The fundamental equation of information on open domain

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Introduction

The fundamental equation of information

$$f(x)+(1-x)f\left(\frac{y}{1-x}\right)=f(y)+(1-y)f\left(\frac{x}{1-y}\right) \tag{1}$$

receives much attention in the study of measures of information, and is the subject of Chapter 3 of the book [1] by ACZÉL and DARÓCZY. The general solution is known when it is supposed to hold for all $x, y \in [0, 1[$ with $x+y \leqq 1$. In the study of information measures of higher dimensions Aczél observed that it is desirable to know the general solution of (1) when it is supposed to hold in a smaller open domain

$$D^0 = \{(x, y) \mid x, y \in]0, 1[\text{ with } x+y < 1\}.$$

We shall solve (1) on D^0 in the next section. While the deduction will become more elaborate the essential steps are in line with those reported in [1].

The general solution

Theorem. *A function* f: $]0, 1[\to R$ *satisfies the functional equation* (1) *on the open domain* D^0 *if, and only if, it is of the form*

$$f(x) = \varphi(x)+\varphi(1-x)+ax \quad \textit{for all} \quad x \in]0, 1[, \tag{2}$$

where a *is an arbitrary constant and* φ: $]0, \infty[\to R$ *is a function satisfying*

$$\varphi(uv) = u\varphi(v)+v\varphi(u) \quad \textit{for all} \quad u, v > 0. \tag{3}$$

PROOF. Let f be a solution of (1) and define F: $]0, \infty[^2\to R$ by

$$F(u, v) = (u+v)f\left(\frac{v}{u+v}\right) \quad \text{for all} \quad u, v > 0. \tag{4}$$

Obviously F satisfies the homogeneity

$$F(tu, tv) = tF(u, v) \quad \text{for all} \quad t, u, v > 0. \tag{5}$$

We show that F satisfies

(6) $$F(u+v, w)+F(u, v) = F(u+w, v)+F(u, w) \quad \text{for all} \quad u, v, w > 0$$

by the following computations using (1):

$$F(u+v, w)+F(u, v) = (u+v+w)f\left(\frac{w}{u+v+w}\right)+(u+v)f\left(\frac{v}{u+v}\right) =$$

$$= (u+v+w)\left[f\left(\frac{w}{u+v+w}\right)+\left(1-\frac{w}{u+v+w}\right)f\left(\frac{v/(u+v+w)}{1-(w/(u+v+w))}\right)\right] =$$

$$= (u+v+w)\left[f\left(\frac{v}{u+v+w}\right)+\left(1-\frac{v}{u+v+w}\right)f\left(\frac{w/(u+v+w)}{1-(v/(u+v+w))}\right)\right] =$$

$$= (u+w+v)f\left(\frac{v}{u+w+v}\right)+(u+w)f\left(\frac{w}{u+w}\right) = F(u+w, v)+F(u, w).$$

Now we define h: $]0, \infty[\to R$ and G: $]0, \infty[^2\to R$ by

(7) $$h(u) = F(u, 1)-F(1, u) \quad \text{for all} \quad u > 0,$$

and

(8) $$G(u, v) = F(u, v)+h(v) \quad \text{for all} \quad u, v > 0.$$

Evidently h satisfies

(9) $$h(1) = 0.$$

We claim that G satisfies the following equations:

(10) $$G(u, v) = G(v, u) \quad \text{for all} \quad u, v > 0,$$

(11) $$G(u+v, w)+G(u, v) = G(u+w, v)+G(u, w) \quad \text{for all} \quad u, v, w > 0.$$

To support the above, we fix $w=1$ in (6) and then subtract from it the equation obtained by interchanging u and v to get

$$F(u, v)-F(v, u) = F(u+1, v)+F(u, 1)-F(v+1, u)-F(v, 1).$$

On the other hand using (6) we have $F(u+1, v)-F(v+1, u)=F(1, v)-F(1, u)$, and so the above equation leads to $F(u, v)-F(v, u)=F(1, v)-F(1, u)+F(u, 1)-$ $-F(v, 1)$, proving (10). Adding $h(v)+h(w)$ to the two sides of (6) we get (11). Next, we proceed to show that the function H: $]0, \infty[^2\to R$ defined by

(12) $$H(u, v) = G(u, v)+h(u+v)-h(u)-h(v) \quad \text{for all} \quad u, v > 0$$

satisfies

(13) $$H(u, v) = H(v, u) \quad \text{for all} \quad u, v > 0,$$

(14) $$H(u+v, w)+H(u, v) = H(u+w, v)+H(u, w) \quad \text{for all} \quad u, v, w > 0,$$

and

(15) $$H(tu, tv) = tH(u, v) \quad \text{for all} \quad t, u, v > 0.$$

Equations (13) and (14) follow easily from (10) and (11). In order to get (15), we first observed that since F satisfies the homogeneity (5) and so

$$G(tu, tv)-tG(u, v) = F(tu, tv)+h(tv)-tF(u, v)-th(v) = h(tv)-th(v).$$

Since the left hand side is symmetric in u and v, we get $h(tv)-th(v)=h(tu)-th(u)$. By fixing $u=1$ and using (9) we get

$$(16) \qquad h(tv)-th(v) = h(t) \quad \text{for all} \quad t, v > 0.$$

Thus $G(tu, tv)-tG(u, v)=h(t)$. But then we have

$$H(tu, tv)-tH(u, v) =$$
$$= G(tu, tv)+h(tu+tv)-h(tu)-h(tv)-t[G(u, v)+h(u+v)-h(u)-h(v)] =$$
$$= h(t)+[h(tu+tv)-th(u+v)]-[h(tu)-th(u)]-[h(tv)-th(v)] =$$
$$= h(t)+h(t)-h(t)-h(t) = 0,$$

proving (15). By a result of JESSEN, KARPF and THORUP [2] we know that H satisfies (13), (14) and (15) if, and only if, it is of the form

$$(17) \qquad H(u, v) = \varphi(u)+\varphi(v)-\varphi(u+v) \quad \text{for all} \quad u, v > 0,$$

where φ is a solution of (3).

To obtain the explicit form of h, we solve (16) as follows: Using the symmetry of $h(tv)$ in t and v, we get from (16) $h(t)+th(v)=h(v)+vh(t)$. By fixing in it $t=2$ this implies

$$(18) \qquad h(v) = -av+a \quad \text{for all} \quad v > 0,$$

where $a=-h(2)$ is a constant. With these known forms of H and h we can determine f, F and G. In fact,

$$f(x) = F(1-x, x) = G(1-x, x)-h(x) =$$
$$= H(1-x, x)-h(1)+h(1-x)+h(x)-h(x) = \varphi(1-x)+\varphi(x)-\varphi(1)+h(1-x) =$$
$$= \varphi(1-x)+\varphi(x)+ax$$

which is the asserted from (2).

The converse is straightforward.

References

[1] J. ACZÉL and Z. DARÓCZY, On Measures of Information and their Characterizations *(Academic Press, New York,* 1975*)*.

[2] B. JESSEN, J. KARPF and A. THORUP, Some Functional Equations in Groups and Rings, Math. Scand. **22** (1968), 257—265.

(Received January 26, 1983)