq X_n] = \sum_{n=1}^{\infty} P[|X_n| \geqq n] =$$

$$= \sum_{n=1}^{\infty} E\{I_{[|X_n|\geqq n]} I_{[|X_n|\geqq a_n]} + I_{[|X_n|<a_n]} I_{[|X_n|\geqq n]}\} \leqq$$

$$\leqq \sum_{n=1}^{\infty} P[|X_n| \geqq a_n] + 2\sum_{n=1}^{\infty} E\frac{\varphi^{2r}(|X_n|)}{\varphi^{2r}(n)+\varphi^{2r}(|X_n|)} I_{[|X_n|<a_n]} \leqq$$

$$\leqq \sum_{n=1}^{\infty} P[|X_n| \geqq a_n] + 2\sum_{n=1}^{\infty} \varphi^{r}(a_n) E\frac{\varphi^{r}(|X_n|)}{\varphi^{2r}(n)+\varphi^{2r}(|X_n|)}$$

for any $r\geqq 1$. Putting $r=2$ and $r=1$ in the case (a) and (b) respectively, we get $\sum_{n=1}^{\infty} P[X_n'\neq X_n] < \infty$, which completes the proof of Theorem 1.

**Corollary 1.** *If* (B) *and* ($B_1$) *are replaced by the condition* $n^{-1}\sum_{i=1}^{n} iE\big(\varphi(|X_i|)/\varphi(i)\big) = o(1)$, *then* $\{X_n,\ n\geqq 1\}$ *satisfies the strong law of large numbers, i.e.* $S_n/n\to 0$ *a.s. as* $n\to\infty$.

For a sequence $\{X_n,\ n\geqq 1\}$ of independent random variables having moments only of the first order one can establish the following.

**Corollary 2.** *Let* $\{X_n,\ n\geqq 1\}$ *be a sequence of independent random variables with* $EX_n=0,\ n\geqq 1$. *If*

$$\sum_{j=2}^{\infty} j^{-2}E\frac{|X_j|}{j+|X_j|}\sum_{i=1}^{j-1} i^2E\frac{|X_i|}{i+|X_i|}<\infty,\quad n^{-2}\sum_{i=1}^{n} i^2E\frac{|X_i|}{i+|X_i|}=o(1)$$

*and* $\sum_{n=1}^{\infty} P[|X_n|\geqq a_n]<\infty$ *for some positive numerical sequence* $\{a_n, n\geqq 1\}$ with $\sum_{n=1}^{\infty} a_nE\frac{|X_n|}{n^2+|X_n|^2}<\infty$ *then* (1) *holds.*

To prove the given assertion it is enough to use the case (a) of Theorem 1 with $\varphi(x)=|x|^{1/2}$.

Corollary 2 allows us to state for instance that a sequence $\{X_k,\ k\geqq 1\}$ of independent random variables such that

$$P[X_n=\pm 2^{k/(1+\delta)}/k^{1/(1+\delta)}]=1/(n^{1+\alpha}2^{k+1}),\quad k=1,2,\ldots,\quad 0<\delta\leqq 1,\quad \alpha>0,$$
$$P[X_n=0]=1-1/n^{1+\alpha},\quad n=1,2,\ldots,$$

satisfies the strong law of large numbers.

To give a generalization of Theorem 1 we need the following lemma [1] p. 329.

**Lemma 1.** *If* $\{x_j,\ 1\leqq j\leqq n\}$, *are real numbers,* $S_n=\sum_{j=1}^{n} x_j$, *and*

$$Q_{k,n}=\sum_{1\leqq i_1<\ldots<i_k\leqq n} x_{i_1}x_{i_2}\ldots x_{i_k},\quad 1\leqq k\leqq n,$$

*then for* $n\geqq k\geqq 2$, $Q_{k,n}=\sum_{j=k}^{n} x_jQ_{k-1,j-1}$, *and* $S_n^k=k!\ Q_{k,n}+c_k$, *where* $c_k$ *is a generic designation for a finite linear combination (coefficients independent of n) of terms* $\prod_{i=1}^{m}\sum_{j=1}^{n} x_j^{h_i}$ *of order* $k$, *that is,* $\sum_{i=1}^{m} h_i=k,\ 1\leqq h_i\leqq k,\ 1\leqq m<k$.

**Theorem 2.** *Let* $\{X_n,\ n\geqq 1\}$ *be a sequence of independent random variables with* $EX_n=0,\ n\geqq 1$, *and* $\varphi$ *be a function from Theorem* 1. *Suppose that in the case* (a)

$$\text{(A}'\text{)}\qquad \sum_{j_k=k}^{\infty} j_k^{-2(k-1)}E\frac{\varphi^2(|X_{j_k}|)}{\varphi^2(j_k)+\varphi^2(|X_{j_k}|)}\sum_{j_{k-1}=k-1}^{j_k-1} j_{k-1}^2\cdot$$
$$\cdot E\frac{\varphi^2(|X_{j_{k-1}}|)}{\varphi^2(j_{k-1})+\varphi^2(|X_{j_{k-1}}|)}\cdots\sum_{j_1=1}^{j_2-1} j_1^2E\frac{\varphi^2(|X_{j_1}|)}{\varphi^2(j_1)+\varphi^2(|X_{j_1}|)}<\infty,$$

(B), *and* (C)—(D) *are satisfied; or in the case* (b)

$$\text{(A}_1'\text{)}\qquad \sum_{j_k=k}^{\infty} j_k^{-2(k-1)}E\frac{\varphi(|X_{j_k}|)}{\varphi(j_k)+\varphi(|X_{j_k}|)}\sum_{j_{k-1}=k-1}^{j_k-1} j_{k-1}^2\cdot$$
$$\cdot E\frac{\varphi(|X_{j_{k-1}}|)}{\varphi(j_{k-1})+\varphi(|X_{j_{k-1}}|)}\cdots\sum_{j_1=1}^{j_2-1} j_1^2E\frac{\varphi(|X_{j_1}|)}{\varphi(j_1)+\varphi(|X_{j_1}|)}<\infty,$$

($B_1$), *and* (C)—($D_1$) *are satisfied. Then* (1) *holds.*

PROOF. Put as previous by $X_i'=X_iI_{[|X_i|<i]}$. Then by Lemma 1, there exist finite constants $c_k;\ c_1, c_2, \dots, c_{k-2}$ such that

(5) $$\Big(n^{-1}\sum_{i=1}^{n}(X_i'-EX_i')\Big)^k = \sum_{h=1}^{k-2} c_h\Big(n^{-1}\sum_{i=1}^{n}(X_i'-EX_i')\Big)^h A_{k-h,n}+c_kA_{k,n}+k!\,n^{-k}Q_{k,n},$$

where $0\leqq h\leqq k-2$, $A_{k-h,n}$ is a finite linear combination of terms $\prod_{i=1}^{m}\big(n^{-h_i}\sum_{j=1}^{n}(X_j'-EX_j')^{h_i}\big)$ satisfying $h_i\geqq 2$ for $1\leqq i<m<k$, $\sum_{i=1}^{m}h_i=k-h$, and

$$Q_{o,n} = 1,\quad Q_{k,n} =$$

$$= \sum_{1\leqq i_1<i_2<\dots<i_k\leqq n}(X_{i_1}'-EX_{i_1}')(X_{i_2}'-EX_{i_2}')\dots(X_{i_k}'-EX_{i_k}') = \sum_{j_k=k}^{n}(X_{j_k}'-EX_{j_k}')Q_{k-1,j-1} =$$

$$= \sum_{j_k=k}^{n}(X_{j_k}'-EX_{j_k}')\sum_{j_{k-1}=k-1}^{j_k-1}(X_{j_{k-1}}'-EX_{j_{k-1}}')\dots\sum_{j_1=1}^{j_2-1}(X_{j_1}'-EX_{j_1}').$$

Note now that for $h_i\geqq 2$

$$\Big|\prod_{i=1}^{m}n^{-h_i}\sum_{j=1}^{n}(X_j'-EX_j')^{h_i}\Big| \leqq \prod_{i=1}^{m}\Big[n^{-2}\sum_{j=1}^{n}(X_j'-EX_j')^2\Big]^{h_i/2}\to 0$$

a.s. as $n\to\infty$, since $n^{-2}\sum_{i=1}^{n}(X_i'-EX_i')^2\to 0$ a.s. as $n\to\infty$, (this fact has been established in the proof of Theorem 1). Thus for $0\leqq h\leqq k-2$,

(6) $$A_{k-h,n}\to 0 \quad \text{a.s. as} \quad n\to\infty.$$

Put

$$U_{k,j_k} = (X_{j_k}'-EX_{j_k}')\sum_{j_{k-1}=k-1}^{j_k-1}(X_{j_{k-1}}'-EX_{j_{k-1}}')\dots\sum_{j_1=1}^{j_2-1}(X_{j_1}'EX_{j_1}').$$

Then $EU_{k,j_k}=0$, $k=1,2,\dots$, and moreover, by (A′) and the properties of the function $\varphi$ of the case (a), we get

$$\sum_{j_k=k}^{\infty}E(U_{k,j_k}/j_k^k)^2\leqq$$

$$\sum_{j_k=k}^{\infty}j_k^{-2k}E(X_{j_k}'-EX_{j_k}')^2\dots\sum_{j_{k-1}=k-1}^{j_k-1}E(X_{j_{k-1}}'-EX_{j_{k-1}}')^2\dots\sum_{j_1=1}^{j_2-1}E(X_{j_1}'-EX_{j_1}')^2\leqq$$

$$\leqq \sum_{j_k=k}^{\infty}j_k^{-2k}E(X_{j_k}')^2\sum_{j_{k-1}=k-1}^{j_k-1}E(X_{j_{k-1}}')^2\dots\sum_{j_1=1}^{j_2-1}E(X_{j_1}')^2=$$

$$= \sum_{j_k=k}^{\infty}j_k^{-2k}E(X_{j_k}')^2\sum_{j_{k-1}=k-1}^{j_k-1}j_{k-1}^2\frac{E(X_{j_{k-1}}')^2}{j_{k-1}^2}\dots\sum_{j_1=1}^{j_2-1}j_1^2\frac{E(X_{j_1}')^2}{j_1^2}\leqq$$

$$\leqq \sum_{j_k=k}^{\infty}j_k^{-2k+1}\frac{E\varphi^2(|X_{j_k}|)}{\varphi^2(j_k)}\sum_{j_{k-1}=k-1}^{j_k-1}j_{k-1}^2\frac{E\varphi^2(|X_{j_{k-1}}|)}{\varphi^2(j_{k-1})}\dots\sum_{j_1=1}^{j_2-1}j_1^2\frac{E\varphi^2(|X_{j_1}|)}{\varphi^2(j_1)}\leqq$$

$$\leqq 2^k\sum_{j_k=k}^{\infty}j_k^{-2k+1}E\frac{\varphi^2(|X_{j_k}|)}{\varphi^2(j_k)+\varphi^2(|X_{j_k}|)}\sum_{j_{k-1}=k-1}^{j_k-1}j_{k-1}^2E\frac{\varphi^2(|X_{j_{k-1}}|)}{\varphi^2(j_{k-1})+\varphi^2(|X_{j_{k-1}}|)}\dots$$

$$\dots\sum_{j_1=1}^{j}j_1^2E\frac{\varphi^2(|X_{j_1}|)}{\varphi^2(j_1)+\varphi^2(|X_{j_1}|)}<\infty$$

while in the case (b)

$$\sum_{j_k=k}^{\infty} E(U_{k,j_k}/j_k^k)^2 \leqq \sum_{j_k=k}^{\infty} j_k^{-2k+1}\frac{E(X'_{j_k})^2}{j_k^2}\sum_{j_{k-1}=k-1}^{j_k-1} j_{k-1}^2\frac{E(X'_{j_{k-1}})^2}{j_{k-1}^2}\cdots\sum_{j_1=1}^{j_2-1} j_1^2\frac{E(X'_{j_1})^2}{j_1^2}\leqq$$

$$\leqq 2^k\sum_{j_k=k}^{\infty} j_k^{-2k+1}E\frac{\varphi(|X_{j_k}|)}{\varphi(j_k)+\varphi(|X_{j_k}|)}\sum_{j_{k-1}=k-1}^{j_k-1} j_{k-1}^2E\frac{\varphi(|X_{j_{k-1}}|)}{\varphi(j_{k-1})+\varphi(|X_{j_{k-1}}|)}\cdots$$

$$\cdots\sum_{j_1=1}^{j_2-1} j_1^2E\frac{\varphi(|X_{j_1}|)}{\varphi(j_1)+\varphi(|X_{j_1}|)}<\infty.$$

But $\{U_{k,j_k},\ k\geqq 2\}$ is a martingale difference sequence, so that $\sum_{j_k=k}^{\infty} j_k^{-k}U_{k,j_k}<\infty$ a.s., and by Kronecker's lemma

(7)

$$n^{-k}\sum_{j_k=k}^{n} U_{k,j_k} = n^{-k}\sum_{j_k=k}^{n}(X'_{j_k}-EX'_{j_k})\cdots\sum_{j_1=1}^{j_2-1}(X'_{j_1}-EX'_{j_1})\to 0 \quad \text{a.s.} \quad \text{as} \quad n\to\infty.$$

Thus according to (5) $n^{-1}\sum_{i=1}^{n}(X'_i-EX'_i)$ is a root of a $k$-th degree polynomial in which the leading coefficient is unity and the remaining coefficients, by (6) and (7), converge almost surely to zero. Therefore the conclusion

$$n^{-1}\sum_{i=1}^{n}(X'_i-EX'_i)\to 0 \quad \text{a.s.,} \quad \text{as} \quad n\to\infty,$$

follows from the well-known relations between the roots and coefficients of a polynomial. But, as in the proof of Theorem 1, for $r=2$ and $r=1$ in the case (a) and (b) respectively

$$\sum_{n=1}^{\infty} P[X'_n\neq X_n]\leqq\sum_{n=1}^{\infty} P[|X_n|\geqq a_n]+2\sum_{n=1}^{\infty}\varphi^r(a_n)E\frac{\varphi^r(|X_n|)}{\varphi^{2r}(n)+\varphi^{2r}(|X_n|)}<\infty$$

which proves (1).

**Corollary 3.** *Let* $\{X_n,\ n\geqq 1\}$ *be a sequence of independent random variables with* $EX_n=0,\ n\geqq 1$. *If*

$$\sum_{j_k=k}^{\infty} j_k^{-2k+1}E\frac{|X_{j_k}|}{j_k+|X_{j_k}|}\sum_{j_{k-1}=k-1}^{j_k-1} j_{k-1}^2E\frac{|X_{j_{k-1}}|}{j_{k-1}+|X_{j_{k-1}}|}\cdots\sum_{j_1=1}^{j_2-1} j_1^2E\frac{|X_{j_1}|}{j_1+|X_{j_1}|}<\infty,$$

$$n^{-2}\sum_{i=1}^{n} i^2E\frac{|X_i|}{i+|X_i|}=o(1),$$

*and* $\sum_{n=1}^{\infty} P[|X_n|\geqq a_n]<\infty$ *for some positive numerical sequence* $\{a_n,\ n\geqq 1\}$ *with* $\sum_{n=1}^{\infty} n^{-2}a_nE\big(|X_n|/(n^2+X_n^2)\big)<\infty$, *then* (1) *holds.*

In [3] JEGOROV proved that if $\{b_n,\ n\geqq 1\}$ is a positive, increasing numerical sequence with $\lim_{n\to\infty} b_n=\infty$ and $\{X_n,\ n\geqq 1\}$ is a sequence of independent random variables with $EX_n=0,\ n\geqq 1$, such that

$$\sum_{j_k=k}^{\infty} b_{j_k}^{-2k}\sigma_{j_k}^2 \sum_{j_{k-1}=k-1}^{j_k-1} \sigma_{j_{k-1}}^2 \cdots \sum_{j_1=1}^{j_2-1} \sigma_{j_1}^2 < \infty \quad \text{and}$$

$\bigwedge_{\varepsilon>0} \sum_{n=1}^{\infty} P[|X_n|>\varepsilon b_n]<\infty$, then $\{X_n,\ n\geqq 1\}$ satisfies the strong law of large numbers. The conditions given by Jegorov are more general than Teicher's conditions.

In that way we can generalize Theorem 1 and Theorem 2 from this note. Then we get the following:

**Theorem 2'.** *Let* $\{b_n,\ n\geqq 1\}$ *be a positive, increasing numerical sequence with* $\lim_{n\to\infty} b_n=\infty$ *and* $\{X_n,\ n\geqq 1\}$ *be a sequence of independent random variables with* $EX_n=0,\ n\geqq 1$ *and* $\bigwedge_{\varepsilon>0} \sum_{n=1}^{\infty} P[|X_n|\geqq\varepsilon b_n]<\infty$. *Suppose that in the case* (a)

$$\sum_{j_k=k}^{\infty} b_{j_k}^{-2k+1} E\frac{\varphi^2(|X_{j_k}|)}{\varphi^2(b_{j_k})+\varphi^2(|X_{j_k}|)} \sum_{j_{k-1}=k-1}^{j_k-1} b_{j_{k-1}}^2 E\frac{\varphi^2(|X_{j_{k-1}}|)}{\varphi^2(b_{j_{k-1}})+\varphi^2(|X_{j_{k-1}}|)} \cdots$$
$$\cdots \sum_{j_1=1}^{j_2-1} b_{j_1}^2 E\frac{\varphi^2(|X_{j_1}|)}{\varphi^2(b_{j_1})+\varphi^2(|X_{j_1}|)} < \infty$$

*or in the case* (b)

$$\sum_{j_k=k}^{\infty} b_{j_k}^{-2k+1} E\frac{\varphi(|X_{j_k}|)}{\varphi(b_{j_k})+\varphi(|X_{j_k}|)} \sum_{j_{k-1}=k-1}^{j_k-1} b_{j_{k-1}}^2 E\frac{\varphi(|X_{j_{k-1}}|)}{\varphi(b_{j_{k-1}})+\varphi(|X_{j_{k-1}}|)} \cdots$$
$$\cdots \sum_{j_1=1}^{j_2-1} b_{j_1}^2 E\frac{\varphi(|X_{j_1}|)}{\varphi(b_{j_1})+\varphi(|X_{j_1}|)} < \infty.$$

*Then* (1) *holds.*

## References


[1] Y. S. CHOW, H. TEICHER, Probability Theory, Independence, Interchangeability, Martingales, 1978 *Springer-Verlag, New York.*

[2] K. L. CHUNG, Note on some strong laws of large numbers, *Amer. J. Math.* (1947), No. **1**, 189—192.

[3] B. A., Егоров Об усиленном законе больших чисел для последовательности независимых случайных беличин, *Теория бероятш. и её примен* **15** (1970), 520—527.

[4] H. TEICHER, Some new conditions for the strong law, *Proc. Nat. Acad. Sci. U. S. A.* **59** (1968), 705—707.

[5] A. KUCZMASZEWSKA, D. SZYNAL, On Chung—Teicher type conditions for the strong law of large numbers, *Lit. Mat. Sb.* **2** (1982), 88—96.



ANNA KUCZMASZEWSKA
TECHNICAL UNIVERSITY
UL. DĄBROWSKIEGO 13
20-109 LUBLIN
POLAND

DOMINIK SZYNAL
INSTITUTE OF MATHEMATICS
MARIA CURIE-SKŁODOWSKA UNIVERSITY
PL. MARII CURIE-SKŁODOWSKIEJ 1
20-031 LUBLIN
POLAND


*(Received April 2, 1984.)*