On defining the distribution $\delta^{(r)}(f(x))$ for summable f

By BRIAN FISHER (Leicester)

In the following we let N be the neutrix, see van der CORPUT [1], having domain $N'=\{1, 2, \dots, n, \dots\}$ and range N'' the real numbers with negligible functions linear sums of the functions n^λ for $\lambda>0$ and all functions which converge to zero as n tends to infinity.

It follows that if

$$f(n)=f_1(n)+f_2(n),$$

where $f_1(n)$ is a negligible function and the limit as n tends to infinity of $f_2(n)$ exists, then the neutrix limit as n tends to infinity of $f(n)$ exists and

$$N\text{-}\lim_{n\to\infty} f(n)=\lim_{n\to\infty} f_2(n).$$

Now let ϱ be a fixed infinitely differentiable function having the properties

(i) $\varrho(x)=0$ for $|x|\geqq 1$,

(ii) $\varrho(x)\geqq 0$,

(iii) $\varrho(x)=\varrho(-x)$,

(iv) $\int_{-1}^{1}\varrho(x)\,dx=1$.

We define the function δ_n by

$$\delta_n(x)=n\varrho(nx)$$

for $n=1, 2, \dots$.

The following definition was given in [2].

Definition 1. Let f be an infinitely differentiable function. We say that the distribution $\delta^{(r)}(f(x))$ exists and is equal to h on the open interval (a, b) if

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}\delta_n^{(r)}(f(x))\varphi(x)\,dx=(h(x), \varphi(x))$$

for all test functions φ with compact support contained in the interval (a, b).

The following theorem was proved in [2] and shows that definition 1 is in agreement with the definition of $\delta^{(r)}(f(x))$ given by GELFAND and SHILOV [3].

Theorem 1. *Let f be an infinitely differentiable function and suppose that the equation $f(x)=0$ has a single simple root at the point $x=x_1$ in the open interval (a, b). Then the distribution $\delta^{(r)}(f(x))$ exists and*

$$\delta^{(r)}(f(x)) = \frac{1}{|f'(x_1)|}\left[\frac{1}{f'(x)}\frac{d}{dx}\right]^r \delta(x-x_1)$$

on the interval (a, b).

The next definition is an extension of definition 1 and was also given in [2].

Definition 2. Let f be an infinitely differentiable function. We say that the distribution $\delta^{(r)}(f(x))$ exists and is equal to h on the open interval (a, b) if

$$N\text{-}\lim_{n\to\infty} \int_{-\infty}^{\infty} \delta_n^{(r)}(f(x))\varphi(x)\,dx = (h(x), \varphi(x))$$

for all test functions φ with compact support contained in the interval (a, b).

The following theorem was then proved.

Theorem 2. *Let F be an infinitely differentiable function and suppose that the equation $F(x)=0$ has a single simple root at the point $x=x_1$ in the open interval (a, b). Then if $f=F^s$, the distribution $\delta^{(r)}(f(x))$ exists and*

$$\frac{d}{dx}\delta^{(r)}(f(x)) = f'(x)\delta^{(r+1)}(f(x))$$

on the interval (a, b) for $r=0, 1, 2, \ldots$ and $s=1, 2, \ldots$. In particular

$$\delta^{(r)}(f(x)) = 0$$

on the interval (a, b) for $r=0, 1, 2, \ldots$ and $s=2, 4, \ldots$ and

$$\delta^{(r)}((x-x_1)^s) = \frac{r!}{s(rs+s-1)!}\delta^{(rs+s-1)}(x-x_1) \tag{1}$$

on the real line for $r=0, 1, 2, \ldots$ and $s=1, 3, 5, \ldots$.

We now extend definition 2 with the following definition.

Definition 3. Let f be a summable function. We say that the distribution $\delta^{(r)}(f(x))$ exists and is equal to h on the open interval (a, b) if

$$N\text{-}\lim_{n\to\infty} \int_{-\infty}^{\infty} \delta_n^{(r)}(f(x))\varphi(x)\,dx = (h(x), \varphi(x))$$

for all test functions φ with compact support contained in the interval (a, b).

Now let f be a summable function. We define the summable functions f_+ and f_- by

$$f_+(x) = \begin{cases} f(x), & x \geqq 0, \\ 0, & x < 0 \end{cases}$$

and

$$f_-(x) = \begin{cases} f(x), & x \leqq 0, \\ 0, & x > 0. \end{cases}$$

However, in accordance with the usual practice, we define the summable functions x_+^s and x_-^s by

$$x_+^s = \begin{cases} x^s, & x \geqq 0, \\ 0, & x < 0 \end{cases} \quad \text{and} \quad x_-^s = \begin{cases} |x|^s, & x \leqq 0, \\ 0, & x > 0 \end{cases}$$

for $s=1, 2, \ldots$.

Theorem 3. *Let f be a summable function and suppose that f is continuous and $f(x) \neq 0$ on the closed interval $[a, b]$, where $a<0<b$. Then the distribution $\delta^{(r)}(f_+(x-x_1))$ exists and*

$$\delta^{(r)}(f_+(x-x_1)) = 0$$

on the interval $(-\infty, b+x_1)$ for $r=0, 1, 2, \ldots$. In particular

$$\delta^{(r)}(H(x-x_1)) = 0$$

on the interval $(-\infty, \infty)$ for $r=0, 1, 2, \ldots$, where H denotes Heaviside's function.

Proof. For simplicity of notation we will assume that $x_1=0$. The more general results will follow by translation.

Let φ be an arbitrary test function with compact support contained in the interval $(-\infty, b)$. Then

$$\int_{-\infty}^{\infty} \delta_n^{(r)}(f_+(x))\varphi(x)\,dx = \int_{-\infty}^{0} \delta_n^{(r)}(0)\varphi(x)\,dx + \int_{0}^{\infty} \delta_n^{(r)}(f(x))\varphi(x)\,dx =$$

$$= n^{r+1}\varrho^{(r)}(0) \int_{-\infty}^{0} \varphi(x)\,dx + n^{r+1} \int_{0}^{b} \varrho^{(r)}(nf(x))\varphi(x)\,dx,$$

where

$$n^{r+1}\varrho^{(r)}(0) \int_{-\infty}^{0} \varphi(x)\,dx$$

is either negligible or zero. Further, since f is continuous and non-zero on the closed interval $[0, b]$, we can find an integer N such that

$$|nf(x)| \geqq 1$$

for $n>N$. It then follows that $\varrho^{(r)}(nf(x))=0$ for $n>N$. Thus

$$\underset{n\to\infty}{N\text{-lim}} \int_{-\infty}^{\infty} \delta_n^{(r)}(f_+(x))\varphi(x)\,dx =$$

$$= \underset{n\to\infty}{N\text{-lim}}\, n^{r+1}\varrho^{(r)}(0) \int_{-\infty}^{0} \varphi(x)\,dx + \lim_{n\to\infty} n^{r+1} \int_{0}^{b} \varrho^{(r)}(nf(x))\varphi(x)\,dx = 0 = (0, \varphi)$$

and so

$$\delta^{(r)}(f_+(x)) = 0$$

for $r=0, 1, 2, \dots$. This completes the proof of the theorem.

Corollary 3.1. *Let f be a summable function and suppose that f is continuous and $f(x)\neq 0$ on the closed interval $[a, b]$, where $a<0<b$. Then the distribution $\delta^{(r)}(f_-(x-x_1))$ exists and*

$$\delta^{(r)}(f_-(x-x_1)) = 0$$

on the interval $(a+x_1, \infty)$ for $r=0, 1, 2, \dots$.

Proof. Let $g(x)=f(-x)$. Then g is continuous and non-zero for $-b\leqq x\leqq -a$ and so by the theorem

$$\delta^{(r)}(g_+(x)) = 0$$

on the interval $(-\infty, -a)$ for $r=0, 1, 2, \dots$. Replacing x by $-x$ we see that

$$\delta^{(r)}(g_+(-x)) = \delta^{(r)}(f_-(x)) = 0$$

on the interval (a, ∞) and the result of the corollary follows.

Corollary 3.2. *Let f be a summable function and suppose that f is continuous and $f(x)\neq 0$ on the closed interval $[a, b]$, where $a<0<b$. Then the distributions $\delta^{(r)}(f(x-x_1))$ and $\delta^{(r)}(f_+(x-x_1)-f_-(x-x_1))$ exist and $\delta^{(r)}(f(x-x_1)) = \delta^{(r)}(f_+(x-x_1)-f_-(x-x_1))=0$ on the interval $(a+x_1, b+x_1)$ for $r=0, 1, 2, \dots$.*

Proof. Let φ be an arbitrary test function with support contained in the interval (a, b). Then

$$\int_{-\infty}^{\infty} \delta_n^{(r)}(f(x))\varphi(x)\,dx = \int_{-\infty}^{0} \delta_n^{(r)}(f(x))\varphi(x)\,dx + \int_{0}^{\infty} \delta_n^{(r)}(f(x))\varphi(x)\,dx$$

and

$$\int_{-\infty}^{\infty} \delta_n^{(r)}(f_+(x)-f_-(x))\varphi(x)\,dx = \int_{-\infty}^{0} \delta_n^{(r)}(-f(x))\varphi(x)\,dx + \int_{0}^{\infty} \delta_n^{(r)}(f(x))\varphi(x)\,dx.$$

It was also proved above that

$$\lim_{n\to\infty} \int_{0}^{\infty} \delta_n^{(r)}(f(x))\varphi(x)\,dx = 0$$

and similarly

$$\lim_{n\to\infty} \int_{-\infty}^{0} \delta_n^{(r)}(f(x))\varphi(x)\,dx = \lim_{n\to\infty} \int_{-\infty}^{0} \delta_n^{(r)}(-f(x))\varphi(x)\,dx = 0.$$

It follows that

$$N\text{-}\lim_{n\to\infty} \int_{-\infty}^{\infty} \delta_n^{(r)}(f(x))\varphi(x)\,dx = N\text{-}\lim_{n\to\infty} \int_{-\infty}^{\infty} \delta_n^{(r)}(f_+(x)-f_-(x))\varphi(x)\,dx = (0, \varphi)$$

and the result of the corollary follows.

As an *example* consider $f(x)=\cos x$. Then it follows immediately from theorems 1 and 3 that

$$\delta(\cos_+ x) = \sum_{k=0}^{\infty} \delta\left(x-k\pi-\frac{1}{2}\pi\right)$$

and it follows from theorem 1 and corollary 3.2 that

$$\delta(\cos_+ x-\cos_- x) = \sum_{k=0}^{\infty} \delta\left(x-k\pi-\frac{1}{2}\pi\right)+\sum_{k=1}^{\infty} \delta\left(x+k\pi-\frac{1}{2}\pi\right).$$

Further, from theorems 2 and 3, it follows that

$$\delta^{(r)}(\cos_+^2 x) = 0$$

for $r=0, 1, 2, \ldots$.

It was proved in [2] that

$$\delta(\sin^3 x) = \sum_{k=-\infty}^{\infty} \frac{1}{6}[\delta(x-k\pi)+\delta''(x-k\pi)]$$

and it follows by translation that

$$\delta(\cos^3 x) = \sum_{k=-\infty}^{\infty} \frac{1}{6}\left[\delta\left(x-k\pi-\frac{1}{2}\pi\right)+\delta''\left(x-k\pi-\frac{1}{2}\pi\right)\right].$$

It now follows from theorem 3 that

$$\delta(\cos_+^3 x) = \sum_{k=0}^{\infty} \frac{1}{6}\left[\delta\left(x-k\pi-\frac{1}{2}\pi\right)+\delta''\left(x-k\pi-\frac{1}{2}\pi\right)\right].$$

All the above results of course hold on the real line.

Theorem 4. *Let F be a summable function which is $ms+s+1$ times continuously differentiable on the closed interval $[a, b]$, where $a<0<b$. Suppose that the equation $F(x)=0$ has a single simple root at the point $x=0$ in the interval $[a, b]$. Then if $f=F^s$, the distribution $\delta^{(r)}(f_+(x-x_1))$ exists on the interval $(-\infty, b+x_1)$ for $r=0, 1, \ldots, m$ and $s=1, 2, \ldots$. In particular*

$$\delta^{(r)}((x-x_1)_+^s) = \frac{(-1)^{rs+r+s-1}r!}{2s(rs+s-1)!}\delta^{(rs+s-1)}(x-x_1)$$

on the interval $(-\infty, \infty)$ for $r=0, 1, 2, \ldots$ and $s=1, 2, \ldots$.

PROOF. We will again prove the theorem for the case $x_1=0$.

Since $x=0$ is a simple root of the equation $F(x)=0$, it follows that $F'(x)\neq 0$ on the closed interval $[0, c]$, where $0<c\leqq b$. The equation $F(x)=y$ will therefore have an inverse $x=g(y)$ on the interval $[0, c]$ and the function g will be $ms+s+1$ times continuously differentiable.

Now let φ be an arbitrary test function with compact support contained in the interval $(-\infty, c)$. Then

$$\int_{-\infty}^{\infty} \delta_n^{(r)}(f_+(x))\varphi(x)\,dx = \int_{-\infty}^{0} \delta_n^{(r)}(0)\varphi(x)\,dx + \int_{0}^{\infty} \delta_n^{(r)}(f(x))\varphi(x)\,dx =$$

$$= n^{r+1}\varrho^{(r)}(0) \int_{-\infty}^{0} \varphi(x)\,dx + \int_{0}^{\infty} \delta_n^{(r)}(f(x))\varphi(x)\,dx,$$

where again

$$n^{r+1}\varrho^{(r)}(0) \int_{-\infty}^{0} \varphi(x)\,dx$$

it either negligible or zero. Further, on making the substitution $t^{1/s} = F(x)$ or $x = g(t^{1/s})$ we have

$$\int_{0}^{\infty} \delta_n^{(r)}(f(x))\varphi(x)\,dx = \frac{1}{s}\int_{0}^{\infty} \delta_n^{(r)}(t)\varphi\big(g(t^{1/s})\big)|g'(t^{1/s})|\,t^{1/s-1}\,dt.$$

The function

$$\psi(y) = \varphi\big(g(y)\big)|g'(y)|$$

is $ms+s$ times continuously differentiable and so since $r \leqq m$, we have by Taylor's theorem

$$\psi(y) = \sum_{i=0}^{rs+s-1} \frac{\psi^{(i)}(0)}{i!}\,y^i + \frac{\psi^{(rs+s)}(\xi y)}{(rs+s)!}\,y^{rs+s},$$

where $0 \leqq \xi \leqq 1$. Thus

$$s\int_{0}^{\infty} \delta_n^{(r)}(f(x))\varphi(x)\,dx = \sum_{i=0}^{rs+s-2} \frac{\psi^{(i)}(0)}{i!} \int_{0}^{1} n^r \varrho^{(r)}(u)(u/n)^{(i+1)/s-1}\,du +$$

$$+ \frac{\psi^{(rs+s-1)}(0)}{(rs+s-1)!}\int_{0}^{1} \varrho^{(r)}(u)u^r\,du + \int_{0}^{1} \frac{\psi^{(rs+s)}\big(\xi(u/n)^{1/s}\big)}{(rs+s)!}\,n^r\varrho^{(r)}(u)(u/n)^{r+1/s}\,du,$$

where the substitution $nt = u$ has been made. It follows that

$$\underset{n\to\infty}{N\text{-lim}} \int_{0}^{\infty} \delta_n^{(r)}(f(x))\varphi(x)\,dx = \frac{\psi^{(rs+s-1)}(0)}{s(rs+s-1)!}\int_{0}^{1} \varrho^{(r)}(u)u^r\,du =$$

$$= \frac{(-1)r!\,\psi^{(rs+s-1)}(0)}{2s(rs+s-1)!} = \underset{n\to\infty}{N\text{-lim}} \int_{-\infty}^{\infty} \delta_n^{(r)}(f_+(x))\varphi(x)\,dx.$$

This proves the existence of $\delta^{(r)}\big(f_+(x)\big)$ on the interval $(-\infty, c)$ for $r = 0, 1, \dots, m$ and $s = 1, 2, \dots$. We of course have $\delta^{(r)}\big(f_+(x)\big) = 0$ on the interval $(1/2\,c, b)$ by corollary 3.2.

In the particular case when $F=x$, F is infinitely differentiable, $f_+=x^s$ and ψ is identical to φ. It follows that

$$(\delta^{(r)}(x_+^s), \varphi(x)) = \frac{(-1)^r r!\, \varphi^{(rs+s-1)}(0)}{2s(rs+s-1)!} = \frac{(-1)^{rs+r+s-1} r!}{2s(rs+s-1)!}(\delta^{(rs+s-1)}(x), \varphi(x))$$

and so

$$\delta^{(r)}(x_+^s) = \frac{(-1)^{rs+r+s-1} r!}{2s(rs+s-1)!}\delta^{(rs+s-1)}(x)$$

for $r=0, 1, 2, \ldots$ and $s=1, 2, \ldots$. This completes the proof of the theorem.

Corollary 4.1. *Let F be a summable function which is $ms+s+1$ times continuously differentiable on the closed interval $[a, b]$, where $a<0<b$. Suppose that the equation $F(x)=0$ has a single simple root at the point $x=0$ in the interval $[a, b]$. Then if $f=F^s$, the distribution $\delta^{(r)}(f_-(x-x_1))$ exists on the interval $(a+x_1, \infty)$ for $r=0, 1, \ldots, m$ and $s=1, 2, \ldots$. In particular*

$$\delta^{(r)}((x-x_1)_-^s) = \frac{(-1)^r r!}{2s(rs+s-1)!}\delta^{(rs+s-1)}(x-x_1)$$

on the interval $(-\infty, \infty)$ for $r=0, 1, 2, \ldots$ and $s=1, 2, \ldots$.

PROOF. We note that

$$\delta^{(r)}(f_-(x)) = \delta^{(r)}(f_+(-x)),$$

the right hand side existing by the theorem.

In particular we have

$$\delta^{(r)}(x_+^s) = \frac{(-1)^{rs+r+s-1} r!}{2s(rs+s-1)!}\delta^{(rs+s-1)}(x)$$

and replacing x by $-x$ we have

$$\delta^{(r)}(x_-^s) = \frac{(-1)^{rs+r+s-1} r!}{2s(rs+s-1)!}\delta^{(rs+s-1)}(-x) = \frac{(-1)^r r!}{2s(rs+s-1)!}\delta^{(rs+s-1)}(x).$$

Corollary 4.2. *Let F be a summable function which is $ms+s+1$ times continuously differentiable on the closed interval $[a, b]$, where $a<0<b$. Suppose that the equation $F(x)=0$ has a single simple root at the point $x=0$ in the interval $[a, b]$. Then if $f=F^s$, the distributions $\delta^{(r)}(f(x))$ and $\delta^{(r)}(f_+(x)-f_-(x))$ exist on the interval $(a+x_1, b+x_1)$ for $r=0, 1, \ldots, m$ and $s=1, 2, \ldots$. In particular*

$$\delta^{(r)}(\operatorname{sgn}(x-x_1)(x-x_1)^s) = 0 \tag{2}$$

on the interval $(-\infty, \infty)$ for $r=0, 2, 4, \ldots$ and $s=2, 4, \ldots$

$$\delta^{(r)}(|x-x_1|^s) = 0 \tag{3}$$

on the interval $(-\infty, \infty)$ for $r, s=1, 3, \ldots,$

$$\delta^{(r)}(\operatorname{sgn}(x-x_1)(x-x_1)^s) = \frac{r!}{s(rs+s-1)!}\delta^{(rs+s-1)}(x-x_1) \tag{4}$$

on the interval $(-\infty, \infty)$ *for* $r=1, 3, \dots$ *and* $s=2, 4, \dots$ *and*

(5) $$\delta^{(r)}(|x-x_1|^s) = \frac{r!}{s(rs+s-1)!}\,\delta^{(rs+s-1)}(x-x_1)$$

on the interval $(-\infty, \infty)$ *for* $r=0, 2, 4, \dots$ *and* $s=1, 3, \dots$.

PROOF. We have

$$\int_{-\infty}^{\infty} \delta_n^{(r)}(f(x))\varphi(x)\,dx = \int_{-\infty}^{0} \delta_n^{(r)}(f(x))\varphi(x)\,dx + \int_{0}^{\infty} \delta_n^{(r)}(f(x))\varphi(x)\,dx =$$

$$= \int_{-\infty}^{\infty} \delta_n^{(r)}(f_-(x))\varphi(x)\,dx + \int_{-\infty}^{\infty} \delta_n^{(r)}(f_+(x))\varphi(x)\,dx - \int_{-\infty}^{\infty} \delta_n^{(r)}(0)\varphi(x)\,dx,$$

the last term being either negligible or zero. It follows that

$$N\text{-}\lim_{n\to\infty} \int_{-\infty}^{\infty} \delta_n^{(r)}(f(x))\varphi(x)\,dx = (\delta^{(r)}(f_+(x)) + \delta^{(r)}(f_-(x)), \varphi(x)),$$

so that in fact

$$\delta^{(r)}(f(x)) = \delta^{(r)}(f_+(x)) + \delta^{(r)}(f_-(x)).$$

Similarly we can prove that

$$\delta^{(r)}(f_+(x) - f_-(x)) = \delta^{(r)}(f_+(x)) + \delta^{(r)}(-f_-(x)) =$$
$$= \delta^{(r)}(f_+(x)) + (-1)^r \delta^{(r)}(f_-(x)).$$

Equations (2), (3), (4) and (5) now follow from these results on using theorem 4 and corollary 4.1.

We note that the following more general results hold.

$$\delta^{(r)}((x-x_1)_+^s + (x-x_1)_-^m) = \frac{(-1)^{rs+r+s-1} r!}{2s(rs+s-1)!}\,\delta^{(rs+s-1)}(x-x_1) +$$

$$+ \frac{(-1)^r r!}{2m(rm+m-1)!}\,\delta^{(rm+m-1)}(x-x_1),$$

$$\delta^{(r)}((x-x_1)_+^s - (x-x_1)_-^m) = \frac{(-1)^{rs+r+s-1} r!}{2s(rs+s-1)!}\,\delta^{(rs+s-1)}(x-x_1) +$$

$$+ \frac{r!}{2m(rm+m-1)!}\,\delta^{(rm+m-1)}(x-x_1)$$

on the interval $(-\infty, \infty)$ for $r=0, 1, 2, \dots$ and $s, m=1, 2, \dots$. Equations (1), (2), (3), (4) and (5) are of course particular cases of these results.

As a final example consider $F(x)=\sin x$ and $f(x)=\sin^2 x$. It follows from theorem 2 that

$$\delta^{(r)}(\sin^2 x) = 0$$

on the interval $(-\infty, \infty)$ for $r=0, 1, 2, \ldots$. Further, for arbitrary test function φ

$$\psi(y) = \varphi(\sin^{-1} y)(1-y^2)^{-1/2}$$

and it can be shown that

$$\psi'(0) = \varphi'(0),$$

$$\psi'''(0) = 4\varphi'(0) + \varphi'''(0).$$

It follows from theorem 4 and its corollaries that

$$\big(\delta(\sin^2_+ x), \varphi(x)\big) = \frac{1}{4}\psi'(0) = \frac{1}{4}\varphi'(0),$$

$$\big(\delta'(\sin^2_+ x), \varphi(x)\big) = -\frac{1}{24}\psi'''(0) = -\frac{1}{6}\varphi'(0) - \frac{1}{24}\varphi'''(0)$$

and so

$$\delta(\sin^2_+ x) = -\frac{1}{4}\delta'(x),$$

$$\delta(\sin^2_- x) = \frac{1}{4}\delta'(x),$$

$$\delta(\operatorname{sgn} x \cdot \sin^2 x) = -\frac{1}{2}\delta'(x),$$

$$\delta'(\sin^2_+ x) = \frac{1}{6}\delta'(x) + \frac{1}{24}\delta'''(x),$$

$$\delta'(\sin^2_- x) = -\frac{1}{6}\delta'(x) - \frac{1}{24}\delta'''(x).$$

$$\delta'(\operatorname{sgn} x \cdot \sin^2 x) = \frac{1}{3}\delta'(x) + \frac{1}{12}\delta'''(x)$$

on the interval $(-\infty, \infty)$.

References

[1] J. G. van der Corput, Introduction to the neutrix calculus, *J. Analyse Math.* **7** (1959—60), 291—398.
[2] B. Fisher, On defining the distribution $\delta^{(r)}(f(x))$, *Rostock. Math. Kolloq.* **23** (1983), 73—80.
[3] I. M. Gelfand and G. E. Shilov, Generalized functions, *Vol. I*, (1964), *Academic Press.*

DEPARTMENT OF MATHEMATICS
THE UNIVERSITY
LEICESTER
LE1 7RH
U. K.

(Received March 16, 1984.)