The Fourier transform of exponential polynomials

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1. Introduction. In this paper we introduce the Fourier transform of exponential polynomials which is a natural generalization of the Fourier transform of almost periodic functions and it seems to be useful to determine all exponential polynomial solutions of some functional equations, linear differential and difference equations with polynomial coefficients, some types of partial differential equations, etc. Here we indicate some possible applications too.

In what follows $\mathbf{C}$ denotes the set of complex numbers, and if f is a function defined on an Abelian group, then the functions $\tau_y f$ and $\check{f}$ will be defined by

$$(\tau_y f)(x) = f(x+y), \quad \check{f}(x) = f(-x).$$

First of all we list some notions and results concerning polynomials and exponential polynomials which we shall use in the sequel and which can be found in [1], [2], [4], [6], [7].

Let G be an Abelian topological group and V a complex topological linear space. By a V-valued polynomial on G we mean a function $p\colon G \to V$ of the form $p = \sum_{k=0}^{n} A_k$ where $A_k\colon G \to V$ is the diagonalization of a continuous k-additive, symmetric function from G^k into V. Here A_k is called a monomial of degree k if $A_k \neq 0$. The degree of polynomials is defined in a natural way. A basic fact on polynomials is that the above representation of p is unique (see e.g. [2], [4], [6]).

The continuous homomorphisms of G into the multiplicative group of nonzero complex numbers are called exponentials. By a V-valued exponential polynomial on G we mean a function $f\colon G \to V$ of the form $f = \sum_{k=0}^{n} p_k \cdot m_k$, where p_k is a V-valued polynomial and m_k is an exponential on G. Similarly to polynomials, this representation of f is unique, if the exponentials are different (see [5], [6]).

2. The Fourier transform. In this paragraph G and V will denote a topological Abelian group, and a complex topological linear space, respectively. We introduce a polynomial-valued linear operator M on the space of V-valued exponential polynomials on G by the formula $M(f) = p_0$, if p_0 is the polynomial coefficient of the exponential $m_0 = 1$ in the unique representation of the exponential polynomial f by means of polynomials and different exponentials. We remark that M is analoguous to the invariant mean on almost periodic functions, and in fact, they coincide on trigonometric polynomials. The basic properties of M are summarized in the next theorem, which is a consequence of the definition.

Theorem 2.1. *The operator M defined above is linear and has the following properties:*

(i) $M(p) = p$,

(ii) $M(pf) = pM(f)$,

(iii) $M(\tau_y f) = \tau_y[M(f)]$,

(iv) $M(\check{f}) = [M(f)]\check{}$

for all exponential polynomials f, polynomials p, and y in G.

We remark, that properties (ii)—(iii) characterize M in some sense; namely, if a polynomial-valued linear operator on the space of all exponential polynomials is homogeneous with respect to polynomials and commutes with all translations, then it is a constant multiple of M.

For a V-valued exponential polynomial f on G we define the polynomial-valued function $\hat{f}$ on the exponentials by the formula

$$\hat{f}(m) = M(f \cdot \check{m}).$$

Clearly $f \cdot \check{m}$ is an exponential polynomial and we realize $\hat{f}(m)$ as the polynomial coefficient of m in the unique representation of f. The function $\hat{f}$ will be called the Fourier transform of f. This coincides with the usual Fourier transform on trigonometric polynomials.

The fundamental properties of the map $f \to \hat{f}$ are summarized in the next two theorems which can be proved easily by theorem 2.1.

Theorem 2.2. ("Inversion theorem".) *Let $f\colon G \to V$ be an exponential polynomial. Then*

$$f = \sum \hat{f}(m)\, m$$

where the sum is taken over all exponentials m.

Theorem 2.3. *The map $f \to \hat{f}$ defined above is linear and has the following properties:*

(i) $\hat{p}(m) = 0 \quad \text{for} \quad m \neq 1$,

(ii) $(pf)\hat{}\,(m) = p \cdot \hat{f}(m)$,

(iii) $(\tau_y f)\hat{}\,(m) = m(y) \cdot (\tau_y \hat{f})(m)$,

(iv) $(\check{f})\hat{}\,(m) = [\hat{f}(\check{m})]\check{}$

for all exponential polynomials f, polynomials p, exponentials m and y in G.

In the next theorems we list further properties concerning differentiation of the Fourier-transform in the cases $G = \mathbf{R}^n$ and $G = \mathbf{R}$ which are fundamental in view of ordinary and partial differential equations. Here ∂ denotes the vector valued operator $\operatorname{grad} = (\partial_1, \ldots, \partial_n)$ on $\mathbf{R}^n$ and in the case $n = 1$ we write D instead of ∂_1. Further, I denotes the identity operator.

Theorem 2.4. *Let P be a complex polynomial in n variables and $f\colon \mathbf{R}^n \to \mathbf{C}^M$ an exponential polynomial. Then, for all exponentials m on $\mathbf{R}^n$ we have*

$$(P(\partial)f)^{\wedge}(m) = P(\partial + \partial m(0)\cdot I)\hat{f}(m).$$

PROOF. It is enough to show that $(\partial^{\alpha} f)^{\wedge}(m) = (\partial + \partial m(0)\cdot I)^{\alpha}\hat{f}(m)$ holds for each multiindex α.

By induction on $|\alpha|$, it is enough to prove that for $1 \leqq j \leqq n$ we have $(\partial_j f)^{\wedge}(m) = (\partial_j + \partial_j m(0) I)\hat{f}(m)$. But this follows from the equation:

$$(\partial_j f)^{\wedge}(m) = M(\partial_j f\cdot \check{m}) = M[\partial_j(f\cdot\check{m}) - f\cdot\partial_j\check{m}] =$$
$$= M[\partial_j(f\cdot\check{m})] - M(f\partial_j\check{m}) = \partial_j M(f\cdot\check{m}) - M[f\cdot(-\partial_j m(0))\check{m}] =$$
$$= \partial_j\hat{f}(m) + \partial_j m(0)\cdot\hat{f}(m) = (\partial_j + \partial_j m(0))\cdot\hat{f}(m)$$

as M clearly commutes with partial differentiation.

Corollary 2.5. *Let $f\colon \mathbf{R} \to \mathbf{C}^M$ be an exponential polynomial. Then, for all exponentials m on $\mathbf{R}$ and nonnegative integers k we have:*

$$(D^k f)^{\wedge}(m) = (D + m'(0)\cdot I)^k\cdot\hat{f}(m).$$

3. Applications. The general properties of the map $f \to \hat{f}$ listed in theorem 2.3 give us the possibility to determine all solutions of functional equations of the form

$$f(x+y) + g(x-y) = \sum_{i=1}^{n} h_i(x)\,k_i(y).$$

Namely, by the results of [3], [6], all solutions f, g are exponential polynomials, and in case of the linear independence of the functions $h_1, \ldots, h_n$, and also of the functions $k_1, \ldots, k_n$ the same is valid for h_i, k_i $(i=1, \ldots, n)$. Then, taking the Fourier-transform of both sides of the equation as a function of x, we have for all exponentials m and x, y in G

$$m(y)\hat{f}(m)(x+y) + m(-y)\hat{g}(m)(x-y) = \sum_{i=1}^{n} \hat{h}_i(m)(x)\cdot k_i(y).$$

Here, however we know that $\hat{f}(m)$, $\hat{g}(m)$ and $\hat{h}_i(m)$ are polynomials and comparing the monomials of the highest degree in their representations we get relations between m, $\check{m}$ and the functions k_i $(i=1, \ldots, n)$. Repeating this argument we easily obtain all relations between the representations of the unknown functions.

The property of $f \to \hat{f}$ expressed in Corollary 2.5 gives us the possibility to determine without integration all solutions of linear differential equations with constant coefficients if the right hand side is any exponential polynomial. Namely, we can reduce the problem to the determination of all polynomial solutions of equations of similar type with a polynomial right hand side by taking the Fourier transform of both sides. As polynomial solutions can be found very easily by comparing the coefficients, this latter problem is trivial. By the same method we can find a particular solution of linear differential equations with polynomial coefficients, if the right hand side is any exponential polynomial. In concrete cases the computations are very simple and systematic (see [8]).

The property of $f \to \hat{f}$ expressed in theorem 2.4 shows that by the above method we can find all exponential polynomial solutions of inhomogeneous linear partial differential equations with constant or polynomial coefficients if the inhomogeneous term is an exponential polynomial. In particular, we derive explicit formulas for the solutions of Cauchy problems for evolution-type partial differential equations if the initial functions and the inhomogeneous term are exponential polynomials.

In the case $G=\mathbf{R}^n$ all exponentials have the form $x \to e^{\langle \lambda, x \rangle}$ for some $\lambda \in \mathbf{C}^n$ (where $\langle \lambda, x \rangle = \sum_{i=1}^{n} \lambda_i x_i$) and for all exponential polynomials f, the value of the Fourier-transform of f at the exponential function $m(x)=e^{\langle \lambda, x \rangle}$ will be denoted by $\hat{f}(\lambda)$ instead of $\hat{f}(m)$.

Let P be a complex polynomial on $\mathbf{R}^n$. Then, by theorem 2.4 for all complex exponential polynomials f and $\lambda \in \mathbf{C}^n$ we have $(P(\partial)f)^{\wedge}(\lambda) = P(\partial+\lambda) \cdot \hat{f}(\lambda)$ where $\partial+\lambda$ denotes the operator $(\partial_1+\lambda_1, \ldots, \partial_n+\lambda_n)$.

Now we study the Cauchy-problem for the heat equation, that is the problem

$$\text{(1)} \qquad \begin{cases} \partial_t u = a^2 \Delta u \\ u(x, 0) = u_0(x). \end{cases}$$

We suppose that $u_0\colon \mathbf{R}^n \to \mathbf{C}$ is an exponential polynomial, and we try to find an exponential polynomial $u_0\colon \mathbf{R}^n \times \mathbf{R} \to \mathbf{C}$ which is a solution of the above problem. It is known that our problem has at most one solution. We remark, that ∂_t denotes the differential operator ∂_{n+1} on $\mathbf{R}^n \times \mathbf{R}$, and $\Delta = \sum_{i=1}^{n} \partial_i^2$.

Observing, that all exponentials on $\mathbf{R}^n \times \mathbf{R}$ have the form $(x, t) \to e^{\langle \lambda, x \rangle + \mu t}$ with some $\lambda \in \mathbf{C}^n$, $\mu \in \mathbf{C}$, we denote the value of the Fourier-transform of u at the exponential $m(x, t) = e^{\langle \lambda, x \rangle + \mu t}$ by $\hat{u}(\lambda, \mu)$. Then by taking the Fourier-transform of both sides of the first equation in (1) we obtain

$$\text{(2)} \qquad \partial_t \hat{u}(\lambda, \mu) + \mu \cdot \hat{u}(\lambda, \mu) = a^2(\Delta + 2\langle \lambda, \partial \rangle + \langle \lambda, \lambda \rangle)\hat{u}(\lambda, \mu).$$

Here $\langle \lambda, \partial \rangle = \sum_{i=1}^{n} \partial_i \lambda_i$. We know that $\hat{u}(\lambda, \mu)$ is a polynomial for all $\lambda \in \mathbf{C}^n$, $\mu \in \mathbf{C}$. For a fixed pair λ, μ let

$$\hat{u}(\lambda, \mu)(x, t) = a_N(x) t^N + \ldots + a_0(x),$$

where a_k is a polynomial $(k=0, 1, \ldots, N)$ and $a_N \neq 0$. Substituting into (2) and comparing the coefficients of t^N we have $\mu = a^2 \langle \lambda, \lambda \rangle$. Then comparing the coefficients of t^k for $k=0, 1, \ldots, N-1$, we have

$$a_{k+1}(x) = \frac{a^2}{k+1}(\Delta + 2\langle \lambda, \partial \rangle) a_k(x).$$

Obviously $a_0(x) = \hat{u}_0(\lambda)(x)$, and hence

$$a_k(x) = \frac{a^{2k}}{k!}(\Delta + 2\langle \lambda, \partial \rangle)^k \hat{u}_0(\lambda)(x) \quad (k = 0, 1, \ldots, N).$$

Here N denotes the smallest nonnegative integer for which $a_N \neq 0$, $a_{N+1} = 0$. The existence of such an N follows from the fact that $\hat{u}_0(\lambda)$ is a polynomial. (We supposed here, that $\hat{u}_0(\lambda)\neq 0$.)

Using the inversion formula we have

Theorem 3.1. *Let* u_0: $\mathbf{R}^n\to\mathbf{C}$ *be an exponential polynomial. Then the unique solution of the Cauchy-problem* (1) *can be written in the form:*

$$u(x,t)=\sum_{\lambda\in\mathbf{C}^n}\sum_{N=0}^{\infty}\frac{[a^2(\Delta+2\langle\lambda,\partial\rangle)]^N}{N!}\hat{u}_0(\lambda)(x)\cdot t^N\cdot e^{\langle\lambda,x\rangle+a^2\langle\lambda,\lambda\rangle t}$$

for all $x\in\mathbf{R}^n$, $t\in\mathbf{R}$.

(Here both sums are actually finite and the values of u can be computed easily without integration.)

A straightforward extension of our result is the following:

Theorem 3.2. *Let* u_0: $\mathbf{R}^n\to\mathbf{C}$ *be an exponential polynomial. Then the unique solution of the Cauchy-problem*

$$\begin{cases}\partial_t u &= a^2\Delta u\\ u(x,t_0) &= u_0(x),\end{cases}\tag{3}$$

can be written in the form

$$u(x,t)=\sum_{\lambda\in\mathbf{C}^n}\sum_{N=0}^{\infty}\frac{[a^2(\Delta+2\langle\lambda,\partial\rangle)]^N}{N!}\hat{u}_0(\lambda)(x)(t-t_0)^N e^{\langle\lambda,x\rangle+a^2\langle\lambda,\lambda\rangle(t-t_0)}\tag{4}$$

for all $x\in\mathbf{R}^n$, $t\in\mathbf{R}$.

The next step is to solve the inhomogeneous Cauchy-problem

$$\begin{cases}\partial_t u &= a^2\Delta u+f(x,t),\\ u(x,0) &= u_0(x).\end{cases}\tag{5}$$

Here we suppose, that u_0: $\mathbf{R}^n\to\mathbf{C}$ and f: $\mathbf{R}^n\times\mathbf{R}\to\mathbf{C}$ are exponential polynomials, and we seek an exponential polynomial u: $\mathbf{R}^n\times\mathbf{R}\to\mathbf{C}$ which is a solution of (5). Of course the solution is unique again, and we can reduce the problem to the problem with homogeneous initial data

$$\begin{cases}\partial_t u &= a^2\Delta u+f(x,t),\\ u(x,0) &= 0.\end{cases}\tag{6}$$

In order to solve (6) we make a simple observation: if the function $(x,t)\to v(x,t,t_0)$ is a solution of (3) with $u_0(x)=f(x,t_0)$, then the function $(x,t)\to\int_0^t v(x,t,\tau)\,d\tau$ is a solution of (6).

Using (4) we see, that the function

$$u(x,t)=\sum_{\lambda\in\mathbf{C}^n}\sum_{\mu\in\mathbf{C}}\sum_{N=0}^{\infty}\int_0^t\frac{[a^2(\Delta+2\langle\lambda,\partial\rangle)]^N}{N!}\hat{f}(\lambda,\mu)(x,\tau)e^{\mu\tau+a^2(t-\tau)\langle\lambda,\lambda\rangle}(t-\tau)^N\,d\tau\cdot e^{\langle\lambda,x\rangle}$$

is a solution of (6). If we use the Taylor-formula

$$\hat{f}(\lambda,\mu)(x,\tau)=\sum_{k=0}^{\infty}\frac{\partial_{n+1}^{k}\hat{f}(\lambda,\mu)(x,t)}{k!}(-1)^{k}(t-\tau)^{k}$$

which is a finite sum as $\hat{f}(\lambda,\mu)$ is a polynomial, then we see that the only task is to compute integrals of the form

$$\int_0^t (t-\tau)^{N+k}e^{[a^2\langle\lambda,\lambda\rangle-\mu](t-\tau)}d\tau.$$

We now put

$$I_M(\alpha)=\int_0^t s^M e^{\alpha s}\,ds$$

for each nonnegative integer M and $\alpha\in\mathbf{C}$. The following theorem is an easy consequence of 3.1 and 3.2.

Theorem 3.3. *Let $f\colon \mathbf{R}^n\times\mathbf{R}\to\mathbf{C}$, $u_0\colon \mathbf{R}^n\to\mathbf{C}$ be exponential polynomials. Then the unique solution of the Cauchy-problem* (5) *can be written in the form $u=u_1+u_2$ where*

$$u_1(x,t)=\sum_{\lambda\in\mathbf{C}^n}\sum_{N=0}^{\infty}\frac{[a^2(\Delta+2\langle\lambda,\partial\rangle)]^N}{N!}\hat{u}_0(\lambda)(x)t^N\cdot e^{\langle\lambda,x\rangle},\tag{7}$$

$$u_2(x,t)=\sum_{\lambda\in\mathbf{C}^n}\sum_{\mu\in\mathbf{C}}\sum_{N=0}^{\infty}\sum_{k=0}^{\infty}(-1)^k\frac{[a^2(\Delta+2\langle\lambda,\partial\rangle)]^N}{N!\,K!}\partial_{n+1}^{k}\hat{f}(\lambda,\mu)(x,t)\cdot \cdot I_{N+k}(a^2\langle\lambda,\lambda\rangle-\mu)\,e^{\langle\lambda,x\rangle+\mu t}\tag{8}$$

for all $x\in\mathbf{R}^n$, and $t\in\mathbf{R}$, where for $\alpha\neq 0$

$$I_M(\alpha)=e^{\alpha t}\sum_{j=0}^{M}(-1)^j\frac{M!}{(M-j)!}\frac{t^{M-j}}{\alpha^{j+1}}-(-1)^M\frac{M!}{\alpha^{M+1}},\quad I_M(0)=\frac{t^{M+1}}{M+1}.$$

(All sums are actually finite.)

The preceding results can easily be extended for evolution equations of more general type, as for instance the Cauchy-problem

$$\begin{cases}\partial_t u &= P(\partial)u+f(x,t),\\ u(x,0)&=u_0(x)\end{cases}\tag{9}$$

where P is a polynomial in n variables, $\partial=(\partial_1,\ldots,\partial_n)$ and $\partial_t=\partial_{n+1}$, further $f\colon\mathbf{R}^n\times\mathbf{R}\to\mathbf{C}$ and $u_0\colon\mathbf{R}^n\to\mathbf{C}$ are exponential polynomials. By the same method we can produce an exponential polynomial solution, which is the only solution if uniqueness is guaranteed. As special cases we get explicit (solutions) of the Cauchy-problem for the Schrödinger, or biharmonic and other equations.

Theorem 3.4. *Let $f\colon \mathbf{R}^n\times\mathbf{R}\to\mathbf{C}$, $u_0\colon\mathbf{R}^n\to\mathbf{C}$ be exponential polynomials. Then* *the exponential polynomial* $u=u_1+u_2$, *where* u_1, u_2 *are defined by*

$$u_1(x,t)=\sum_{\lambda\in\mathbf{C}^n}\sum_{N=0}^{\infty}\frac{[P(\partial+\lambda)-P(\lambda)]^N}{N!}\hat{u}_0(\lambda)(x)\,t^N\cdot e^{\langle\lambda,x\rangle+P(\lambda)t}, \tag{10}$$

$$u_2(x,t)= \tag{11}$$

$$=\sum_{\lambda\in\mathbf{C}^n}\sum_{\mu\in\mathbf{C}}\sum_{N=0}^{\infty}\sum_{k=0}^{\infty}(-1)^k\frac{[P(\partial+\lambda)-P(\lambda)]^N}{N!\,k!}\,\partial_{n+1}^k\hat{f}(\lambda,\mu)(x,t)\,I_{N+k}\big(P(\lambda)-\mu\big)e^{\langle\lambda,x\rangle+\mu t}$$

for $x\in\mathbf{R}^n$, $t\in\mathbf{R}$ *is a solution of the Cauchy-problem* (9).

PROOF. The proof proceeds either by direct computation, as the sums are finite (which follows from the fact, that u_0, f are exponential polynomials, and hence $\hat{u}_0(\lambda)$, $\hat{f}(\lambda,\mu)$ are polynomials), or repeating the arguments used in proving Theorem 3.2.

Example 3.5. As an illustration we solve the Cauchy-problem for the Schrödinger-equation

$$\begin{cases}\partial_t u = i\Delta u+x\cos t-y^2\sin t,\\ u(x,y,0)=x^2+y^2.\end{cases}$$

Using the notations of theorem 3.1 we have here:

$$P(\lambda_1,\lambda_2)=i(\lambda_1^2+\lambda_2^2),$$

$$\hat{u}_0(\lambda_1,\lambda_2)(x,y)=\begin{cases}x^2+y^2 & \text{for } \lambda_1=\lambda_2=0\\ 0 & \text{otherwise,}\end{cases}$$

$$\hat{f}(\lambda_1,\lambda_2,\mu)(x,y,t)=\begin{cases}\dfrac{x}{2}-\dfrac{y^2}{2i} & \text{for } \lambda_1=\lambda_2=0,\quad \mu=i,\\[2mm] \dfrac{x}{2}+\dfrac{y^2}{2i} & \text{for } \lambda_1=\lambda_2=0,\quad \mu=-i,\\[2mm] 0 & \text{otherwise.}\end{cases}$$

Finally,

$$I_0(\pm i)=\frac{e^{\pm i}-1}{\pm i},\quad I_1(\pm i)=\frac{te^{\pm it}}{\pm i}+e^{\pm it}-1,$$

and hence by (10) and (11) we obtain

$$u_1(x,y,t)=x^2+y^2+4it,$$

$$u_2(x,y,t)=x\sin t+y^2(\cos t-1)-2it+2i\sin t.$$

Thus the solution is

$$u(x,y,t)=x\sin t+x^2+y^2\cos t+2i(t+\sin t).$$

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