On topological projective planes I

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Abstract

In this paper we determine conditions under which a topology on the set of points (or dually on the set of lines) of a projective plane gives already a topological projective plane.

1. Introduction

A topological projective plane is a projective plane in which the set of points $\mathcal{P}$ and the set of lines $\mathcal{L}$ are topological spaces and the operations of joining and intersecting are continuous in both variables. We assume that the topologies on $\mathcal{P}$ and $\mathcal{L}$ are Hausdorff. In this paper we prove that given a topology on the space of points (lines) of a projective plane π satisfying certain conditions, there exists a unique topology on the space of lines (points) so that π becomes a topological projective plane with these topologies. We use the following known result (Pickert, Page 264). In a topological pojective plane the relative topologies on any two affine rays are homeomorphic and the set of points of an affine plane is homeomorphic to the product of an affine ray with itself. We also use the notion of defining a topology by specifying a convergence scheme for nets (Kelley, Page 73).

2. Main results

We first observe that in any topological projective plane the following statements are valid.

(A): Let $P_\alpha, Q_\alpha, R_\alpha$ and S_α be nets in $\mathcal{P}$ directed by a set D converging to P, Q, R and S respectively. Let $P_\alpha Q_\alpha \neq R_\alpha S_\alpha$ for all $\alpha \in D$ and $PQ \neq RS$. Then $P_\alpha Q_\alpha \cap R_\alpha S_\alpha$ converges to $PQ \cap RS$.

(B): If P_α, Q_α and R_α are nets in $\mathcal{P}$ directed by a set D converging to three distinct points P, Q and R respectively and if $P_\alpha, Q_\alpha, R_\alpha$ are collinear for every $\alpha \in D$ then P, Q and R are collinear.

We denote by (dA) and (dB) the dual of the statements (A) and (B).

Lemma 1. *Let τ_1 be a Hausdorff topology on the set of points $\mathcal{P}$ of a projective plane. If condition* (A) *is valid in $(\mathcal{P}, \tau_1)$ then condition* (B) *is also valid and the set of all points on any line is closed.*

PROOF. Let P_α, Q_α and R_α be nets in $\mathscr{P}$ directed by a set D converging to three distinct points P, Q and R respectively and let $P_\alpha, Q_\alpha, R_\alpha$ be collinear for every $\alpha\in D$. Let S be a point not lying on PQ and distinct from each R_α and R. By (A), $R_\alpha=P_\alpha Q_\alpha\cap R_\alpha S_\delta$ converges to $PQ\cap RS$. Hence $PQ\cap RS=R$ so that P, Q and R are collinear. It follows easily from (B) that the set of all points on any line is closed.

Theorem 1. *Let τ_1 be a Hausdorff topology on the set of points $\mathscr{P}$ of a projective plane π and let* (A) *be valid in $(\mathscr{P}, \tau_1)$. Then there exists a unique topology τ_2 on the set of lines $\mathscr{L}$ such that π is a topological projective plane with these topologies.*

PROOF. We define a convergence scheme in $\mathscr{L}$ as follows. Let a, b, c be three non-concurrent lines. If l is a line different from a, b and c, a net l_α is said to converge to l iff $l_\alpha\neq a, b, c$ for all α and $l_\alpha\cap a\to l\cap a$, $l_\alpha\cap b\to l\cap b$ and $l_\alpha\cap c\to l\cap c$ in $(\mathscr{P}, \tau_1)$. If l coincides with one of the lines a, b, c say $l=a$, then l_α is said to converge to l iff $l_\alpha\neq b, c$ for all α and $l_\alpha\cap b\to l\cap b$ and $l_\alpha\cap c\to l\cap c$. It is easily seen that this convergence scheme defines a topology τ_2 on $\mathscr{L}$. We now prove that π is a topological projective plane with these topologies.

Let $P_\alpha\to P$ and $Q_\alpha\to Q$ in $(\mathscr{P}, \tau_1)$. We assume that $PQ\neq a$ so that either $P\notin a$ or $Q\notin a$. Let $P\notin a$. By lemma 1 the set of all points on the line a is closed and hence we may assume that $P_\alpha\notin a$ for all α so that $P_\alpha Q_\alpha\neq a$. If X and Y are two distinct points on a it follows from (A) that $P_\alpha Q_\alpha\cap XY=P_\alpha Q_\alpha\cap a\to PQ\cap a$. Similarly $P_\alpha Q_\alpha\cap b\to PQ\cap b$ and $P_\alpha Q_\alpha\cap c\to PQ\cap c$ so that $P_\alpha Q_\alpha\to PQ$ in $(\mathscr{L}, \tau_2)$. Hence joining is a continuous operation. The continuity of intersection follows from (A).

Finally let τ_2' be any other topology on $\mathscr{L}$ which together with τ_1 on $\mathscr{P}$ makes π into a topological projective plane. Let $l_\alpha\to l$ in τ_2. Then at least two of the nets $l_\alpha\cap a$, $l_\alpha\cap b$ and $l_\alpha\cap c$ converge respectively to $l\cap a$, $l\cap b$ and $l\cap c$ in τ_1 and hence by the continuity of join $l_\alpha\to l$ in τ_2'. Also if $l_\alpha\to l$ in τ_2' it follows from the continuity of intersection that $l_\alpha\to l$ in τ_2. Hence $\tau_2=\tau_2'$.

Dually, we have

Theorem 2. *Let τ_2 be a Hausdorff topology on the space of lines $\mathscr{L}$ of a projective plane π and let* (dA) *be valid in $(\mathscr{L}, \tau_2)$. Then there exists a unique topology τ_1 on $\mathscr{P}$ such that π is a topological projective plane with these topologies.*

Example. We now give an example to show that in theorem 1 condition (A) cannot be replaced by the weaker condition (B). Let R_u denote the set of real numbers with the upper limit topology which has the set of all semi-closed intervals of the form $(a, b]$, $a, b\in R$ and $a<b$, as its basis. Let τ denote the usual topology on the space of points $\mathscr{P}$ of the real projective plane. Let $OEUV$ denote the fundamental quadrangle with UV as the line at infinity. Consider the affine plane $\mathscr{P}-UV$ with the product topology of $R_u\times R_u$ which we denote by τ' and the ray of points UV with the subspace topology of $(\mathscr{P}, \tau)$ which we denote by τ''. We now define another topology τ_1 on $\mathscr{P}$ by declaring a subset A of $\mathscr{P}$ to be open iff $A\cap(\mathscr{P}-UV)$ and $A\cap UV$ are open in τ' and τ'' respectively. Since the upper limit topology on R is finer than the usual topology on R it follows that τ_1 is finer than τ. Further since the real projective plane is a topological plane with usual topology, condition (B) is valid in $(\mathscr{P}, \tau)$ and hence it is valid in $(\mathscr{P}, \tau_1)$ also. However in $(\mathscr{P}, \tau_1)$ the relative topology on the affine ray $OU-\{U\}$ is homeomorphic to R_u whereas the relative

topology on the affine ray $UV-\{V\}$ is homeomorphic to R with usual topology. Hence there is no topology on the space of lines which makes the real projective plane into a topological plane with these topologies.

We shall now prove that if τ_1 is compact condition (A) in theorem 1 can be replaced by (B).

Theorem 3. *Let τ_1 be a compact Hausdorff topology on the space of points $\mathscr{P}$ of a projective plane π and let* (B) *be valid in $(\mathscr{P}, \tau_1)$. Then there exists a unique topology τ_2 on the space of lines $\mathscr{L}$ such that π is a topological projective plane with these topologies.*

PROOF. Let $P_\alpha, Q_\alpha, R_\alpha, S_\alpha$ be nets in $\mathscr{P}$, directed by a set D, converging to P, Q, R and S respectively and let $P_\alpha Q_\alpha \neq R_\alpha S_\alpha$ for all α and $PQ \neq RS$. It follows from (B) that every convergent subnet of $P_\alpha Q_\alpha \cap R_\alpha S_\alpha$ converges to $PQ \cap RS$. Since τ_1 is a compact this implies that the net $P_\alpha Q_\alpha \cap R_\alpha S_\alpha$ converges to $PQ \cap RS$. Thus (A) is valid in $(\mathscr{P}, \tau_1)$ and the result follows from Theorem 1.

Dually, we have

Theorem 4. *Let τ_2 be a compact Hausdorff topology on the space of lines $\mathscr{L}$ of a projective plane π and let* (dB) *be valid in $(\mathscr{L}, \tau_2)$. Then there exists a unique topology τ_1 on $\mathscr{P}$ such that π is a topological projective plane with these topologies.*

Acknowledgement

The work in this paper represents part of author's doctoral thesis submitted to the Madurai—Kamaraj University. The author is thankful to Dr. R. P. BURN for his guidance and to Prof. M. VENKATARAMAN for his helpful suggestions.

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(Received August 18, 1983.)