Inhomogeneous discriminant form and index form equations and their applications

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1. Introduction

Let L, K be algebraic number fields with $L \subset K$, $[L:\mathbf{Q}]=l$ and $[K:L]=n \geqq 3$. Let $\alpha_1=1, \alpha_2, \ldots, \alpha_k$ be algebraic integers of K, linearly independent over L and let $0 \neq \beta \in \mathbf{Z}_L$. [1]) In our previous papers [2], [3] we considered inhomogeneous norm form equations of type

$$N_{K/L}(x_1+\alpha_2 x_2+\ldots+\alpha_k x_k+\lambda)=\beta \tag{1}$$

where $x_1, \ldots, x_k \in \mathbf{Z}_L$ are dominating variables, and $\lambda \in \mathbf{Z}_K$ is a non-dominating variable such that[2]) $\overline{|\lambda|}<(\max_{1 \leqq i \leqq k} \overline{|x_i|})^{1-\xi}$ (ξ is a given small positive constant). Under certain assumptions concerning $\alpha_2, \ldots, \alpha_k$, in [2], [3] we gave effective upper bounds for the sizes of all solutions $x_1, \ldots, x_k$, λ of equation (1). Our theorems generalized some results of SPRINDŽUK [23], GYŐRY and PAPP [20], [21] and KOTOV [22] concerning norm form equations.

The purpose of the present paper is to generalize discriminant form and index form equations to the inhomogeneous case in a similar way, as we generalized norm form equations. We shall derive effective upper bounds for the solutions of inhomogeneous discriminant form and index form equations. Further, we shall present some applications to algebraic integers with given discriminant and with given index, respectively.

Our results will be deduced from an effective theorem of GYŐRY [8] concerning algebraic integers with given degree and given discriminant.

2. Results

Let $L \subset K$ be algebraic number fields as above with $l=[L:\mathbf{Q}]$ and let $[K:L]=n \geqq 2$. Further let $\alpha_0=1, \alpha_1, \ldots, \alpha_k$ be algebraic integers in K, linearly independent over L, and let $0 \neq \delta \in \mathbf{Z}_L$. Let us consider the discriminant form equation

$$D_{K/L}(\alpha_1 x_1+\ldots+\alpha_k x_k)=\delta \tag{2}$$

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[1]) $\mathbf{Z}_M$ will denote the ring of integers of an arbitrary algebraic number field M.

[2]) $\overline{|\alpha|}$ denotes the size of an algebraic number α, i.e. the maximum absolute value of its conjugates.

in $x_1, \dots, x_k \in \mathbf{Z}_L$. The discriminant form $D_{K/L}(\alpha_1 X_1 + \dots + \alpha_k X_k)$ in equation (2) is a decomposable form of degree $n(n-1)$ with coefficients in $\mathbf{Z}_L$. Effective bounds for all solutions $x_1, \dots, x_k$ of equation (2) were given by Győry [6], [7] and later by Győry and Papp [19], [20]. For $\mathfrak{p}$-adic extensions and certain generalizations see Győry and Papp [19] and Győry [10], [11], [13], [14], [16], [18].

As an inhomogeneous generalization of equation (2) we may consider the equation

$$D_{K/L}(\alpha_1 x_1 + \dots + \alpha_k x_k + \lambda) = \delta \tag{3}$$

where the dominating variables are $x_1, \dots, x_k \in \mathbf{Z}_L$ and $\lambda \in \mathbf{Z}_K$ is a non-dominating variable such that $\overline{|\lambda|} < C_0 \max_{1 \le i \le k} \overline{|x_i|}$ with a given small positive constant C_0.

In order to formulate our Theorem 1, we need some further notation. Suppose that in (3) $\overline{|\alpha_i|} \le A$ $(i=1, \dots, k)$ and let $|N_{L/Q}(\delta)| \le d$ $(d \ge 2)$. Let us denote by D_L the absolute value of the discriminant of L. Let

$$C_0 = \left(4k^{(kl+1)/2}(2A)^{kl-1}\right)^{-1}$$

and let

$$C_1 = n\,\overline{|\delta|}^{1/(n-1)} \exp\{(5\ln^3)^{30\ln^3}[(dD_L^n)^{3/2}(\log dD_L)^{ln}]^{3n^2}\}.$$

Theorem 1. *If* $x_1, \dots, x_k$, λ *is a solution of equation* (3) *with* $x_1, \dots, x_k \in \mathbf{Z}_L$, $\lambda \in \mathbf{Z}_K$ *and* $\overline{|\lambda|} < C_0 \max_{1 \le i \le k} \overline{|x_i|}$ *then*

$$\max_{1 \le i \le k} \overline{|x_i|} < C_0^{-1} C_1. \tag{4}$$

In the special case $\lambda = 0$ our theorem provides an effective upper bound for the solutions of equation (2). Our Theorem 1 includes e.g. Theorem 4 of Győry and Papp [20] (with another estimate).

Let O be an order in the field extension K/L (i.e. let O be a subring of $\mathbf{Z}_K$ containing $\mathbf{Z}_L$ that has the full dimension n as a $\mathbf{Z}_L$-module), and suppose that O has a relative integral basis of the form $\{1, \alpha_2, \dots, \alpha_n\}$ over L with $\overline{|\alpha_i|} \le A$ $(i=2, \dots, n)$. Let $x_2, \dots, x_n \in \mathbf{Z}_L$ and $\lambda \in O$. Then the discriminant of $\alpha_2 x_2 + \dots + \alpha_n x_n \in O$ over L can be written in the form

$$D_{K/L}(\alpha_2 x_2 + \dots + \alpha_n x_n) = [F(x_2, \dots, x_n)]^2 D_{K/L}(1, \alpha_2, \dots, \alpha_n)$$

where $F(X_2, \dots, X_n)$ denotes the index form of the basis $\{1, \alpha_2, \dots, \alpha_n\}$ of O over L, which is a decomposable form of degree $\frac{n(n-1)}{2}$ in $X_2, \dots, X_n$. Similarly

$$D_{K/L}(\alpha_2 x_2 + \dots + \alpha_n x_n + \lambda) = [G(x_2, \dots, x_n, \lambda)]^2 D_{K/L}(1, \alpha_2, \dots, \alpha_n)$$

where $G(x_2, \dots, x_n, \lambda) \in \mathbf{Z}_L$ is the index of $\alpha_2 x_2 + \dots + \alpha_n x_n + \lambda \in O$ over L with respect to the basis $\{1, \alpha_2, \dots, \alpha_n\}$ of O. Since

$$G(X_2, \dots, X_n, 0) = F(X_2, \dots, X_n)$$

hence $G(X_2, \dots, X_n, \lambda)$ may be called an "inhomogeneous" index form of the basis $\{1, \alpha_2, \dots, \alpha_n\}$ of O over L.

Let $0 \neq \beta \in \mathbf{Z}_L$. Effective bounds for all solutions $x_2, \ldots, x_n \in \mathbf{Z}_L$ of the index form equation

$$F(x_2, \ldots, x_n) = \beta \tag{5}$$

were given by GYŐRY [6], [7] and GYÖRGY and PAPP. These results were extended to the p-adic case by TRELINA [24], GYŐRY and PAPP [19] and GYŐRY [10], [11], [13], [14]. We remark that recently GYŐRY [16], [18] obtained effective results on discriminant form and index form equations also in the more general case, when the ground ring is an arbitrary integral domain, finitely generated over $\mathbf{Z}$.

Let us consider the equation

$$G(x_2, \ldots, x_n, \lambda) = \beta \tag{6}$$

with variables $x_2, \ldots, x_n \in \mathbf{Z}_L$ and $\lambda \in O$, which may be called an inhomogeneous index form equation.

Theorem 2. *If* $x_2, \ldots, x_n$, λ *is a solution of equation* (6) *with* $x_2, \ldots, x_n \in \mathbf{Z}_L$, $\lambda \in O$ *and* $\overline{|\lambda|} < C_0 \max_{2 \leq i \leq n} \overline{|x_i|}$ *then*

$$\max_{2 \leq i \leq k} \overline{|x_i|} < \exp\{(5\, ln^3)^{33\, ln^3} [(\overline{|\beta|}^{2l} A^{ln} D_L^n)^{3/2} (\log \overline{|\beta|}\, A D_L)^{ln}]^{3n^2}\}. \tag{7}$$

In the special case $\lambda = 0$ our Theorem 2 gives Theorem 4 of GYŐRY and PAPP [19] (with a different estimate).

Discriminant form and index form equations have several applications (see e.g. GYŐRY [12]). We shall show that their inhomogeneous versions have also applications.

Let L, l and D_L be as above. We shall call the algebraic integers α and α^* $\mathbf{Z}_L$-equivalent if $\alpha - \alpha^* \in \mathbf{Z}_L$. In this case it is clear that $D_{L(\alpha)/L}(\alpha) = D_{L(\alpha^*)/L}(\alpha^*)$. In a series of papers Győry [4], [5], [6], [7], [8], [9], [15] examined polynomials with integer coefficients and given discriminant. As an application of his results Győry proved that there are only finitely many pairwise $\mathbf{Z}_L$-inequivalent algebraic integers α with given degree and with given non-zero discriminant over L, and such a system of algebraic integers can be effectively determined. Győry obtained a similar result for algebraic integers with given index. Further, he extended these results to the more general case when the ground ring is an arbitrary integral domain, finitely generated over $\mathbf{Z}$ (see [17]).

In the remaining part of this paper we shall give certain inhomogeneous versions of these theorems concerning algebraic integers of given discriminant and of given index, respectively. We prove in an effective way that there are only finitely many pairwise $\mathbf{Z}_L$-inequivalent algebraic integers α of given degree (over L) such that $\alpha + \lambda$ is of given degree and of given discriminant over L for some algebraic integer λ which is "small" compared to α in a certain sense. In the case $L = \mathbf{Q}$ we have as a consequence a similar result for algebraic integers of the form $\alpha + \lambda$ with a given index.

Theorem 3. *Suppose that* α *and* λ *are algebraic integers,* α *is of degree* $m \geq 2$ *over* L *and* $K = L(\alpha + \lambda)$ *is of degree* $n \geq 2$ *over* L. *If* $\overline{|\lambda|} < \frac{1}{4} \overline{|D_{L(\alpha)/L}(\alpha)|}^{1/m(m-1)}$ *and*

$$D_{K/L}(\alpha + \lambda) = \delta \tag{8}$$

where $0\neq\delta\in\mathbf{Z}_L$ *with* $|N_{L/Q}(\delta)|\leqq d$ *then there exists an algebraic integer* α^*, $\mathbf{Z}_L$-*equivalent to* α *such that*

$$\overline{|\alpha^*|} < 2C_1 \tag{9}$$

with the constant C_1 *occurring in Theorem 1.*

We remark that our theorem implies that $\overline{|\lambda|}$ is also bounded and (9) makes possible to derive an effective upper bound for the size of $D_{L(\alpha)/L}(\alpha)=D_{L(\alpha^*)/L}(\alpha^*)$ which gives a bound for $\overline{|\lambda|}$.

In the special case $\lambda=0$ our Theorem 3 includes e.g. Theorem 3A of GYŐRY [8] (whit another estimate).

Finally, for algebraic integers of given index we have the following theorem:

Theorem 4. *Let K be an algebraic number field of degree* $n\geqq 2$, *let* $\alpha,\lambda\in\mathbf{Z}_K$ *and suppose that* α *is of degree* $m\geqq 2$ *over* $\mathbf{Q}$. *If* $\overline{|\lambda|}<\frac{1}{4}|D_{\mathbf{Q}(\alpha)/\mathbf{Q}}(\alpha)|^{1/m(m-1)}$ *and the index of* $\alpha+\lambda$ *is*

$$I(\alpha+\lambda) = I \tag{10}$$

where $0\neq I\in\mathbf{Z}$ *then there exists an algebraic integer* α^*, $\mathbf{Z}$-*equivalent to* α *such that*

$$\overline{|\alpha^*|} < \exp\{(6n^3)^{30n^3}[(I^2D_K)^{3/2}(\log|I|D_K)^n]^{3n^2}\} \tag{11}$$

where D_K *denotes the absolute value of the discriminant of K.*

In the special case $\lambda=0$ this theorem gives Corollaire 3.2 of GYŐRY [6] (with another bound).

3. Proofs

Our theorems can be easily deduced from the following result of Győry [8].

Theorem A. *If* α *is an algebraic integer of degree* $n\geqq 2$ *over L with* $D_{L(\alpha)/L}(\alpha)=\delta\neq 0$ *and* $|N_{L/\mathbf{Q}}(\delta)|\leqq d$, *then there exists an algebraic integer* α^*, $\mathbf{Z}_L$-*equivalent to* α *such that*

$$\overline{|\alpha^*|} < C_1$$

with the constant C_1 *occurring in Theorem 1.*

This theorem is Theorem 3A of GYŐRY [8] (for $n=2$ see also the remark after Theorem 1 in [8]). We remark that the proof of this theorem involves Baker's method (see e.g. [1]).

PROOF OF THEOREM 1. This short proof of Theorem 1 was suggested by K. Győry.

Let $x_1,\dots,x_k\in\mathbf{Z}_L$ and $\lambda\in\mathbf{Z}_K$ be a solution of equation (3) with $\overline{|\lambda|}<C_0X$ where $X=\max_{1\leqq i\leqq k}\overline{|x_i|}$. Denote by γ_i $(i=1,\dots,n)$ the conjugates of any $\gamma\in K$ over L and let $l(\underline{x})=\alpha_1x_1+\dots+\alpha_kx_k$. Applying Theorem A to equation (3) we obtain

$$l(\underline{x})+\lambda = \alpha^*+a \tag{12}$$

where $\alpha^*\in\mathbf{Z}_K$ with $\overline{|\alpha^*|}<C_1$ and $a\in\mathbf{Z}_L$. Put $l_{ij}(\underline{x})=l_i(\underline{x})-l_j(\underline{x})$, $\lambda_{ij}=\lambda_i-\lambda_j$ and $\alpha_{ij}^*=\alpha_i^*-\alpha_j^*$ for any $i\neq j$, $1\leqq i,j\leqq n$. With this notation (12) implies

$$l_{ij}(\underline{x})+\lambda_{ij}=\alpha_{ij}^*$$

whence

$$\overline{|l_{ij}(\underline{x})|}\leqq\overline{|\alpha_{ij}^*|}+\overline{|\lambda_{ij}|}\leqq 2C_1+2C_0X \tag{13}$$

for any $i\neq j$. Here C_1, C_0 denote the expressions specified in the theorem. Since by assumption $\alpha_0=1,\alpha_1,\dots,\alpha_k$ are linearly independent over L, hence the equation system $l_{ij}(\underline{x})=0$ $(1\leqq i<j\leqq n)$ has no nontrivial solution in $\mathbf{C}$. Solving the equation system $l_{ij}(\underline{x})=\alpha_{ij}^*-\lambda_{ij}$ $(1\leqq i<j\leqq n)$ by Cramer's rule, and applying (13) we get

$$X<k^{(kl+1)/2}(2A)^{kl-1}(2C_1+2C_0X)=\frac{1}{4}C_0^{-1}(2C_1+2C_0X)=\frac{1}{2}C_0^{-1}C_1+\frac{1}{2}X$$

that is $X<C_0^{-1}C_1$ and thus (4) is proved.

Proof of Theorem 2. If $x_2,\dots,x_n\in\mathbf{Z}_L$ and $\lambda\in O$ is a solution of (6) then

$$D_{K/L}(\alpha_2x_2+\dots+\alpha_nx_n+\lambda)=\beta^2D_{K/L}(1,\alpha_2,\dots,\alpha_n). \tag{14}$$

Now we have $\overline{|\beta^2D_{K/L}(1,\alpha_2,\dots,\alpha_n)|}\leqq\overline{|\beta|}^2n^{n/2}A^n$. Applying our Theorem 1 to (14) with $k=n-1$ and with $\delta=\beta^2D_{K/L}(1,\alpha_2,\dots,\alpha_n)$ and using $\overline{|\delta|}\leqq\overline{|\beta|}^2n^{n/2}A^n$, $d\leqq\overline{|\delta|}^l$, we get (7).

Our Theorem 3 and Theorem 4 could be deduced (with other bounds) from Theorem 1 and Theorem 2 respectively, but it is easier to derive them from Theorem A.

Proof of Theorem 3. Applying Theorem A to (8) we obtain

$$\alpha+\lambda=\alpha^*+a \tag{15}$$

where $\alpha^*\in\mathbf{Z}_K$ with $\overline{|\alpha^*|}<C_1$ and $a\in\mathbf{Z}_L$. Let $K'=L(\alpha,\lambda)$ and let us denote the conjugates of any $\gamma\in K'$ over L by γ_i, $i=1,\dots,n'$, where $n'=[K':L]\leqq nm$. By assumption we have $\overline{|\lambda|}<\frac{1}{4}\overline{|D_{L(\alpha)/L}(\alpha)|}^{1/m(m-1)}$ and thus

$$\overline{|\lambda|}<\frac{1}{4}\overline{|\alpha_i-\alpha_j|} \tag{16}$$

where $\overline{|\alpha_i-\alpha_j|}=\max_{1\leqq i_1,i_2\leqq n'}\overline{|\alpha_{i_1}-\alpha_{i_2}|}$. From (15) we get

$$\alpha_i-\alpha_j+\lambda_i-\lambda_j=\alpha_i^*-\alpha_j^*$$

whence

$$\overline{|\alpha_i-\alpha_j|}\leqq 2\overline{|\lambda|}+2C_1\leqq\frac{1}{2}\overline{|\alpha_i-\alpha_j|}+2C_1$$

that is $\overline{|\alpha_i-\alpha_j|}<4C_1$. Comparing this with (16) we have $\overline{|\lambda|}<C_1$. Now let $\alpha^{**}=$ $=\alpha^*-\lambda$, then $\overline{|\alpha^{**}|}<2C_1$ and by (15) α is $\mathbf{Z}_L$-equivalent to α^{**}, which proves (9).

PROOF OF THEOREM 4. From (10) we get

$$|D_{K/\mathbf{Q}}(\alpha+\lambda)| = I^2 D_K. \tag{17}$$

Applying our Theorem 3 to (17) with $L=\mathbf{Q}$, $l=1$ and $d=|\delta|=I^2 D_K$ we get (11).

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